Thermoacoustic and Photoacoustic Tomography with a variable continuous or discontinuous sound speed

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Based on a joint work with
Jianliang Qian, Gunther Uhlmann and Hongkai Zhao

- It is a linear inverse problem
- Under the "right conditions", it is well-posed (stable)

but...:

- it is formally determined
- the speed is variable, and might be discontinuous (we can include a metric, etc.)
- in many interesting cases, conditions are not "right", hence ill posedness
- we give if and only if conditions for uniqueness, even in the partial data case
- we give if and only if conditions for stability, even in the partial data case
- we write an explicit solution formula in the form of a converging Neumann series (whole boundary, T above the stability threshold)

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Thermo- and photo- acoustic Tomography

In thermo/photo-acoustic tomography, a short electro-magnetic pulse/laser beam is sent through a patient's body. The tissue reacts and emits an ultrasound wave form any point, that is measured away from the body. Then one tries to reconstruct the internal structure of a patient's body form those measurements.

The Mathematical Model

Let $c(x) > 0$ be the acoustic speed. Let u solve the problem

$$\begin{cases} (\partial_t^2 - c^2\Delta)u &= 0 \quad \text{in } (0,T)\times \mathbf{R}^n, \\ u|_{t=0} &= f, \\ \partial_t u|_{t=0} &= 0, \end{cases} \tag{1}$$

where $T > 0$ is fixed.
Assume that f is supported in $\bar{\Omega}$, where $\Omega \subset \mathbf{R}^n$ is some smooth bounded domain. The measurements are modeled by the operator

$$\Lambda f := u|_{[0,T]\times\partial\Omega}.$$

The problem is to reconstruct the unknown f.

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If $T = \infty$, we can just solve a Cauchy problem backwards with zero initial data.

One of the most common methods when $T < \infty$ is to do the same (time reversal). Solve

$$\begin{cases} (\partial_t^2 - c^2\Delta)v_0 &= 0 \quad \text{in } (0,T)\times\Omega,\\ v_0|_{[0,T]\times\partial\Omega} &= \chi h,\\ v_0|_{t=T} &= 0,\\ \partial_t v_0|_{t=T} &= 0, \end{cases} \tag{2}$$

where h will be taken to be $h = \Lambda f$. Here χ cuts off smoothly near $t = T$ so that the 1st order compatibility condition is satisfied.

Then we define the following

Time Reversal

$$f \approx A_0 h := v_0(0,\cdot) \quad \text{in } \bar{\Omega}, \text{ where } h = \Lambda f.$$

Most (but not all) works are in the case of constant coefficients, i.e., when $c = 1$. If n is odd, and $T > \mathrm{diam}(\Omega)$, this is an exact method by the Hyugens' principle.

In that case, this is actually an integral geometry problem because of Kirchoff's formula — recovery of f from integrals over spheres centered at $\partial\Omega$.

When n is even, or when the coefficients are not constant, this is an "approximate solution" only. As $T \to \infty$, the error tends to zero by finite energy decay. When the geometry is non-trapping, the convergence is uniform and exponentially fast for n odd and $O(t^{1-n})$ for n even [HRISTOVA].

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Prior results

KRUGER; AGRANOVSKY, AMBARTSOUMIAN, FINCH, GEORGIEVA-HRISTOVA, JIN, HALTMEIER, KUCHMENT, NGUYEN, PATCH, QUINTO, WANG, XU ...

The time reversal (but not only) is often used for reconstruction. It is exact only when $T = \infty$ but above some critical time T_1, it is a parametrix.

When T is fixed, there is no good control over the error (unless n is odd and $c = \mathrm{const}$). There are other methods, as well, for example a method based on an eigenfunctions expansion; or explicit formulas if $c = \mathrm{const}$ and Ω is a ball (with $T = \infty$ in even dimensions).

Results for variable coefficients existed but not so many. FINCH AND RAKESH (2009) proved uniqueness when $T > \mathrm{diam}(\Omega)$, based on Tataru's uniqueness theorem (that we use, too). Reconstructions for finite T have been tried numerically, and they "seem to work" at least for non-trapping geometries.

Another problem of a genuine applied interest is uniqueness and reconstruction with measurements on a part of the boundary. There were no results so far for the variable coefficient case, and there is a uniqueness result in the constant coefficients one by FINCH, PATCH AND RAKESH (2004).

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Ω=ball, constant speed

The simplest case is when $c = 1$ and Ω is the unit ball. Let also $n = 3$. Then there are explicit reconstruction formulas (Finch, Haltmeier, Kunyansky, Nguen, Patch, Rakesh, Xu, Wang). Let $g(x,t) = \Lambda f$ be the data, $x \in S^{n-1}$. Then, in 3D,

$$f(x) = -\frac{1}{8\pi^2}\Delta_x \int_{|y|=1} \frac{g(y,|x-y|)}{|x-y|}\mathrm{d}S_y.$$

Also,

$$f(x) = -\frac{1}{8\pi^2}\int_{|y|=1}\left(\frac{1}{t}\frac{d^2}{dt^2}g(y,t)\right)\Big|_{t=|y-x|}\mathrm{d}S_y.$$

Yet another one:

$$f(x) = \frac{1}{8\pi^2}\nabla_x \cdot \int_{|y|=1}\left(\nu(y)\frac{1}{t}\frac{d}{dt}\frac{g(y,t)}{t}\right)\Big|_{t=|y-x|}\mathrm{d}S_y.$$

The latter is a partial case of an explicit formula in any dimension (Kunyansky).

