The geodesic X-ray transform with conjugate points

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Joint work with François Monard and Gunther Uhlmann

Formulation of the Problem

Let (M, g) be a Riemannian manifold, and let γ_0 be a fixed geodesic on it *with possible conjugate points*. More general curves are allowed, as well. Let $\kappa \neq 0$ be a fixed weight function on TM.

Main Problem

What information about the singularities of f can we recover, given

$$Xf(\gamma) = \int \kappa(\gamma(t), \dot{\gamma}(t)) f(\gamma(t))\, \mathrm{d}t$$

known for all geodesics γ near γ_0?

We assume here that $\operatorname{supp} f$ is disjoint from the endpoints of γ_0.

In particular, if $Xf(\gamma) = 0$ (or is smooth) near γ_0, what do we get for $\mathrm{WF}(f)$?

This is a linearized version of the problem to recover a unknown speed from travel times.

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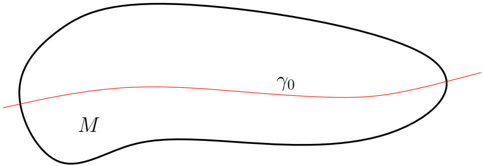
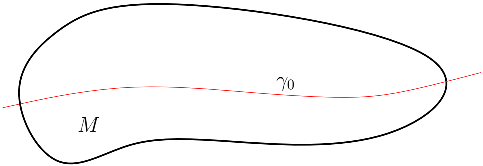


Figure: $Xf(\gamma)$ known for all γ near a fixed geodesic γ_0.

Why do we want to know that?

- In some applications in medical imaging and geophysics, this is all we want to know, to recover the "features" of the image.
- If we can prove uniqueness somehow (possible even with conjugate points in some cases), we immediately say if we have stability or not.
- We can think of this as stability analysis even if uniqueness might not hold.



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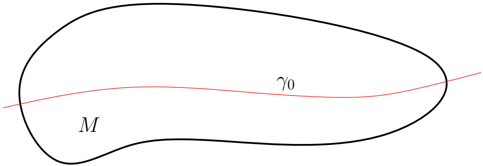

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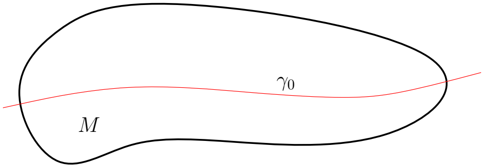

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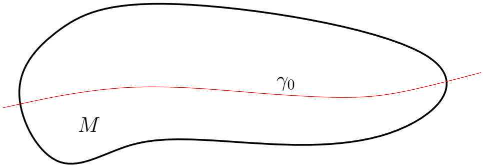

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If no conjugate points, we can do the best thing possible:

No conjugate points

We can recover singularities conormal to γ_0 (and close to those); i.e., $\mathrm{WF}(f)$ near $N^*\gamma_0$.

If we know $Xf(\gamma)$ for all (or for a rich enough set of) geodesics, then

- The problem is Fredholm; hence injectivity implies stability
- if $\kappa = 1$, there is injectivity (Mukhometov et al.)
- Finitely dimensional smooth kernel
- Works also for general curves, tensors, incomplete data
- If the metric (the family of curves) is real analytic, then we have injectivity (hence stability).
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Conjugate Points

If there are conjugate points, things change quite a bit.

- $n \geq 3$: the problem is overdetermined and we could use an open subset of geodesics. If they do not have conjugate points, and their conormals cover T^*M, we are fine [S-Uhlmann, 2008].
- $n \geq 3$, [Uhlmann–Vasy, 2013]: Under a foliation condition (allowing conjugate points), we can do layer stripping.
- $n \geq 3$, [S-Uhlmann, 2012]: Conjugate points of fold type might not be a problem, if a certain non-degeneracy condition holds. Hard to verify, and there is no geodesic example (but there are non-geodesic ones).
- $n = 2$, [S-Uhlmann, 2012]: If there are conjugate points of fold type, there is always mild instability at least (loss of 1/4 derivative) for the local problem (γ near γ_0).
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How is this work different?

We give a complete answer of what we can recover (and what we cannot) from knowing Xf in a neighborhood of one geodesic γ_0.

