Recovering Anisotropic Metrics from Travel Times

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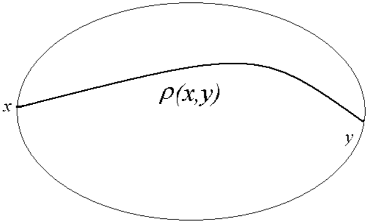
The Boundary Rigidity Problem (Inverse Kinematic Problem, Travel Time Tomography)

Let M be a bounded domain (manifold) with boundary. Let $g = \{g_{ij}\}$ be a Riemannian metric on M. Let $\rho(x, y)$ be the distance between any two boundary points x, y (in the metric g).



Bondary Rigidity: *Does ρ, known on $\partial M \times \partial M$, determine uniquely g?*

The distance $\rho(x, y)$ is equal to the travel time of a signal coming from x and measured at y under the following *simplicity conditions*:

- $\forall(x, y) \in \partial M \times \partial M$, there is unique geodesics connetcing x, y, depending smoothly on (x, y) (i.e., no caustics);
- ∂M is strictly convex.

Then we call g a *simple* metric.

The answer is negative because for every diffeomorphism ψ fixing ∂M pointwise, the metric $\psi_* g$ has the same data as g! Here,

$$(\psi_* g)_{ij} = g_{kl} \frac{\partial \psi^k}{\partial x_i} \frac{\partial \psi^l}{\partial x_j},$$

and we use the Einstein summation convention.

So the right question to ask is whether we can recover g, up to an isometry as above, from the boundary distance function.

In other words, if $\rho_{\hat g} = \rho_g$, is there a diffeo $\psi : M \to M$, $\psi|_{\partial M} = Id$, such that $\psi_* \hat g = g$?

Assume that g is isotropic, i.e., $g_{ij}(x) = c(x)\delta_{ij}$. Physically, this corresponds to a variable wave speed that does not depend on the direction of propagation. Then any ψ that is not identity, will make g anisotropic. Therefore, in the class of the isotropic metrics, we do not have the freedom to apply isometries and we would expect that g is uniquely determined. This is known to be true for simple metrics (Mukhometov, Romanov, et al.)

Our interest is in anisotropic metrics.

If the metric is not simple, the answer is negative.

Travel Time Seismology

This problem was first studied in the beginning of the 20th century by **Herglotz**, and **Wiechert & Zoeppritz** in an attempt to recover the inner structure of the Earth from travel times of seismic waves. They solved explicitly a partial case of this problem: when M is a ball, and g is a radially symmetric isotropic metric, i.e.,

$$ds^2 = a^2(r)dx^2, \quad r := |x|.$$

They imposed a simplicity assumption as well. Travel time seismology is still one of the main methods to study the inner structure of the Earth today.

Other possible applications: in medical imaging.

nice picture: actual travel times of seismic waves

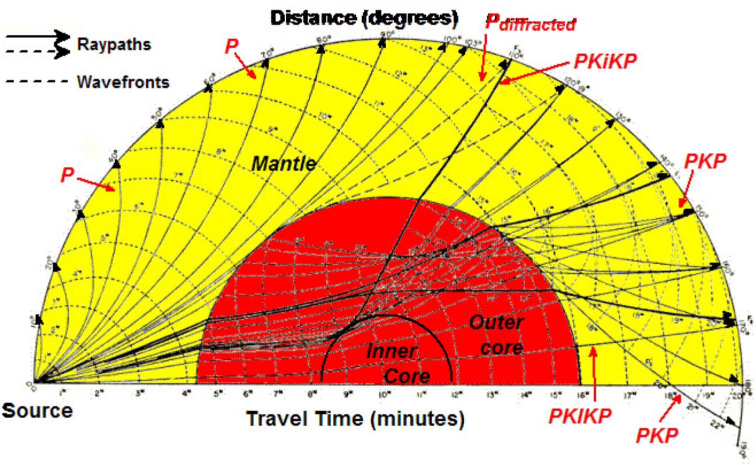
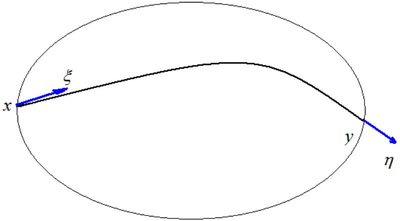


Figure: Travel times of P-waves through Earth. Picture taken from the web page of L. Braile, Purdue University.