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The latter is a partial case of an explicit formula in any dimension (Kunyansky).

When $c =$ const., an n is odd, this is also an integral geometry problem. By the Kirchhoff's formula, up to time derivatives, **in odd dimensions**, what we measure are the spherical means of f centered at point on $\partial\Omega$:

$$\Lambda f \sim \int_{|\omega|=1} f(x + t\omega)\,\mathrm{d}\omega, \quad t \in [0, T],\ x \in \partial\Omega.$$

Now, we have to invert it. This transform can be (and has been) studied with microlocal methods that in particular answer some questions about stability and recovery of singularities, including cases with partial data (but c still constant). One can also use analytic microlocal analysis for uniqueness.

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Uniqueness

The underlying metric is $c^{-2}\mathrm{d}x^2$. Set

$$T_0 = \max_{x\in\bar\Omega} \operatorname{dist}(x, \partial\Omega).$$

Theorem 1

$T \geq T_0 \implies$ *uniqueness.*

$T < T_0 \implies$ *no uniqueness. We can recover $f(x)$ for $\operatorname{dist}(x,\partial\Omega) \leq T$ and nothing else.*

The proof is based on the unique continuation theorem by Tataru.

The explanation is simple. We can recover $f(x)$ on the maximal set that signals from $\partial\Omega$ can reach at times $t \leq T$ (by unique continuation), and nothing else (by finite speed of propagation).

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Stability

Stability should be related to propagation of singularities. As a general principle, it is necessary (and sufficient) to be able to "detect" all singularities. By singularities, we mean elements of the wave front set WF(f). Since $u_t = 0$ for $t = 0$, each singularity (x, ξ) splits into two parts with equal energy and they start to travel in positive (ξ) and negative ($-\xi$) direction. We need to detect one of them, at least.

Let $T_1 \leq \infty$ be the length of the longest (maximal) geodesic through $\bar{\Omega}$. Then the "stability time" is $T_1/2$. One can show that $T_0 \leq T_1/2$. If $T_1 = \infty$, we say that the speed is **trapping** in Ω.

Theorem 2

$T > T_1/2 \implies$ *stability.*

$T < T_1/2 \implies$ *no stability, in any Sobolev norms.*

The second part follows from the fact that Λ is a smoothing FIO on an open conic subset of $T^*\Omega$. In particular, if the speed is trapping, there is no stability, whatever T.

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Reconstruction. Modified time reversal

A modified time reversal, harmonic extension

Given h (that eventually will be replaced by Λf), solve

$$\begin{cases} (\partial_t^2 - c^2\Delta)v &= 0 \quad \text{in } (0,T)\times\Omega, \\ v|_{[0,T]\times\partial\Omega} &= h, \\ v|_{t=T} &= \phi, \\ \partial_t v|_{t=T} &= 0, \end{cases} \tag{3}$$

where ϕ is the harmonic extension of $h(T,\cdot)$:

$$\Delta\phi = 0, \quad \phi|_{\partial\Omega} = h(T,\cdot).$$

Note that the initial data at $t=T$ satisfies compatibility conditions of first order (no jump at $\{T\}\times\partial\Omega$). Then we define the following pseudo-inverse

$$Ah := v(0,\cdot) \quad \text{in } \bar\Omega.$$

Why would we do that? We are missing the Cauchy data at $t = T$; the only thing we know there is its value on $\partial\Omega$. The time reversal methods just replace it by zero. We replace it by that data (namely, by $(\phi, 0)$), having the same trace on the boundary, that minimizes the energy.

Given $U \subset \mathbf{R}^n$, the energy in U is given by

$$E_U(t,u) = \int_U \left(|\nabla u|^2 + c^{-2}|u_t|^2\right) \mathrm{d}x.$$

We define the space $H_D(U)$ to be the completion of $C_0^\infty(U)$ under the Dirichlet norm

$$\|f\|_{H_D}^2 = \int_U |\nabla u|^2 \,\mathrm{d}x.$$

The norms in $H_D(\Omega)$ and $H^1(\Omega)$ are equivalent, so

$$H_D(\Omega) \cong H_0^1(\Omega).$$

The energy norm of a pair $[f,g]$ is given by

$$\|[f,g]\|_{\mathcal{H}(\Omega)}^2 = \|f\|_{H_D(\Omega)}^2 + \|g\|_{L^2(\Omega, c^{-2}\mathrm{d}x)}^2$$

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Consider the "error operator" K. It is straightforward to see that

$$Kf = \text{first component of: } U_{\Omega,D}(-T)\Pi_\Omega U_{\mathbf{R}^n}(T)[f,0],$$

where

- $U_{\mathbf{R}^n}(t)$ is the dynamics in the whole $\mathbf{R}^n$,
- $U_{\Omega,D}(t)$ is the dynamics in Ω with Dirichlet BC,
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That projection is given by $\Pi_\Omega[f,g] = [f|_\Omega - \phi, g|_\Omega]$, where ϕ is the harmonic extension of $f|_{\partial\Omega}$.