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Typical geometry in 2D

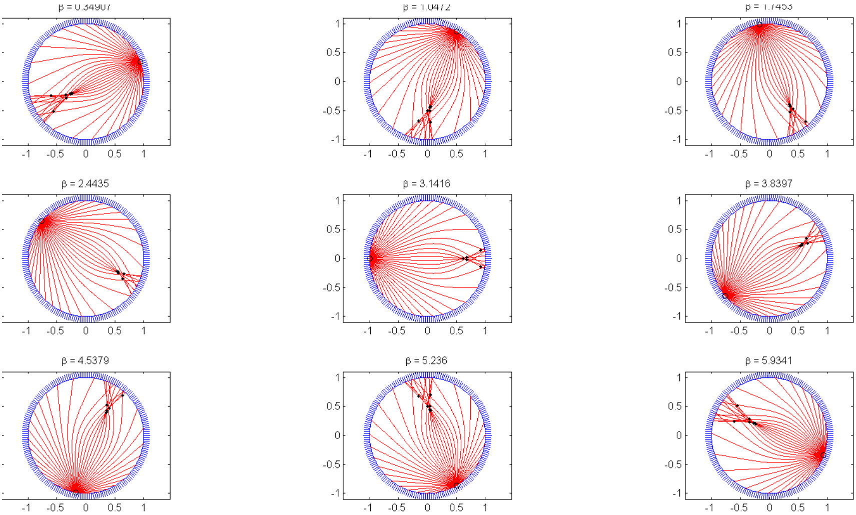

Figure: A cusp and two folds formed by geodesics around a slow region

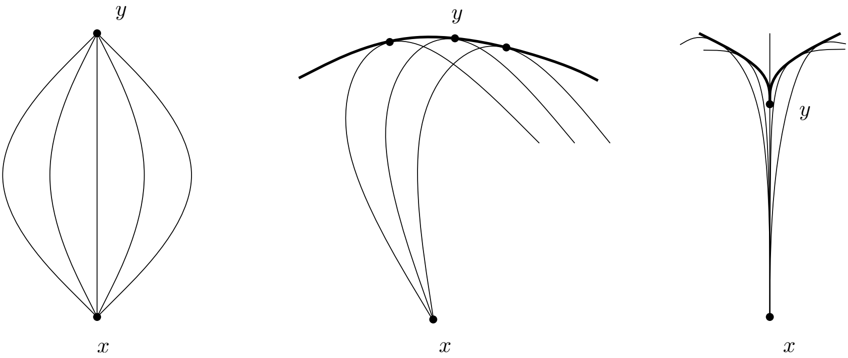

Figure: The three non-degenerate types of conjugate points in the plane together: a blowdown, a fold, and a cusp.

Main results in a nutshell

- Recovery of $\mathrm{WF}(f)$ (for the localized transform, in one direction) is impossible, regardless of the type of the conjugate points — loss of all derivatives at conjugate points!
- If the weight $\kappa(x, \theta)$ is an even function of θ, reversing the direction of t of $\gamma(t)$ does not matter — even knowing $Xf(\gamma)$ for all γ does not help! The problem then is unstable.
- For the attenuated transform with a positive attenuation, if there are no more than two conjugate points along each geodesic, then there is stability for the global problem! Reason: reversing the time gives us an additional equation.
- For the attenuated transform with a positive attenuation, if there are three or more conjugate points along each geodesic, then there is no stability.

Unlike the previous works, we do not study X^*X (but we have results for it, as well). We work directly with X.

Theorem 1

X is a Fourier Integral Operator in the class $I^{-\frac{n}{4}}(\mathcal{M}_0 \times M_0, C')$. It is a ΨDO (of order -1) if and only if the geodesics in $\mathcal{M}_0$ have no conjugate points.

From now on, $n = 2$. Then all manifolds in the microlocal diagram

are of the same dimension, 4. Both projections are local diffeomorphisms (π_M is global) thus so is $\mathcal{C}$. Is $\mathcal{C}$ global? If and only if there are no conjugate points.

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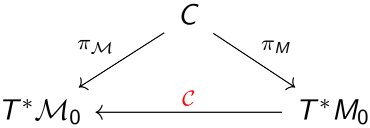

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Theorem 2

$\mathcal{C}(p,\xi) = \mathcal{C}(q,\eta)$ if and only if there is a geodesic $[0,1] \to \gamma \in \mathcal{M}_0$ joining p and q so that
(a) p and q are conjugate to each other,
(b) $\xi = \lambda J'(0)$, $\eta = \lambda J'(1)$, $\lambda \neq 0$, where $J(t)$ is the unique non-trivial, up to rescaling, Jacobi filed with $J(0) = J(1) = 0$.