Multiple Arrival Times: The Lens Rigidity Problem

Define the scattering relation σ and the length (travel time) function ℓ:



$$\sigma : (x, \xi) \to (y, \eta), \quad \ell(x, \xi) \to [0, \infty].$$

Diffeomorphisms preserving ∂M pointwise do not change σ, ℓ!

Lens rigidity: *Do σ, ℓ determine uniquely g, up to an isometry?*

In other words, if $\sigma_{\hat{g}} = \sigma_g$, $\ell_{\hat{g}} = \ell_g$, is there a diffeo $\psi : M \to M$, $\psi|_{\partial M} = Id$, such that $\psi_* \hat{g} = g$?

No, in general but the counterexamples are harder to construct.

The lens rigidity problem and the boundary rigidity one are equivalent for simple metrics! Indeed, then $\rho(x, y)$, known for x, y on ∂M determines σ, ℓ uniquely, and vice-versa.

For non-simple metrics (caustics and/or non-convex boundary), the Lens Rigidity is the right problem to study.

Even for non-simple metrics, one can still recover σ, ℓ from the travel times, but we need multiple arrival times, and a non-degeneracy assumption.

Those travel times are related to propagation of waves in anisotropic media modeled by a Riemannian metric. One sends a signal from a boundary point x and waits for the first signal to arrive at $y \in \partial M$: call that $\rho(x, y)$.

The model:

$$(\partial_t^2 - \Delta_g)u = 0$$

$$u|_{t\le 0} = 0, \quad u|_{\partial M} = f(t, x).$$

We measure $\Lambda f = \partial u/\partial \nu$ on $\mathbf{R}_+ \times \partial M$ (the hyperbolic Dirichlet-to-Neumann map).

If we know the whole Λf, $\forall f$, unique recovery of g is known [**Belishev** and **Kurylev**] but the proof is very unstable (based on unique continuation).

We want to know only the leading singularities of Λ,
and we show that we can recover g in a stable way.

There is a price to pay: we will impose some assumptions.

Brief history, simple metrics:

- **Mukhometov; Mukhometov & Romanov, Bernstein & Gerver, Croke, Gromov, Michel, Pestov & Sharafutdinov**
- Results for g conformal; flat; of negative curvature.
- **S & Uhlmann** '98: for g close to the Euclidean one.
- **Croke, Dairbekov and Sharafutdinov** '00: locally, near metrics with (explicitly) small enough curvature.
- **Lassas, Sharafutdinov & Uhlmann** '03: one metric with (explicitly) small curvature, one close to the Euclidean.
- **Pestov & Uhlmann** '03: $n = 2$, simple metrics (no smallness assumptions)
- **S & Uhlmann** '03-07: the results described below.

Lens Rigidity for non-simple metrics:

Very few results:

- **Croke** '05: If M is lens rigid, a finite quotient is lens rigid, too.
- **Croke & Kleiner** '94: counterexamples (M trapping).

Linearized problem:

Recover a tensor field f_{ij} from the geodesic X-ray transform

$$I_g f(\gamma) = \int f_{ij}(\gamma(t))\dot{\gamma}^i(t)\dot{\gamma}^j(t)\, dt$$

known for all (or some) max geodesics γ in M.

Every tensor admits an orthogonal decomposition into a *solenoidal* part f^s and a *potential* part dv,

$$f = f^s + dv, \quad v|_{\partial M} = 0.$$

where $\delta f^s = 0$.

Here the symmetric differential dv is given by $[dv]_{ij} = (\nabla_i v_j + \nabla_j v_i)/2$, and the divergence δ is given by: $[\delta f]_i = g^{jk}\nabla_k f_{ij}$. We have $I_g(dv) = 0$.

More precise formulation of the linearized problem: Does $I_g f = 0$ imply $f^s = 0$? We will call this *s-injectivity* of I_g.

It should not be surprising that $I_g(dv) = 0$. Take a diffeo

$$\psi = Id + \epsilon v + O(\epsilon^2).$$

Then

$$\psi_* g = g + \epsilon dv + O(\epsilon^2)$$

have the same data. Next, $v|_{\partial M} = 0$ because $\psi|_{\partial M} = Id$.