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Reconstruction, whole boundary

Theorem 3

Let $T > T_1/2$. Then $A\Lambda = I - K$, where $\|K\|_{\mathcal{L}(H_D(\Omega))} < 1$. In particular, $I - K$ is invertible on $H_D(\Omega)$, and the inverse thermoacoustic problem has an explicit solution of the form

$$f = \sum_{m=0}^{\infty} K^m A h, \quad h := \Lambda f.$$

If $T > T_1$, then K is compact.

We have the following estimate on $\|K\|$:

Corollary 4

$$\|Kf\|_{H_D(\Omega)} \le \left(\frac{E_\Omega(u,T)}{E_\Omega(u,0)}\right)^{\frac{1}{2}} \|f\|_{H_D(\Omega)}, \quad \forall f \in H_{D(\Omega)},\ f \neq 0,$$

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Summary: Dependence on T

(i) $T < T_0 \implies$ **no uniqueness**
Λf does not recover uniquely f. $\|K\| = 1$.

(ii) $T_0 < T < T_1/2 \implies$ **uniqueness, no stability**
We have uniqueness but not stability (there are invisible singularities). We do not know if the Neumann series converges. $\|Kf\| < \|f\|$ but $\|K\| = 1$.

(iii) $T_1/2 < T < T_1 \implies$ **stability and explicit reconstruction**
This assumes that c is non-trapping. The Neumann series converges exponentially but maybe not as fast as in the next case (K contraction but not compact). There is stability (we detect all singularities but some with 1/2 amplitude). $\|K\| < 1$

(iv) $T_1 < T \implies$ **stability and explicit reconstruction**
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Iterating the Time Reversal

What if we use Neumann series for the time reversal?

The "error operator" K then is smoothing for $T > T_1$ (good) but not necessarily a contraction (bad). Still, for $T \gg T_1$ and χ with $|\chi'| \ll 1$, it will be a contraction by well known local energy decay estimates (HRISTOVA in the TAT setting). Therefore,

$$(I - K)f = A\chi\Lambda f$$

can be solved by Neumann series, if $T \gg T_1$ and χ are "right".

We cannot give sharp conditions when $T \gg T_1$ and χ are "right".

In contrast, with the "harmonic extension method", $T > T_1$ is right, $T < T_1$ is not, and there is no χ. Also, $\|K\|$ is minimized by that method.

Iterating the Time Reversal

What if we use Neumann series for the time reversal?

The "error operator" K then is smoothing for $T > T_1$ (good) but not necessarily a contraction (bad). Still, for $T \gg T_1$ and χ with $|\chi'| \ll 1$, it will be a contraction by well known local energy decay estimates (HRISTOVA in the TAT setting). Therefore,

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can be solved by Neumann series, if $T \gg T_1$ and χ are "right".

We cannot give sharp conditions when $T \gg T_1$ and χ are "right".

In contrast, with the "harmonic extension method", $T > T_1$ is right, $T < T_1$ is not, and there is no χ. Also, $\|K\|$ is minimized by that method.

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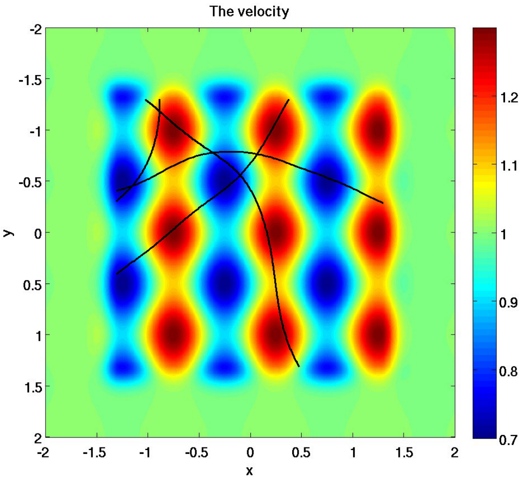

Figure: The speed, $T_0 \approx 1.15$. $\Omega = [-1.28, 1.28]^2$, computations are done in $[-2, 2]^2$

Example 1: Nontrapping speed

Figure: Original

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Figure: Neumann Series reconstruction, $T = 4T_0 = 4.6$, error $= 3.45\%$