The main idea is to use a partition of unity with cutoffs localized near conjugate points.

Then we have X acting on f with small supports, and there are no conjugate points on each piece. So we just need to understand X without conjugate points, that is all. This is easy (in 2D):

Theorem 3

Assume no conjugate points and $n = 2$. Then X is an FIO associated with the canonical diffeomorphism $\mathcal{C}$. It is elliptic at (x,ξ) if and only if $\kappa(x, \xi^\perp/|\xi|) \neq 0$.

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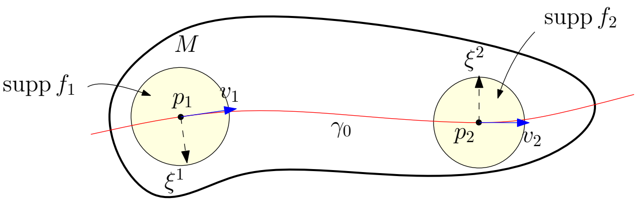
Cancellation of singularities

We are ready to prove one of the main results.



Write $f = f_1 + f_2$, where f_k are microlocalized near (p_k, ξ^k), $k = 1, 2$, where p_1, p_2 are conjugate. Write also $X = X_1 + X_2$. Then

$$Xf = X_1 f_1 + X_2 f_2.$$

But $X_{1,2}$ are elliptic; therefore

$$X_1 f_1 + X_2 f_2 = g \iff f_1 + X_1^{-1} X_2 f_2 = X_1^{-1} g \iff X_2^{-1} X_1 f_1 + f_2 = X_2^{-1} g$$

Theorem 4 (Cancellation of singularities)

Given f_1, one can choose f_2 so that $X(f_1 + f_2) \in C^\infty$ (microlocally).

Indeed, just solve $X_1 f_1 + X_2 f_2 = 0$ for f_2 to get $f_2 = -X_2^{-1} X_1 f_1$.

In other words, there is a huge microlocal kernel, and we can only recover $\mathrm{WF}(f)$ up to that kernel. Basically, we have one equation for two variables.

This can be generalized easily to m conjugate points: choose f_k for all k but one; then one can solve for the remaining one.

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Numerical Example

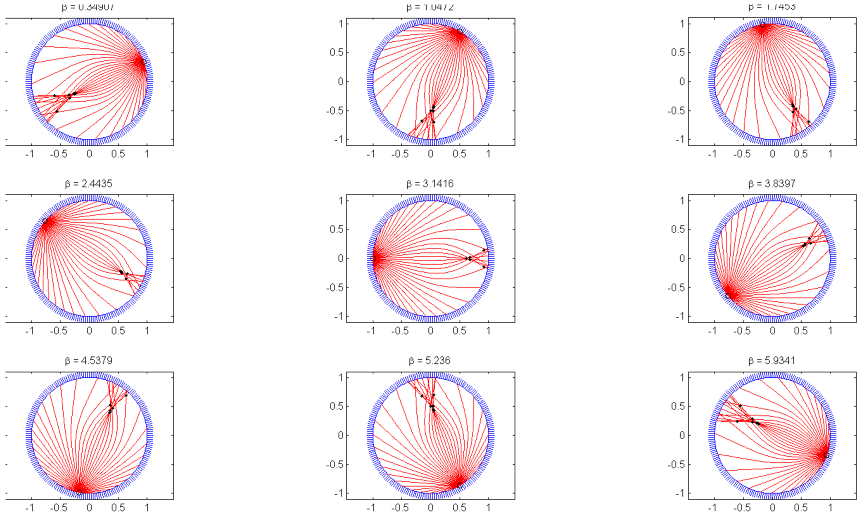


Figure: The geometry of the geodesics

Choose f_1 to be an approximate delta function.

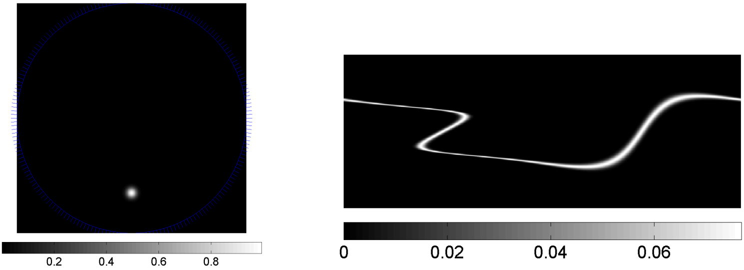


Figure: The function f_1 (left) and Xf_1 (right). The horizontal axis is the initial point on the boundary; the vertical one is the initial angle from 0 to π.

