Is a curved flight path in SAR better than a straight one?

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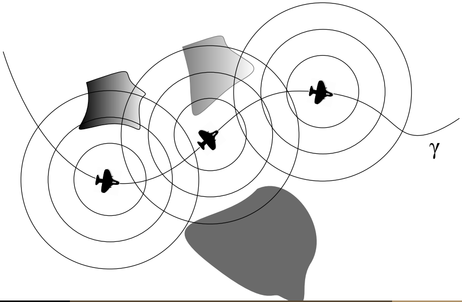
Based on a joint work with Gunther Uhlmann

In Synthetic Aperture Radar (SAR) imaging a plane flies along a curve in $\mathbf{R}^3$ and collects data from the surface, that we consider flat. A simplified model of this is to project the curve on the plane, call it γ; then the data are integrals of a unknown density function on the surface over circles with various radii centered at the curve. Then the model is the inversion of the circular transform

$$R_\gamma f(r,p) = \int_{|x-p|=r} f(x)\,\mathrm{d}\ell(x), \quad p \in \gamma,\ r \geq 0. \tag{1}$$

$\mathrm{d}\ell(x) =$ the Euclidean arc-length measure.



This transform has been studied extensively; injectivity sets for R_γ on C_0^∞ have been described in full (Agranovski and Quinto). In particular, each non-flat curve, does not matter how small, is enough for uniqueness. Without the compact support assumption, there is no injectivity. In view of the direct relation to the wave equation, this transform, and its 3D analog have been studies extensively as well and in particular in thermoacoustic tomography with constant speed, for example by Agranovsky Kuchment, Ambartsoumian, Finch, Rakesh, Haltmeier, Patch.

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The problem we study is the following: what part of the wave front set WF(f) can we recover? Clearly, we can only hope to recover the *visible* singularities: those conormal to the circles involved in the transform.

If γ is a straight line, there is obvious non-uniqueness due to symmetry, called *left-right ambiguity*. Moreover, we can have cancellation of singularities symmetric about that line. More precisely, we can recover the singularities of the even part of f and cannot recover those of the odd part. When using $R_\gamma^* R_\gamma f$ as a method for recovery, we get symmetric images of the actual singularities (artifacts), and we do not not know whether they are real on not.

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Straight line flight path

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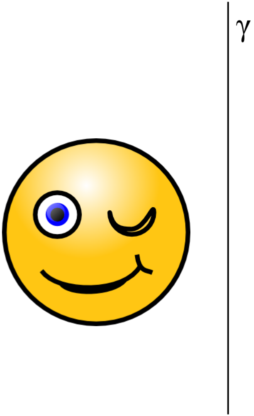

Figure: Original

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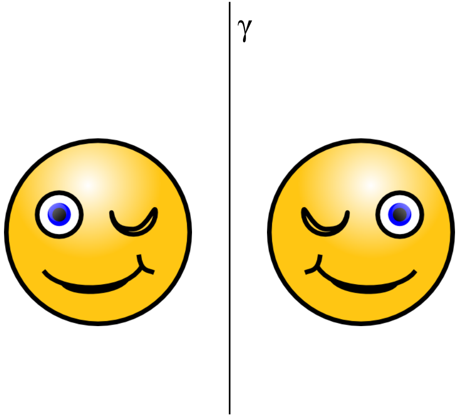

Figure: Original and an "artifact"

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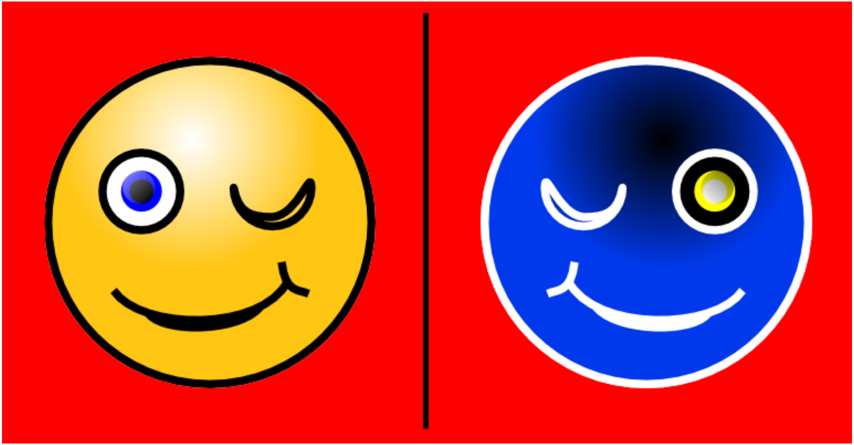