Example 1: Nontrapping speed

Figure: Time Reversal, $T = 4T_0 = 4.6$, error $= 23\%$

Example 2: Trapping speed

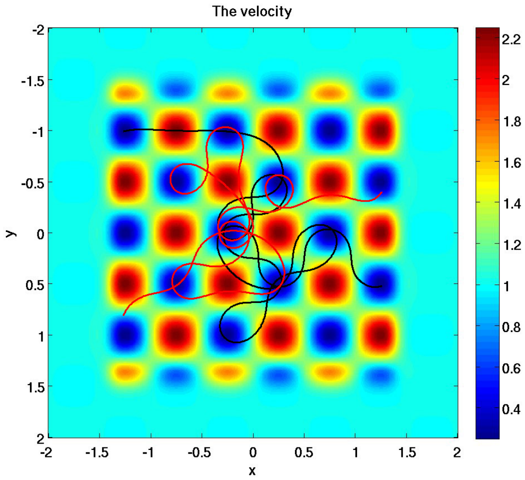


Figure: The speed, $T_0 \approx 1.18$

Example 2: Trapping speed

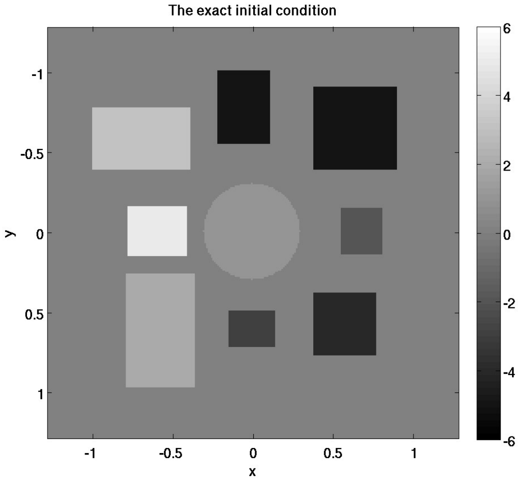


Figure: The original

Example 2: Trapping speed

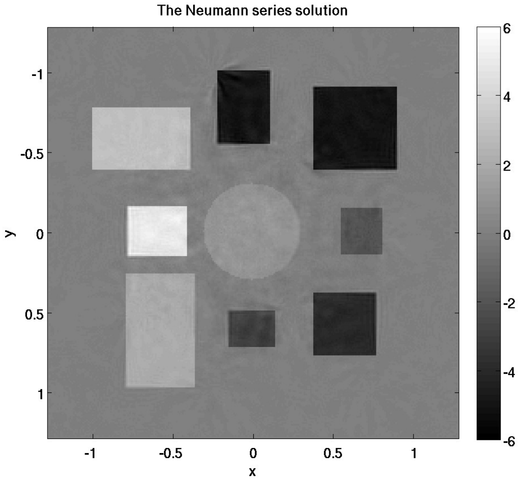


Figure: Neumann Series reconstruction, 10 steps, $T = 4T_0 = 4.7$, error $= 8.75\%$

Example 2: Trapping speed

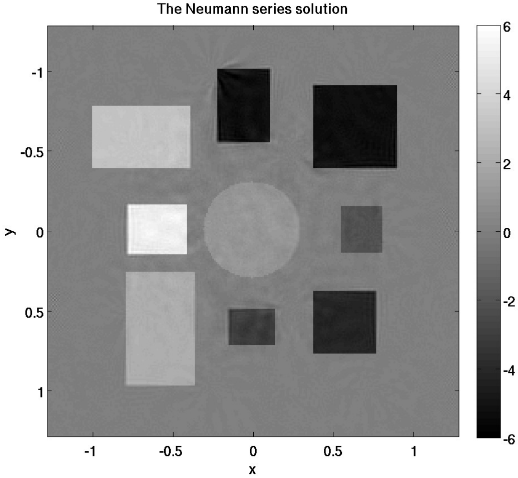


Figure: Neumann Series reconstruction with 10% noise, 15 steps, $T = 4T_0 = 4.7$, error $= 8.72\%$