Geodesics issued from the bottom in a direction close to a vertical one, will hit the blob once. They are plotted around the 0.06 mark. The ones issued from the top at a downward vertical direction or close would hit the blob three times. They are plotted around the 0.02 mark.

Given f_1, we construct f_2 as in the theorem:

$$f_2 = -X_2^{-1} X_1 f_1 \quad \text{microlocally.}$$

For this purpose, we invert X in a smaller domain encompassing the expected location of the artifact (near the conjugate locus).

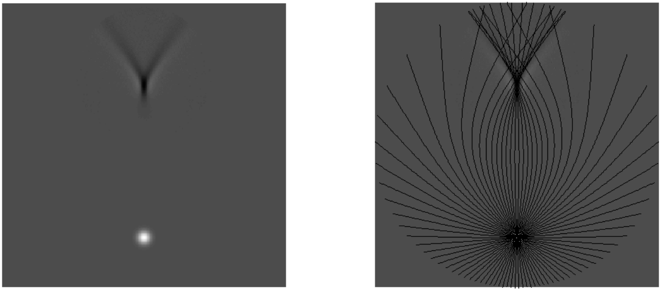
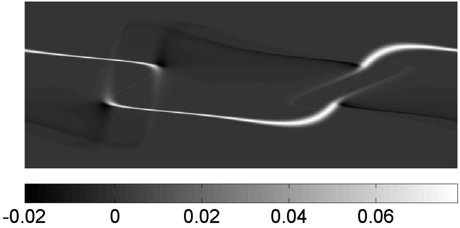


Figure: The function $f = f_1 + f_2$ (left) and the same function with a few superimposed geodesics on it (right). The "artifact" f_2 appears as an approximate conormal distribution to the conjugate locus of the blob that f_1 represents. The gray scale has changed, and black now represents negative values, around -0.5.



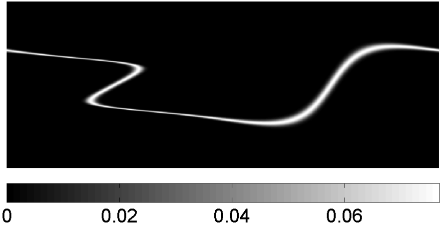


Figure: $X(f_1 + f_2)$ (top) and Xf_1 (bottom). Some singularities of Xf_1 are nearly erased. The gray scale on top is slightly different to allow for the negative values of $X(f_1 + f_2)$. The erased singularities correspond to nearly vertical geodesics.

Corollary: instability

When there are no conjugate points on the geodesics in M, $\forall n$, one has

$$\|f\|_{H^s(M)} \le C\|Xf\|_{H^{s+1/2}(\partial_+ SM_1)} + C_k\|f\|_{H^{-k}(M)}, \quad \forall f \in H_0^s(M)$$

for all $s \ge 0$, where $M_1 \supset\supset M$. When we know that X is injective, for example when the weight is constant; then we can remove the H^{-k} term.

Let $\kappa(x,\theta)$ be even in θ (then integrating over $\gamma(-t)$ does not provide more information). Then, if there are conjugate points, such an estimate does not hold. Moreover, even

$$\|f\|_{H^{s_1}(M)} \le C\|Xf\|_{H^{s_2}(\partial_+ SM_1)} + C\|f\|_{H^{s_3}(M)}$$

does not hold, regardless of the choice of s_1, s_2, s_3.

We therefore have an if and only if condition (up to the borderline case of conjugate points on the boundary) for stability for even weights.