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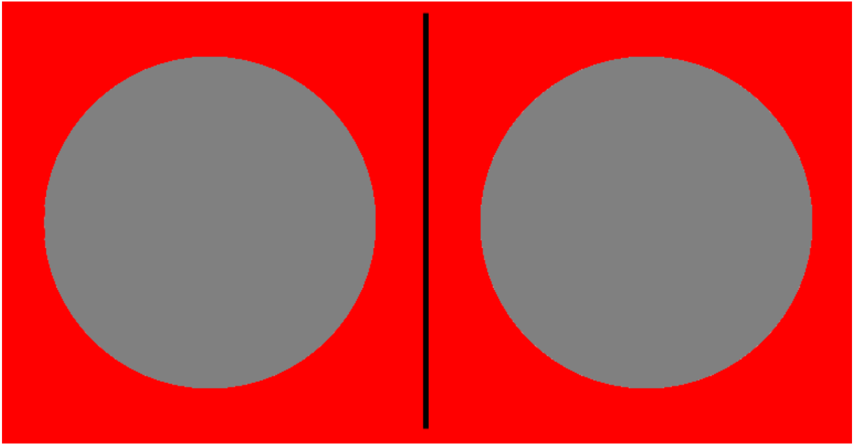

Figure: Cancelation of singularities

Figure: Original

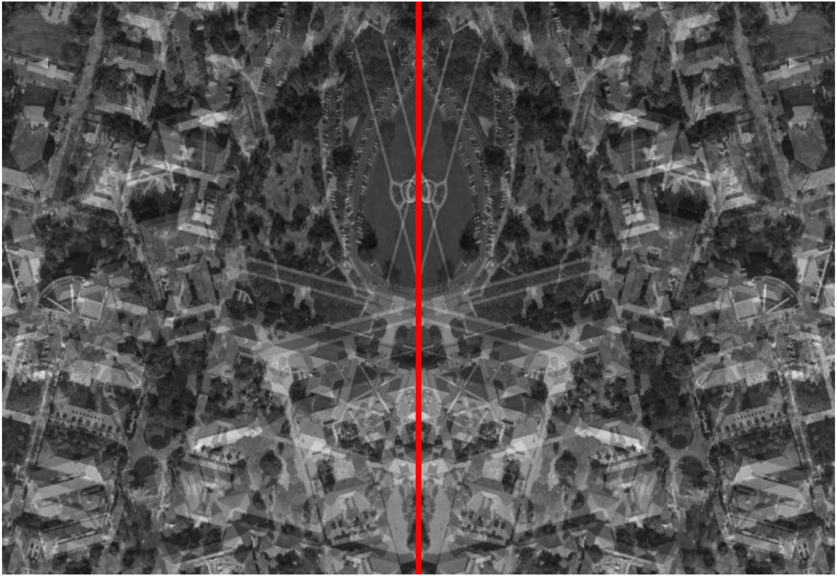

Figure: Left-right ambiguity

Why R^*R?

Why R^*R? If we study injectivity, R^*R and R are injective at the same time (proof: $(R^*Rf, f) = \|Rf\|^2$). There is some microlocal equivalent of that which has not been studied. But even then, we would get that knowing $R_\gamma f$ is equivalent to knowing $R_\gamma^* R_\gamma f$ (mod C^∞). We still have to solve one of those equations however!

If no artifacts, i.e., everything on one side of the curve, then R^*R is a ΨDO elliptic at the visible singularities; problem solved. . . almost — with amplitudes proportional on the number of times the singularity is detected.

In the general case, a typical structure of R^*R is

$$R^*R = \Psi\text{DO} + \text{FIO}$$

and the FIO is the one creating the artifacts. When γ is a straight line, then the FIO is just the symmetry about it. If the FIO is of the right class and a lower order, then the ΨDO dominates and we are done; but often, this is not the case. Then we still want to invert the ΨDO+FIO part.