Example 2: Trapping speed

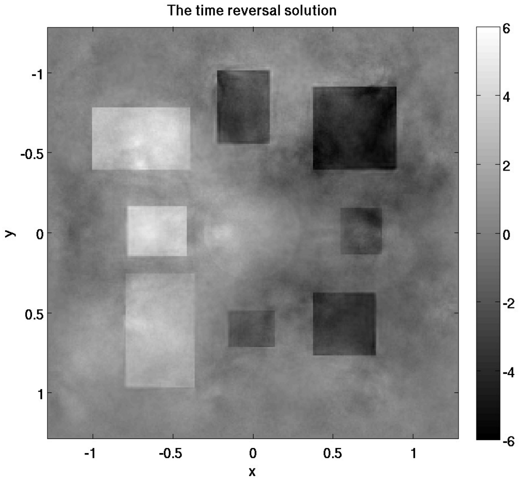


Figure: Time Reversal, $T = 4T_0 = 4.7$, error $= 55\%$

Example 2: Trapping speed

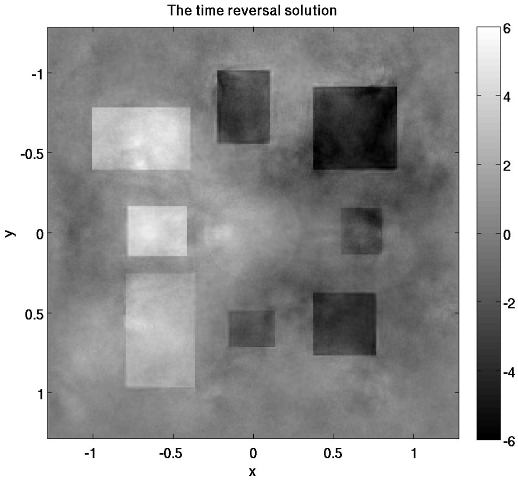


Figure: Time Reversal with 10% noise, $T = 4T_0 = 4.7$, error $= 54\%$

Example 3: The same trapping speed, Barbara

Figure: Original

Example 3: The same trapping speed, Barbara

Figure: Neumann series, $T = 4T_0 = 4.7$, error $= 7.5\%$, 10 steps

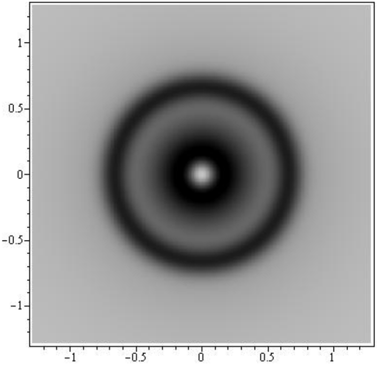
Example 3: The same trapping speed, Barbara

Figure: Time Reversal, $T = 4T_0 = 4.7$, error $= 27.7\%$

Example 3: The same trapping speed, Barbara

Figure: Time Reversal, $T = 12\,T_0 = 14.1$, error $= 99.67\%$

Example 4: a radial trapping speed



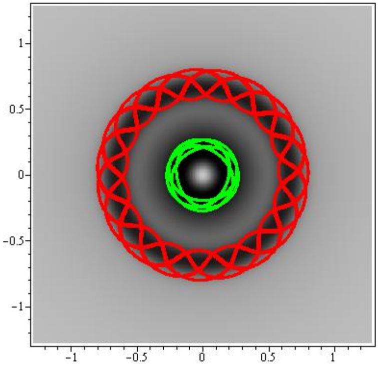


Figure: A trapping speed. Darker regions represent a slower speed. The circles of radii approximately 0.23 and 0.67 are stable periodic geodesics. Left: the speed. Right: the speed with two trapped geodesics

Example 4: a radial trapping speed

Figure: Original, lower resolution than before

Example 4: a radial trapping speed

Figure: Neumann series, 10 steps, $T = 8T_0 = 8.7$, error $= 9.7\%$

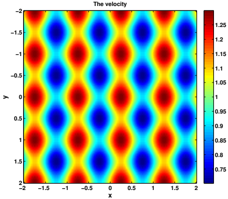
Example 4: a radial trapping speed

Figure: **Iterated** Time Reversal, 10 steps, $T = 8T_0 = 8.7$, error $= 12.1\%$

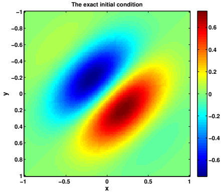
Example 4: a radial trapping speed

Figure: Time Reversal, $T = 8T_0 = 8.7$, error $= 21.7\%$

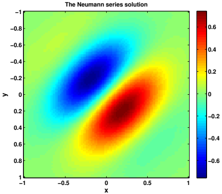
What if the waves can come back to Ω (reflectors)?



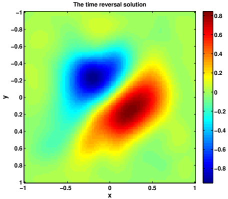
The speed



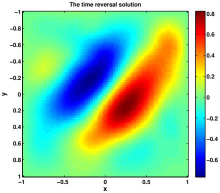
original



NS, $T = 6$



TR, $T = 6$



TR, $T = 26$

Figure: $T_0 \approx 1.2$, $2.9 < T_1 < 3.5$. There are Neumann BC here at the boundary of the larger square! Waves leaving Ω come back without any damping!

Measurements on a part of the boundary

Assume that $c = 1$ outside Ω. Let $\Gamma \subset \partial\Omega$ be a relatively open subset of $\partial\Omega$.

Assume now that the observations are made on $[0, T] \times \Gamma$ only, i.e., we assume we are given

$$\Lambda f|_{[0,T]\times\Gamma}.$$

We consider f's with

$$\operatorname{supp} f \subset \mathcal{K},$$

where $\mathcal{K} \subset \Omega$ is a fixed compact.

Uniqueness?

Stability?

Reconstruction?