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The X^*Xf (backprojection) inversion fails

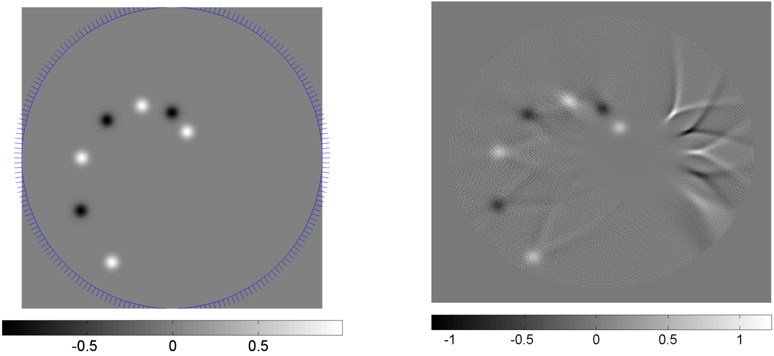


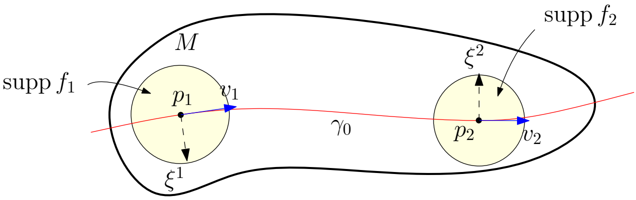
Figure: f_1 (left) and $C\sqrt{-\Delta_g}X^*Xf_1$ (right).

The artifacts are at the conjugate loci to each point. In the notation above, we see a linear combination of f_1 and f_2 in the reconstruction Then f_2 is an artifact.

The attenuated X-ray transform

Assume now that the weight is coming from an attenuation $\sigma(x, v) > 0$:

$$\kappa(x, v) = e^{-\int_0^\infty \sigma(\gamma_{x,v}(s),\dot\gamma_{x,v}(s))\,\mathrm{d}s}.$$



Then the direction along γ matters. Microlocally, to recover singularities near (p_1, ξ^1) and (p_2, ξ_2), we have two equations. If the determinant is not zero, we can solve them!

$$\det \begin{pmatrix} \kappa(p_1, v_1) & \kappa(p_2, v_2) \\ \kappa(p_1, -v_1) & \kappa(p_2, -v_2) \end{pmatrix} \neq 0.$$

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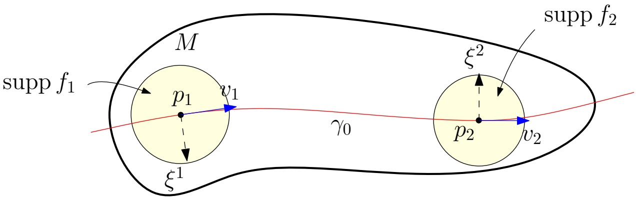

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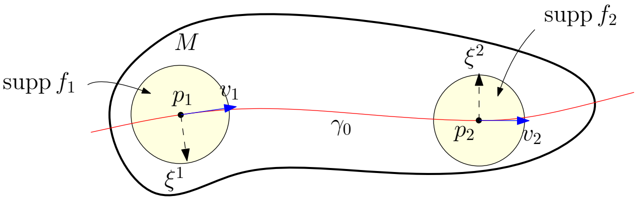

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$$\det \begin{pmatrix} \kappa(p_1, v_1) & < & \kappa(p_2, v_2) \\ \kappa(p_1, -v_1) & > & \kappa(p_2, -v_2) \end{pmatrix} \neq 0.$$

Automatically true! Then we can recover the singularities!

More examples

Reconstruction with the Landweber iteration method. The metric has conjugate points.

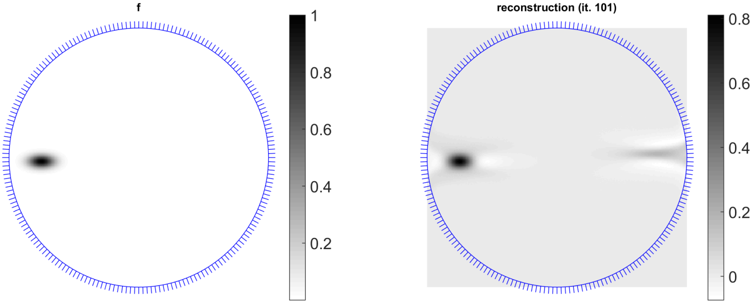

Figure: Attenuation $= 0$. Left: original; right: reconstruction

There is an artifact at the conjugate locus.