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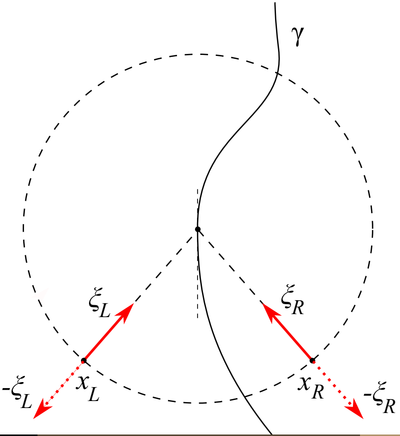
Based on the straight line example, it has been suggested that a curved trajectory γ might be a batter flight path. Nolan and Cheney 2003: numerical examples suggesting that when the curvature of γ is non-zero, the artifacts are "weaker", and increasing of the curvature, they become even weaker. By artifacts, they mean singularities in the wave front set of $R_\gamma^* R_\gamma f$ that are not in $\mathrm{WF}(f)$ located at *mirror points*:



Wave equation model: studied from a point of view of FIOs by Nolan and Cheney 2004 and Felea 2007. The artifacts have been explained in terms of the FIO part of the Lagrangian of $R_\gamma^* R_\gamma$. They are of the same strength, as an order of the corresponding FIO. More precisely, this is true at least away from the set of measure zero consisting of the points whose projections to the base falls on γ (points right below the plane's path, i.e., $r = 0$), and for (x, ξ) such that the line trough it is tangent to γ at some point. The latter set is responsible for existence of a submanifold of the Lagrangian near which the left and right projections are not diffeomorphisms.

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Artifact removal?

In other words, if we use R^*R, there are many works explaining where we may get artifacts. What remains unclear is if those artifacts are unavoidable, i.e.,

- if there is lack of uniqueness, and (if not)
- if they can be resolved constructively by some other method.

In a nutshell, the main result we prove is that

no better!

A curved path is not better!

Cancellation of singularities always occurs, and there is no unique recovery. We describe the microlocal kernel. For simplicity, we stay away from points on γ or singularities that hit γ tangentially.

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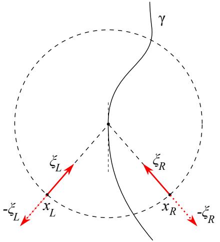
The microlocal kernel of R_γ

We use the subscripts L/R for singularities which hit γ from the Left/Right.



Let $f = f_L + f_R$, where $f_{L,R}$ have left (right) singularities hitting γ only once.

Theorem 1

$$R_\gamma(f_L + f_R) \in C^\infty(\Sigma_\gamma) \iff f_R - Uf_L \in C^\infty(\Sigma_R),$$

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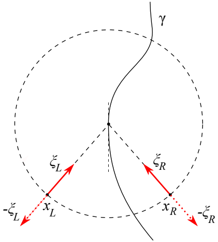

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In other words, regardless of the shape of γ (a line or a curve), singularities can always cancel! The "artifacts" are unavoidable, and they are not weaker, they are unitary images of the original!

Idea of the Proof: We relate $\mathbf{R}_\gamma$ to the solution of the wave equation in $\mathbf{R}^2$. Let u solve the problem

$$\begin{cases} (\partial_t^2 - \Delta)u &= 0 \quad \text{in } \mathbf{R}_t \times \mathbf{R}_x^2, \\ u|_{t=0} &= 0, \\ \partial_t u|_{t=0} &= f. \end{cases}$$

Solution restricted to γ:

$$u_\gamma = (A \otimes I)R_\gamma, \quad Ah(t) = \int_0^t \frac{rh(r)}{\sqrt{t^2 - r^2}}\,\mathrm{d}r, \quad t > 0.$$

Here, $A \otimes I$ just means that we apply A with respect to the t variable only. The presence of A is more of nuisance than a problem because A is an elliptic ΨDO of order $-1/2$ on $\mathbf{R}_+$.

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$$\Lambda f := u|_{\mathbf{R}_+ \times \gamma}.$$

Let $u = u_L + u_R$, corresponding to $f_{L,R}$. We know that $u_L + u_R \in C^\infty$. Given f_L, we want to solve

$$\Lambda(f_L + f_R) \in C^\infty$$

for f_R. This boils down to the question whether Λ is microlocally invertible. Not in principle (which is the whole point of the theorem) but restricted to singularities on the right only, (or on left only), it is. Explicit microlocal inversion can be done by back-projection to one side of γ. Then

$$f_R = -\Lambda_R^{-1} \Lambda_L f_L,$$

where $\Lambda_{L,R}$ are the corresponding restrictions. The unitarity of $U := \Lambda_R^{-1} \Lambda_L$ follows by energy preservation.