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$$T_0(\mathcal{K}) = \max_{x \in \mathcal{K}} \operatorname{dist}(x, \Gamma).$$

The uniqueness condition then should be

$$T \geq T_0(\mathcal{K}). \tag{4}$$

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Let $c = 1$ outside Ω, and let $\partial\Omega$ be strictly convex. Then if $T \geq T_0(\mathcal{K})$, if $\Lambda f = 0$ on $[0, T] \times \Gamma$ and $\operatorname{supp} f \subset \mathcal{K}$, then $f = 0$.

Proof based on Tataru's uniqueness continuation results. Generalizes a similar result for constant speed by Finch, Patch and Rakesh.

As before, without (4), one can recover f on the reachable part of $\mathcal{K}$. Of course, one cannot recover anything outside it, by finite speed of propagation. Therefore,

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$$\forall(x,\xi)\in\mathcal{K}\times S^{n-1}, \text{ the ray (geodesic) through it reaches } \Gamma \text{ at time } |t|<T.$$

We show next that this is an "if and only if" condition (up to replacing an open set by a closed one) for stability. Actually, we show a bit more.

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Assume formally $T=\infty$. Then $\Lambda=\Lambda_++\Lambda_-$, where $\Lambda_\pm$ are elliptic Fourier Integral Operators of zeroth order with canonical relations given by the graphs of the maps

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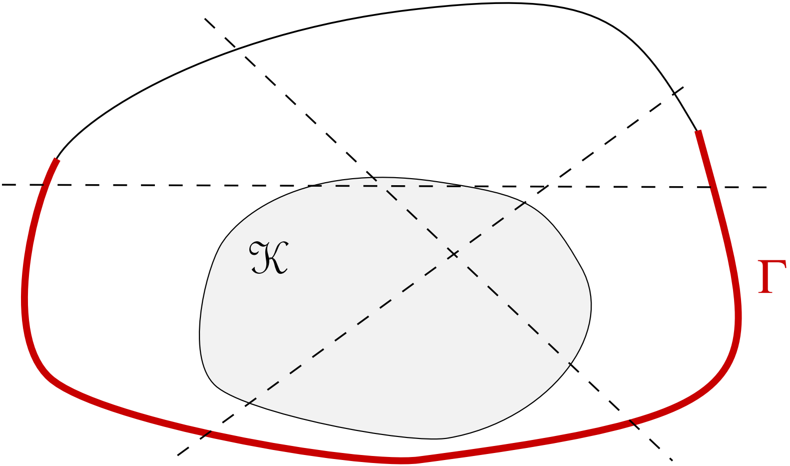
A reformulation of the stability condition

- Every geodesic through $\mathcal{K}$ intersects Γ.
- $\forall (x, \xi) \in \mathcal{K} \times S^{n-1}$, the travel time along the geodesic through it satisfies $|t| < T$.

Let us call the least such time $T_1/2$, then $T > T_1/2$ as before.

In contrast, any small open Γ suffices for uniqueness.



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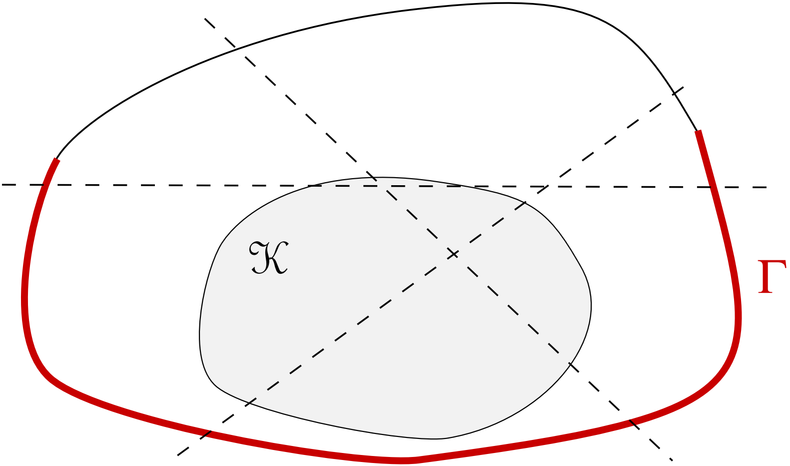

Let A be the "modified time reversal" operator as before. Actually, ϕ will be 0 because of χ below. Let $\chi \in C_0^\infty([0,T] \times \partial\Omega)$ be a cutoff (supported where we have data).

Theorem 7

$A\chi\Lambda$ is a zero order classical ΨDO in some neighborhood of $\mathcal{K}$ with principal symbol

$$\frac{1}{2}\chi(\gamma_{x,\xi}(\tau_+(x,\xi))) + \frac{1}{2}\chi(\gamma_{x,\xi}(\tau_-(x,\xi))).$$

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(c) there exists a constant $C > 0$ so that

$$\|f\|_{H_D(\mathcal{K})} \le C\|\Lambda f\|_{H^1([0,T]\times\Gamma)}.$$

(b) follows by building a parametrix, and (c) follows from (b) and from the uniqueness result.