What is known about the linearized problem for simple metrics

- s-injectivity for functions (can be viewed as scalar multiples of the metric $\alpha(x)g_{ij}(x)$)
- also, for 1-tensors
- also, for metrics with small enough curvature (Pestov and Sharafutdinov)

Theorem 1

- *true for real analytic simple metrics*
- *true for metrics close enough to real analytic simple metrics*
- *Moreover, if I_g is s-injective for some simple g, there is a stability estimate of hypoelliptic type:*

$$\|f^s\|_{L^2(M)} \leq C\|N_g f\|_{\tilde{H}^2(M_1)}, \tag{1}$$

where $M \subset\subset M_1$, and $N_g = I_g^* I_g$; and C can be chosen uniform under small perturbations of g.

Here $H^2 \subset \tilde{H}^2 \subset H^1$, and roughly speaking, $\tilde{H}^2$ is defined as H^2 but second order normal derivatives are excluded.

More about the linear problem...

The main point is that the linear problem behaves like an elliptic one. One can construct explicitly a parametrix to N_g that involves a pseudodifferential operator step, and solving some elliptic BVP.

After that, one gets a Fredholm equation of the kind

$$(Id + K_g)f = h,$$

where K_g is compact (of order -1) and depends continuously on g.

If -1 is not an eigenvalue of K_g (happens when I_g is s-injective), then there is an estimate.

S-injectivity for analytic g is proved by using analytic microlocal analysis.

Results for the non-linear Boundary Rigidity Problem:

- uniqueness for isotropic (simple) metrics
- more generally, if g_{ij} and $\hat{g}_{ij} = \alpha(x) g_{ij}$ have the same data, then $\alpha = 1$

Theorem 2

- *If I_{g_0} is s-injective (g_0 simple), then there is local uniqueness near that g_0.*
- *Moreover, there is Hölder stability*

Corollary 3 (generic local uniqueness)

We have local uniqueness, up to isometry, near any simple g_0 in a generic set.

Results for non-simple manifolds:

We study more general manifolds than the simple ones. The right question to study then is the lens rigidity one (multiple arrival times).

- M does not need to be diffeomorphic to a ball (but some topological restrictions are still needed)
- ∂M does not need to be convex
- Conjugate points are allowed but some non-conjugacy assumptions are still made
- incomplete data

Main Condition:

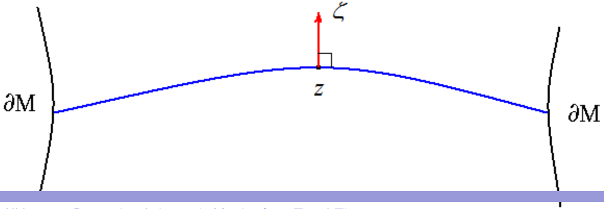
We study the scattering relation and travel times restricted to $(x, \xi) \in \mathcal{D} \subset \partial(SM)$. Here $\mathcal{D}$ is chosen so that the conormal bundle of the geodesics issued from $\mathcal{D}$ covers T^*M, and those geodesics have no conjugate points. Such $\mathcal{D}$ are called *complete*.

Definition 4

We say that $\mathcal{D}$ is **complete** for the metric g, if for any $(z, \zeta) \in T^*M$ there exists a maximal in M, finite length geodesic $\gamma : [0, l] \to M$ through z, normal to ζ, such that

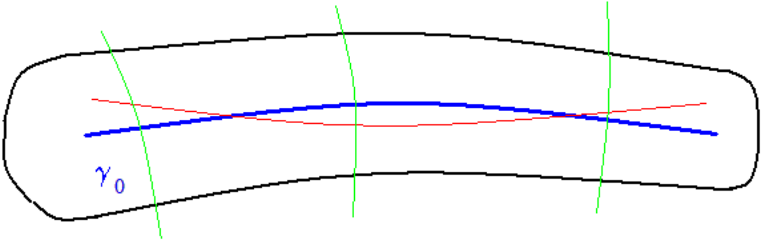
- γ belongs to our data (issued from $\mathcal{D}$);
- there are no conjugate points on γ.

We call g **regular**, if a complete set $\mathcal{D}$ exists, i.e., if the maximal $\mathcal{D}$ is complete.