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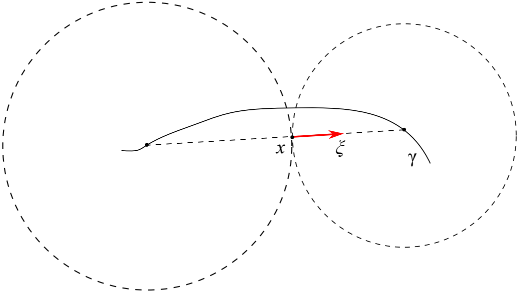
We assumed above that each singularity hits γ once. In some cases (γ *has to be curved for that!*) each singularity leaves two traces on γ. Can we use this extra information to recover the singularities?



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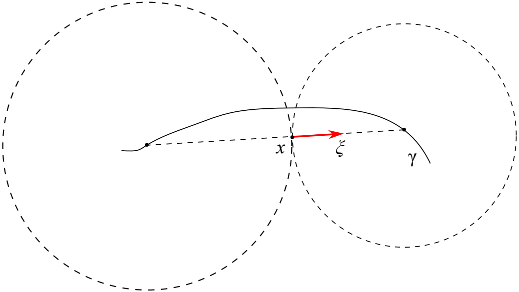

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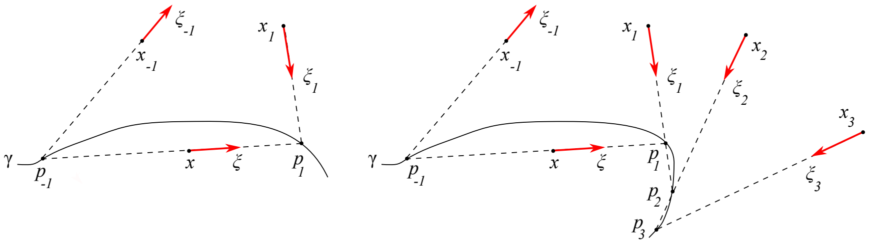

Figure: Singularities that cannot be resolved. Left: (x, ξ) has mirror images (x_{-1}, ξ_{-1}) and (x_1, ξ_1). Singularities at any two of those three points are related by unitary maps. Right: an example with more than three points.

A closed curve γ.

Let $\mathcal{M}(x, \xi)$ be the discrete set of all mirror points to (x, ξ). In the second example above,

$$\mathcal{M}(x, \xi) = \{(x_{-1}, \xi_{-1}), (x, \xi), (x_1, \xi_1), (x_2, \xi_2), (x_3, \xi_3)\}.$$

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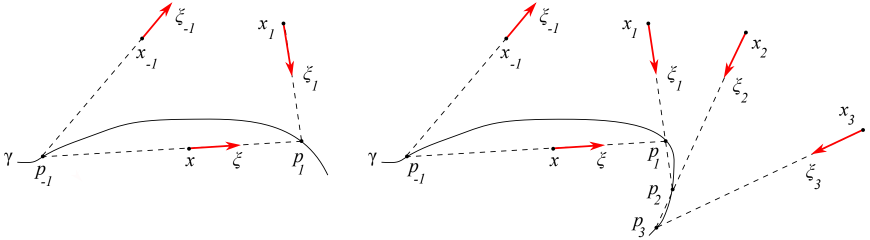

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We then get the following "propagation of singularities theorem".

Theorem 2

Let $\gamma = \partial\Omega$, where $\Omega \subset \mathbf{R}^2$ is a strictly convex domain. Let $f \in \mathcal{D}'(\mathbf{R}^2)$ and assume that $R_\gamma f \in C^\infty$. Then for any $(x,\xi) \in T^*\Omega \setminus 0$, either $\mathcal{M}(x,\xi) \subset \mathrm{WF}(f)$ or $\mathcal{M}(x,\xi) \cap \mathrm{WF}(f) = \emptyset$.

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This is a discrete analog of the corresponding theorem in the Duistermaat-Hörmander theory of ΨDOs of real principal type: Let P be such an operator; $Pu \in C^\infty$, and let $\operatorname{supp} f \subset K$, where K is compact. If K is non-trapping w.r.t. the Hamiltonian flow of P, then $f \in C^\infty$ as well.