In particular, we get that for a fixed $T > T_1$, the classical Time Reversal is a parametrix (of infinite order, actually).

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Reconstruction

One can constructively write the problem in the form

Reducing the problem to a Fredholm one

$$(I-K)f = BA\chi\Lambda f \quad \text{with the r.h.s. given,}$$

i.e., B is an explicit operator (a parametrix), where K is compact with 1 not an eigenvalue.

Constructing a parametrix without the ΨDO calculus.

Assume that the stability condition is satisfied in the interior of $\operatorname{supp}\chi$. Then

$$A\chi\Lambda f = (I-K)f,$$

where $I-K$ is an elliptic ΨDO with $0 \le \sigma_p(K) < 1$. Apply the formal Neumann series of $I-K$ (in Borel sense) to the l.h.s. to get

$$f = (I+K+K^2+\dots)A\chi\Lambda f \quad \text{mod } C^\infty.$$

With a bit of luck, this series may converge or at least give a good approximation with a certain number of finitely many terms.

Reconstruction

One can constructively write the problem in the form

Reducing the problem to a Fredholm one

$$(I-K)f = BA\chi\Lambda f \quad \text{with the r.h.s. given,}$$

i.e., B is an explicit operator (a parametrix), where K is compact with 1 not an eigenvalue.

Constructing a parametrix without the ΨDO calculus.

Assume that the stability condition is satisfied in the interior of $\operatorname{supp}\chi$. Then

$$A\chi\Lambda f = (I-K)f,$$

where $I-K$ is an elliptic ΨDO with $0 \le \sigma_p(K) < 1$. Apply the formal Neumann series of $I-K$ (in Borel sense) to the l.h.s. to get

$$f = (I + K + K^2 + \dots)A\chi\Lambda f \mod C^\infty.$$

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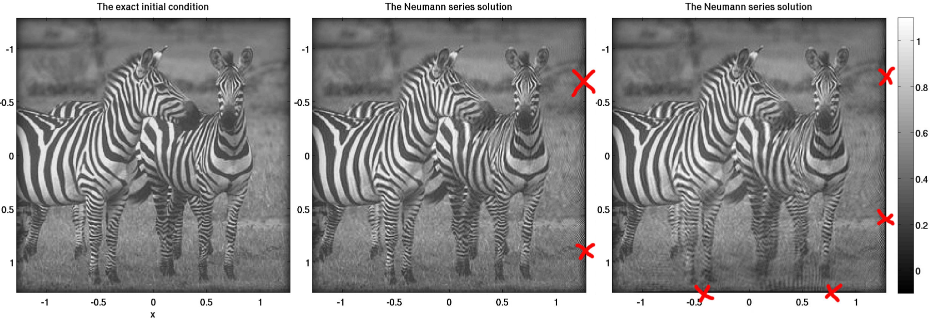
Examples: Non-trapping speed, 1 and 2 sides missing



original NS, 3 sides, error = 7.99% NS, 2 sides, error = 12.2%

Figure: Partial data reconstruction, non-trapping speed, $T = 4T_0$.

Discontinuous speeds, modeling Brain Imaging

The following modification appears in brain imaging and was proposed by Lihong Wang in May 2010, during a Banff meeting. Let c be piecewise smooth with a jump across a smooth closed surface Γ. How much of all that is preserved? The direct problem is a transmission problem, and there are reflected and refracted rays.

In brain imaging, the interface is the skull. The sound speed jumps by about a factor of 2 there. Experiments show that the ray that arrives first carries about 20% of the energy.

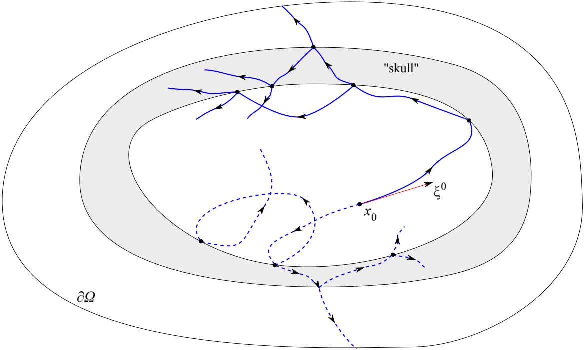


Figure: Propagation of singularities in the “skull” geometry

Propagation of singularities (an example is shown on the previous slide) is well understood away from tangent rays. When a ray approaches Γ from the side with the higher speed, there are always a reflected and a refracted rays. When the ray is coming from a slower to a faster region, we may or may not have a refracted one, but we always have a reflected one. If there is only a reflected one, this is known as full internal reflection. The energy (at high frequencies) naturally splits into fractions of the total one. So a single singularity may exit at several different places with different amplitudes.

There are might be trapping singularities, as well, that remain invisible. But even the visible ones, are visible with a fraction of their amplitudes only! In a way, all singularities inside Γ are partly invisible, some — totally invisible.