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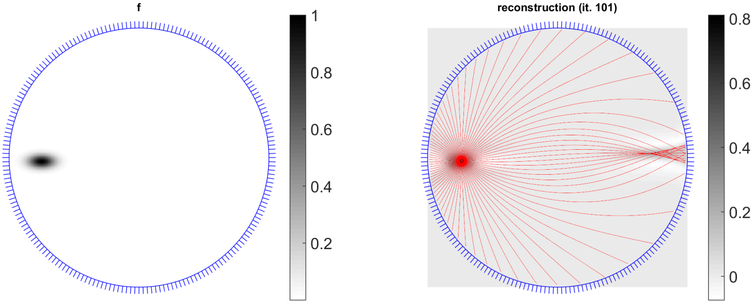

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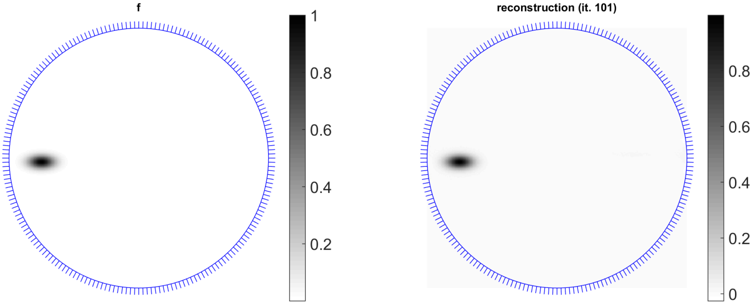


Figure: Variable attenuation with average $= 0.6$. Left: original; right: reconstruction.

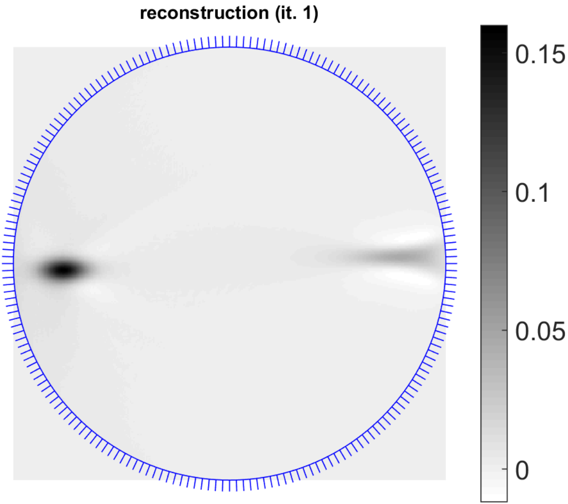


Figure: Variable attenuation with average = 0.6. Iteration #1.

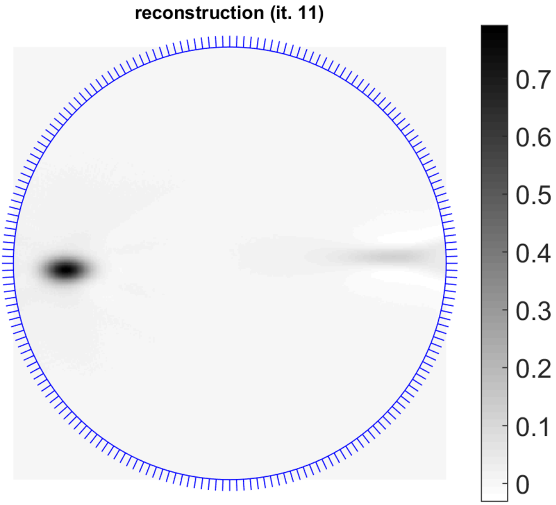


Figure: Variable attenuation with average $= 0.6$. Iteration $\#11$.

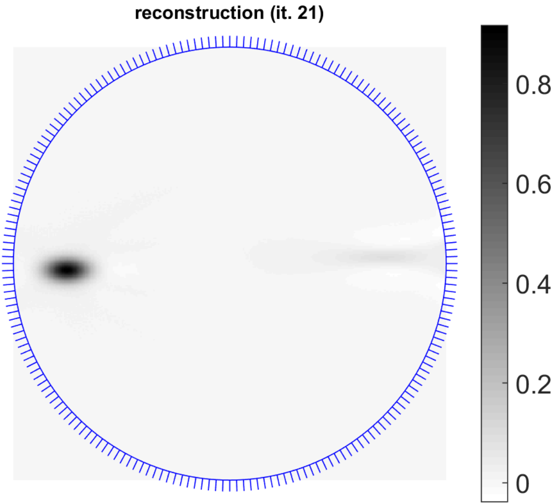
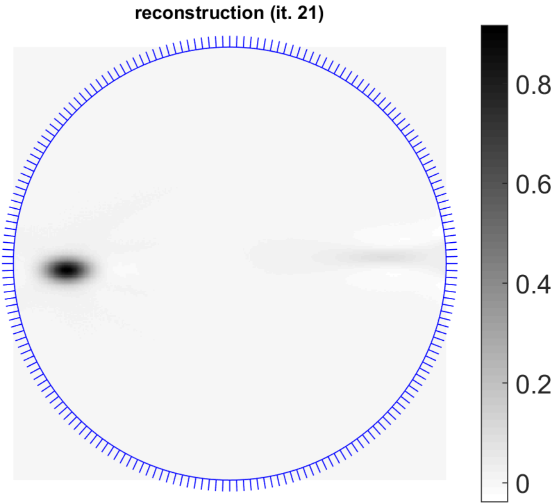