Note that this conclusion is not based on ellipticity; it is based on the fact that if there is a singularity, then the whole zero bicharacteristic through it consists of singularities (by propagation of singularities); but some part of it goes out of $\operatorname{supp} f$; thus there cannot be singularities.

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The second statement of the theorem says that we can take any f singular in Ω, and extend it outside Ω so that its circular transform will be smooth on γ. Therefore, not only it is the case that singularities cannot be detected but any chosen in advance f singular in Ω can be neutralized by choosing suitable extension singular outside Ω.

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Example

Let $\gamma = S^1$ be the unit circle. Take f to be the characteristic function of the circle $|x| < 1/2$. That makes $R_\gamma f$ singular for r near $1/2$; the singularity is of the type $\sqrt{(r-1/2)_+}$. One can easily construct g supported in $|x| \geq 3/2$ so that $R_\gamma(f-g) \in C^\infty$ near $r = 1/2$.

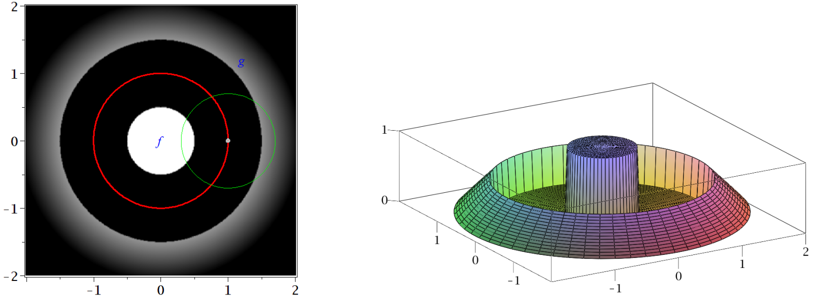

Figure: Left: density plot (white = 1, black = 0); right: a graph of f and g with $R(f-g)$ smooth near $r = 1/2$.

The first three coefficients of g are shown below (h = Heaviside)

$$g = h(t)\left(\frac{\sqrt{3}}{3} - \frac{5}{16}t + \frac{83}{5184}t^2 + O\left(t^3\right)\right), \quad t := |x|^2 - 9/4.$$

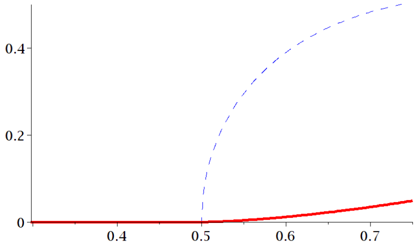

Figure: The thick red line: The graph of $R_\gamma(f-g)$ near $r = 1/2$ computed numerically with three terms in the expansion of g. The blue dotted line: the graph of $R_\gamma f$ having a square root type of singularity.

We could go on to kill all the singularities for all r, not just at $r = 3/2$ by constructing a suitable jump of g at $r = 5/2$, then at $r = 7/2$, etc.

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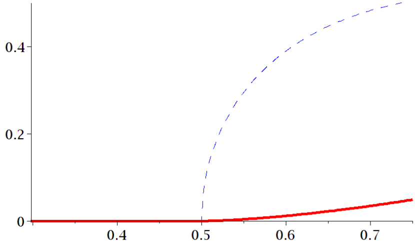

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Would R^*R work?

We showed in Theorem 3 that for $\gamma = \partial\Omega$, Ω convex, if f has singularities known a priori to lie in a compact set K, then there is unique recovery of the visible ones, and in particular the ones in Ω. How to recover them? Would $R_\gamma^* R_\gamma$ work?

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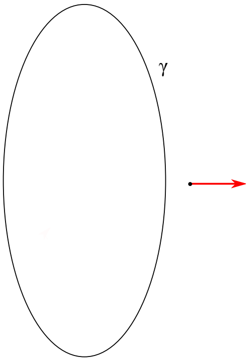

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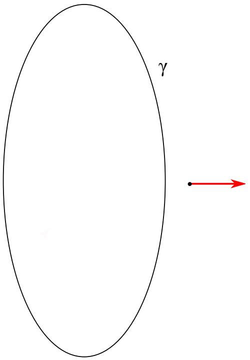