(Completely) trapped singularities are a problem, as before. Let $\mathcal{K} \subset \Omega$ be a compact set such that all rays originating from it are never tangent to Γ and non-trapping. For f satisfying

$$\operatorname{supp} f \subset \mathcal{K}$$

the Neumann series above still converges (uniformly to f).

We need a small modification to keep the support in $\mathcal{K}$ all the time. We use the projection $\Pi_{\mathcal{K}} : H_D(\Omega) \to H_D(\mathcal{K})$ for that purpose.

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Reconstruction

Theorem 8

Let all rays from $\mathcal{K}$ have a path never tangent to Γ that reaches $\partial\Omega$ at time $|t| < T$. Then

$$\Pi_{\mathcal{K}} A\Lambda = I - K \text{ in } H_D(\mathcal{K}), \text{ with } \|K\|_{H_D(\mathcal{K})} < 1.$$

In particular, $I - K$ is invertible on $H_D(\mathcal{K})$, and Λ restricted to $H_D(\mathcal{K})$ has an explicit left inverse of the form

$$f = \sum_{m=0}^{\infty} K^m \Pi_{\mathcal{K}} A h, \quad h = \Lambda f. \tag{5}$$

The assumption $\operatorname{supp} f \subset \mathcal{K}$ means that we need to know f outside $\mathcal{K}$; then we can subtract the known part.

In the numerical experiments below, we do not restrict the support of f, and still get good reconstruction images but the invisible singularities remain invisible.

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Brain imaging of square headed people

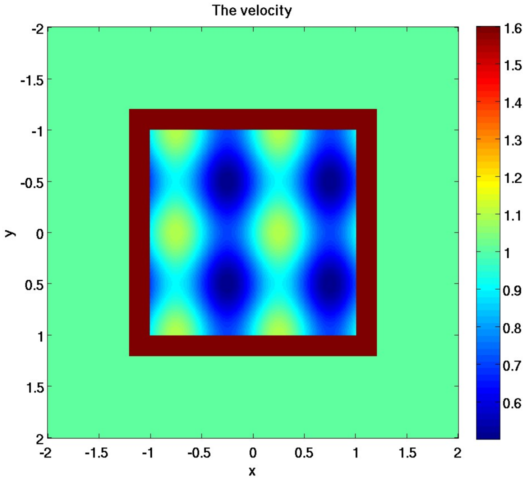
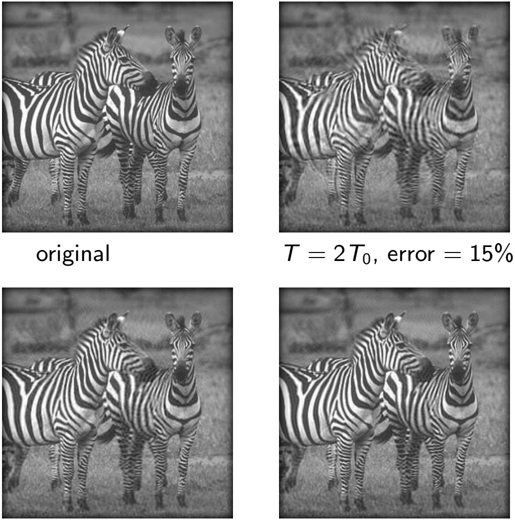


Figure: The speed jumps by a factor of 2 in average from the exterior of the "skull". The region Ω, as before, is smaller: $\Omega = [-1.28, 1.28]^2$.

A “skull” speed, Neumann series



original

$T = 2T_0$, error $= 15\%$

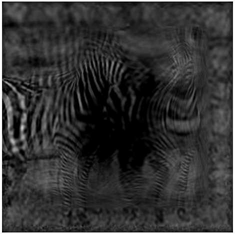
$T = 4T_0$, error $= 9.75\%$

$T = 8T_0$, error $= 7.55\%$

Figure: Neumann Series, 15 steps

A “skull” speed, Time Reversal

original



$T = 2T_0$, error $= 68\%$

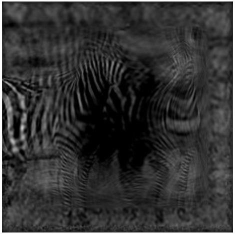
$T = 4T_0$, error $= 23.7\%$

$T = 8T_0$, error $= 78.5\%$

Figure: Time Reversal. There is a lot of “white clipping” in the last image, many values in $[1, 1.6]$

A “skull” speed, Time Reversal

original



$T = 2T_0$, error $= 68\%$

$T = 4T_0$, error $= 23.7\%$

$T = 8T_0$, error $= 78.5\%$

Figure: Time Reversal. The values in last image are compressed from $[0,1]$ to $[-0.05, 1.6]$

Original vs. Neumann Series vs. Time Reversal

original NS, error $= 7.55\%$ TR, error $= 78.5\%$

Figure: $T = 8T_0$. Original vs. Neumann Series vs. Time Reversal (the latter compressed from $[0, 1]$ to $[-0.05, 1.6]$)