Example 1: A cylinder around an arbitrary geodesic

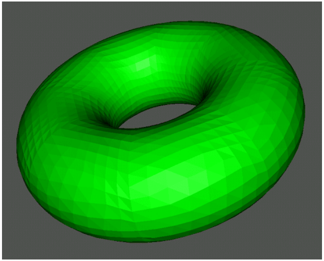
γ_0: a finite length geodesic segment on a Riemannian manifold, conjugate points are allowed.
M: a "cylinder" around γ_0, close enough to it.



One can study the scattering relation only for geodesics almost perpendicular to γ_0, there are no conjugate points on them.

Example 2: The interior of a perturbed torus

$M = S^1 \times \{x_1^2 + x_2^2 \le 1\}$, with g close to the flat one:



We need only geodesics almost perpendicular to the boundary. Note that M is trapping!

More generally, one can consider a tubular neighborhood of any periodic geodesic on any Riemannian manifold.

Even more generally, we can study $M \times N$, where M is simple, and N is arbitrary; and study σ for all geodesics over fixed points of N, and all those close to them. A small enough perturbation of this manifold satisfies our assumptions, and can have a terrible topology and all kinds of trapping rays and conjugate points.

The examples above are of that type.

New results for this class of non-simple manifolds:

About the linearized problem:

Theorem 5

Let g be analytic, $\mathcal{D}$ - open and complete. Then $I_{g,\mathcal{D}}$ is s-injective.

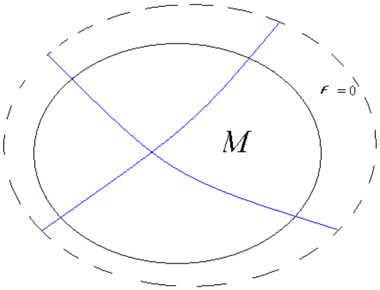
A very brief sketch of the proof

The proof is based on analytic microlocal analysis. We show that $N = I^*_{g,\mathcal{D}} I_{g,\mathcal{D}}$ is an analytic pseudo-differential operator. If $I_{g,\mathcal{D}} f = 0$, then $Nf = 0$ near M. We show that N is not elliptic, but restricted to solenoidal tensors, it is. Then f^s has to be analytic up to the boundary. Next, we show that all derivatives at ∂M vanish. Therefore, $f^s = 0$.

It is actually more complicated than that...

As an easy example, here is how one can prove this theorem for integrals of functions. Note that this is a partial case: if $f(x)$ is a function, not a tensor, then $f(x)g_{ij}$ is a tensor, and

$$\int f(\gamma)g_{ij}\dot{\gamma}^i\dot{\gamma}^j\,dt = \int f(\gamma)\,dt.$$



Extend f as zero outside M. Then Nf is still zero because integrals outside M are zero. Now, $Nf = 0$ implies the extended f is analytic. Since $f = 0$ outside M, then $f = 0$.

More results about the linear problem for non-simple metrics

Theorem 6 (stability)

Let $\mathcal{D}$ be open and complete. Then s-injectivity of $I_{g,\mathcal{D}}$ implies a locally uniform stability estimate.

In other words, injectivity implies stability!

Theorem 7 (generic s-injectivity)

Let $\mathcal{D}$ be open and complete for g in an open set $\mathcal{G}$ of regular metrics. Then there exists an open dense subset $\mathcal{G}_s$ of $\mathcal{G}$ (in the C^k topology, $k \gg 2$), so that $I_{g,\mathcal{D}}$ is s-injective for $g \in \mathcal{G}_s$.

Results about the non-linear lens rigidity problem:

Theorem 8 (local uniqueness if the linear problem is s-injective)

Let g_0 be regular, $\mathcal{D}$ - open and complete for g_0. Assume $\exists \mathcal{D}' \subset\subset D$ be such that $I_{g_0,D'}$ is s-injective.
Then for g, $\hat{g}$ close enough to g_0, the relations

$$\sigma = \hat{\sigma}, \quad \ell = \hat{\ell}, \quad \text{on } \mathcal{D}$$

imply that $\hat{g}$ is isometric to g.

In other words, uniqueness for the linear problem implies local uniqueness for the non-linear one. This implicit function type of theorem heavily depends on the hypoellipticity of the linear map.

By Thm 7, the condition in Thm 8 is generic:

Theorem 9 (generic local uniqueness)

Let $\mathcal{D}' \subset\subset \mathcal{D}$, $\mathcal{G}$, $\mathcal{G}_s$ be as above. Then the conclusion of Theorem 8 holds for any $g_0 \in \mathcal{G}_s$.