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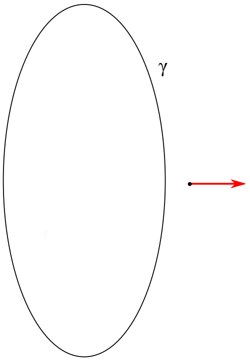

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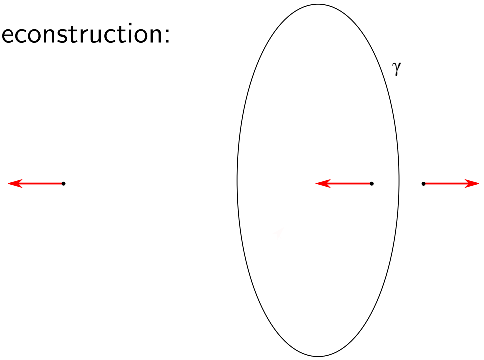

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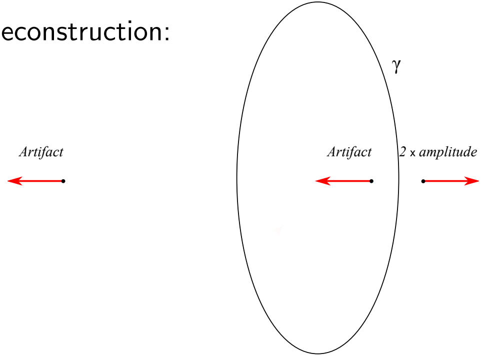

Recovery when f is compactly supported via backprojection

R^*R does not work but one could apply a certain FIO to "displace the artifacts" (Felea 2007). This pushes the most singular part away step by step. Instead, one can simply do the following.

Let $T \gg 1$ so that any singularity of f enters and leaves Ω for time T. We are assuming that we have wave equation data, as explained above. Solve

$$\begin{cases} (\partial_t^2 - \Delta)v &= 0 \quad \text{in } [0,T]\times\Omega, \\ v|_{[0,T]\times\partial\Omega} &= \chi.\text{data}, \\ v|_{t=T} &= 0, \\ \partial_t v|_{t=T} &= 0, \end{cases}$$

where χ cuts smoothly near $t = T$. Then we get

$$f|_\Omega = \partial_t v|_{t=0} \mod C^\infty.$$

Why? Because the difference $u - v$ (here, u is the forward solution with Cauchy data $(0, f)$) solves a mixed problem with smooth data everywhere.

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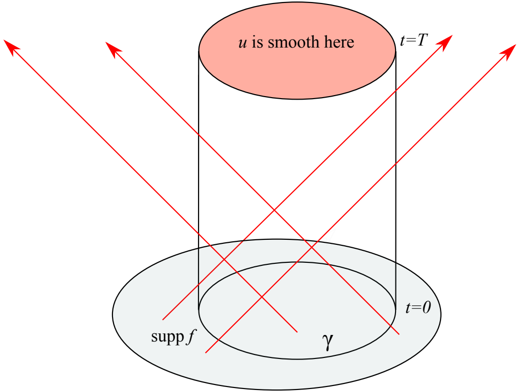

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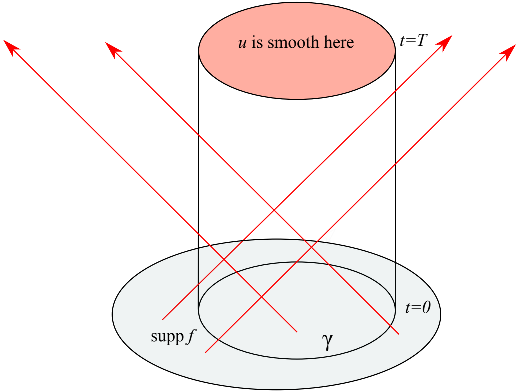

Non closed curves

What if γ is not closed? Let us take an arc which is concave when looking from the left, for example. The some singularities are detected twice, some once, some — never. The R^*R inversion will put different amplitudes (from 1/2 to 1) on the first two kinds (and a zero amplitude on the invisible ones). Close that arc to an closed curve and do a backprojection with zero data on the artificial boundary. Still wrong amplitudes.

Now, iterate, i.e., write the result as $BR_\gamma f = (I - K)f$, where B is the backprojection, and K is the "error". The latter is a ΨDO with a principal symbol between 0 and 1/2. Consider the truncated Neumann series

$$(I + K + K^2 + \cdots + K^N)BR_\gamma$$

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