Figure: Variable attenuation with average = 0.6. Iteration #21.



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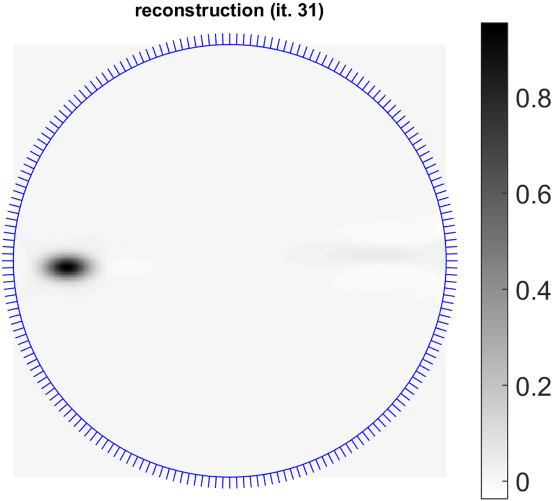

Figure: Variable attenuation with average = 0.6. Iteration #31.

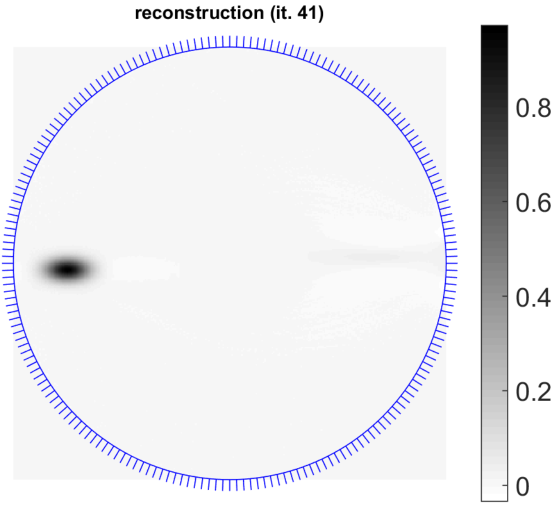

Figure: Variable attenuation with average = 0.6. Iteration #41.

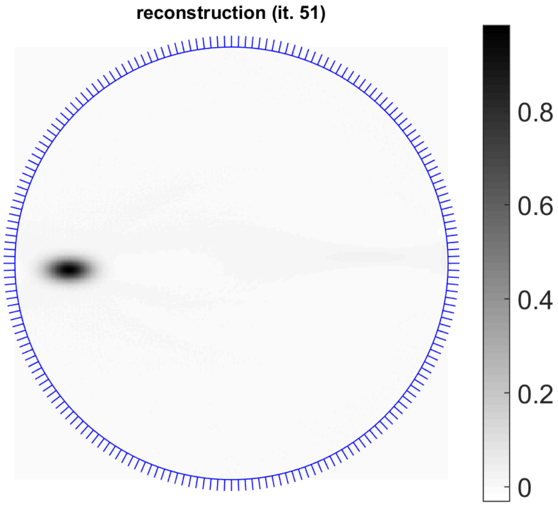

Figure: Variable attenuation with average = 0.6. Iteration #51.

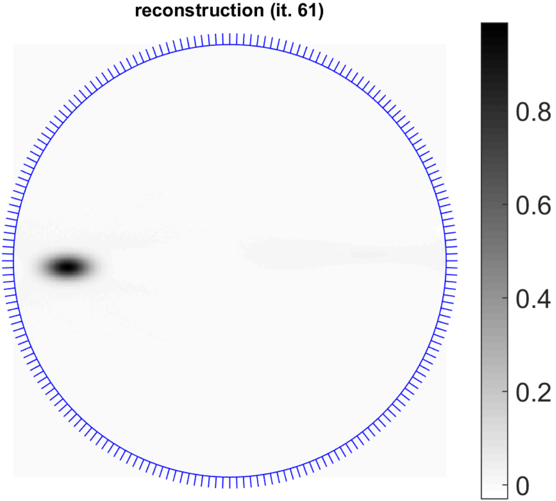

Figure: Variable attenuation with average = 0.6. Iteration #61.