In other words, we get local uniqueness near a generic set of regular metrics (for σ, ℓ restricted to $\mathcal{D}$ that is complete for each $g \in \mathcal{G}$).

We believe that one can prove Hölder stability, too; like in the case of simple metrics. It seems doable but we expect it to be very technical.

Boundary Determination

Finally, a result on boundary determination. It is part of the proof of the theorems above but it is of independent interest as well.

Theorem 10

Let (M, g) be a compact Riemannian manifold with boundary. Let $(x_0, \xi_0) \in S(\partial M)$ be such that the maximal geodesic γ_{x_0,ξ_0} through it is of finite length, and assume that x_0 is not conjugate to any point in $\gamma_{x_0,\xi_0} \cap \partial M$. If σ and ℓ are known on some neighborhood of (x_0, ξ_0), then the jet of g at x_0 in boundary normal coordinates is determined uniquely.

Note that there are no generic or analyticity assumptions here. Previous results required convexity of ∂M. In that case, there is a Lipshitz stability estimate (S&Uhlmann) but no constructive recovery was known.
The proof of Thm 10 is actually constructive, and as such, implies Lipshitz stability.

How stable and constructive is all this?

- First, it is a perturbation method. Given a known g_0, and a unknown g close enough to g_0, we want to recover g from its scattering relation. We linearize, and we want to recover an approximation f to $g - g_0$.
- Recover the derivatives of g at ∂M: constructive, finite number are needed.
- We get $Nf = h$, with h known, and N related to g_0 (aslo known). Construct a parametrix to N of order 1 only.
- Get a Fredholm equation $(Id + K)f = \tilde{h}$ with K compact.
- Solve it, and get f.
- If g is simple, we proved that there is Hölder stability.

The linear problem for other families of curves.

Consider the weighted X-ray transform of *functions* over a general family of curves Γ:

$$If(\gamma) = \int w(\gamma(t), \dot\gamma(t)) f(\gamma(t))\, dt, \quad \gamma \in \Gamma.$$

On can assume that Γ are the solutions of a Newton-type of equation

$$\ddot x = G(x, \dot x)$$

with a generator G. (For example, $G = 0$ gives us lines).

Theorem 11 (Frigyik, S & Uhlmann)

I is injective for generic regular (G, w), including real analytic ones. There is a stability estimate.

Here, G is called regular, if the corresponding curves have no "conjugate points" on $\operatorname{supp} w$, and their conormal bundle (on $\operatorname{supp} w$) covers T^*M.

Magnetic systems.

On (M,g), consider an one form α, and the Hamiltonian

$$H = \frac{1}{2}(\xi+\alpha)_g^2.$$

The corresponding characteristics on the energy level $H = 1/2$ are called unit speed magnetic geodesics. They describe the trajectories of a charged particle in a magnetic field.

The lens rigidity is formulated in a similar way. The boundary rigidity is formulated in terms of the *action* $A(x,y)$, on $\partial M\times\partial M$, not the boundary distance function $\rho(x,y)$. The action $A(x,y)$ is defined by

$$A(x,y) = T(x,y) - \int_{\gamma_{[x,y]}} \alpha,$$

where $T(x,y)$ is the travel time from x to y, and $\gamma_{[x,y]}$ is the unit speed magnetic geodesic connecting x and y (under a simplicity assumption).

Magnetic Systems

In a joint work with Dairbekov, Paternain and Uhlmann, we study simple magnetic systems. We prove analogs of the results above. The linearized problem then reduces to the invertibility of the integral transform

$$I\phi(\gamma) = \int_\gamma \phi(\gamma, \dot\gamma)\, dt$$

for functions $\phi(x, \xi)$ that are quadratic in ξ:

$$\phi(x, \xi) = h_{ij}(x)\xi^i\xi^j + \beta_j(x)\xi^j.$$

Then I is called s-injective, if $I\phi = 0$ implies $h = dv$, $\beta = d\phi - Y(v)$, where $Y(\eta) = ((d\alpha)_i^j \eta_j)$.

The uniqueness of the non-linear problem is possible up to a gauge transformation only

$$g \mapsto \psi^* g, \quad \alpha \mapsto \psi^*\alpha + d\phi.$$