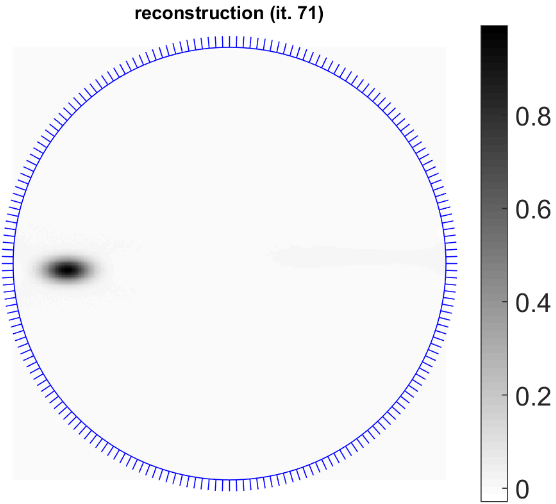

Figure: Variable attenuation with average $= 0.6$. Iteration #71.

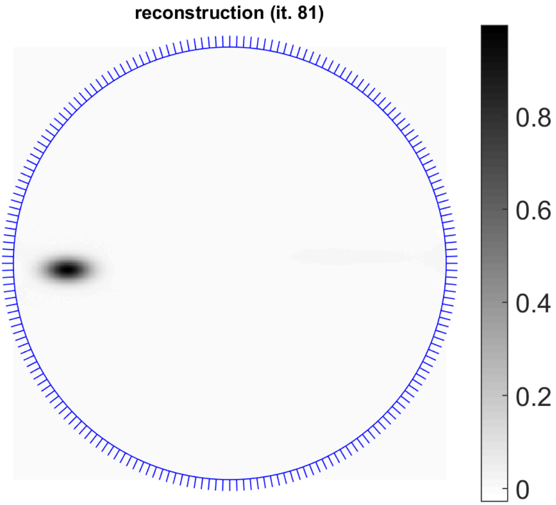

Figure: Variable attenuation with average $= 0.6$. Iteration #81.

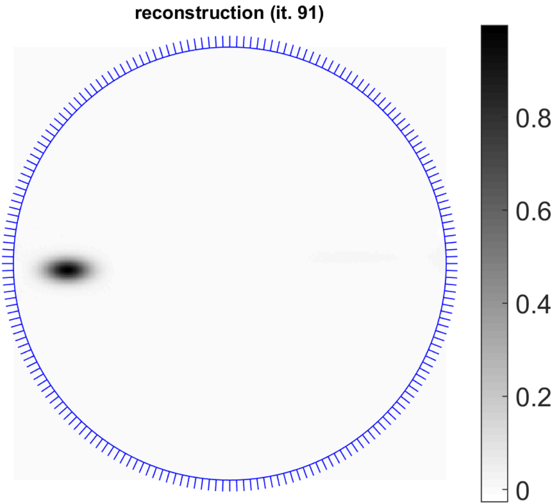

Figure: Variable attenuation with average = 0.6. Iteration #91.

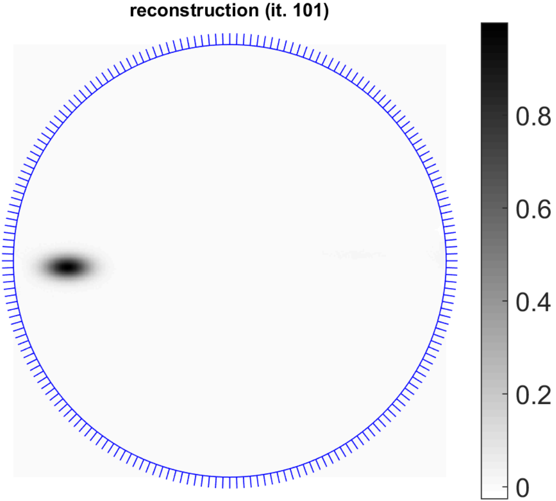

Figure: Variable attenuation with average = 0.6. Iteration #101.