Travel Time Tomography and Tensor Tomography, III

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We now turn our attention to I_g acting on symmetric 2-tensors with $g_{ij} = \delta_{ij}$ (Euclidean). We will work in $\mathbf{R}^n$ first, assuming that f is compactly supported. Parameterize the geodesics (lines) by the direction ω and by the point z on the hyperplane $z^i\omega_i = 0$ where the line crosses that hyperplane. Then

$$I_g f(z,\omega) = \int f_{ij}(z + t\omega)\omega^i\omega^j \, dt.$$

Any $f \in L^2(\mathbf{R}^n)$ can then be orthogonally decomposed uniquely into a solenoidal and potential part (different from the decomposition above!)

$$f = f^s_{\mathbf{R}^n} + dv_{\mathbf{R}^n} \quad \text{in } \mathbf{R}^n,$$

such that $\delta f^s_{\mathbf{R}^n} = 0$ in $\mathbf{R}^n$ and $f^s_{\mathbf{R}^n}$, $dv_{\mathbf{R}^n}$ are in $L^2(\mathbf{R}^n)$. Similarly, we have

$$v_{\mathbf{R}^n} = (\delta d)^{-1}\delta f, \quad f^s_{\mathbf{R}^n} = f - d\,(\delta d)^{-1}\delta f, \tag{1}$$

with δd acting in the whole $\mathbf{R}^n$, and the notation $v_{\mathbf{R}^n}$ indicates that v is defined in the whole $\mathbf{R}^n$ and does not necessarily satisfy boundary conditions if f is supported in $\bar{\Omega}$. The inverse $(\delta d)^{-1}$ is defined through the Fourier transform. Actually, the latter provides a more detailed form of this decomposition.

We have

$$[\widehat{dv}]_{ij} = \frac{\sqrt{-1}}{2}\left(\xi_i \hat v_j + \xi_j \hat v_i\right), \quad \widehat{\delta f}_i = \sqrt{-1}\,\xi^j \hat f_{ij}.$$

Then

$$[\widehat{\delta dv}]_i = -\frac{1}{2}\left(|\xi|^2 \hat v_i + \xi_i \xi^j \hat v_j\right).$$

The operator δd is elliptic (multiply by $\hat v_i$ and take the sum). Solving this for $\hat v$ is easy. As a result, we get

$$(\hat f^s_{\mathbf{R}^n})_{kl} = \lambda^{ij}_{kl}(\xi)\hat f_{ij}(\xi),$$

where

$$\lambda^{ij}_{kl}(\xi) = \left(\delta^i_k - \frac{\xi_k\xi^i}{|\xi|^2}\right)\left(\delta^j_l - \frac{\xi_l\xi^j}{|\xi|^2}\right).$$

Note that $f^s_{\mathbf{R}^n}$ and $dv_{\mathbf{R}^n}$ may not be compactly supported even if f is. It follows from our integral representation that

$$(N_e f)^{kl}(x) = 2f_{ij} * \frac{x^i x^j x^k x^l}{|x|^{n+3}}.$$

Taking into account that $\mathcal{F}|x|^\alpha = (c_n/2)|\xi|^{-\alpha-n}$ with c_n as below, and Fourier transforming the latter, we get

$$\mathcal{F}(N_e f)^{kl}(\xi) = c_n \hat f_{ij}(\xi)\frac{\partial^4}{\partial\xi_i\partial\xi_j\partial\xi_k\partial\xi_l}|\xi|^3, \quad c_n = \frac{\pi^{(n+1)/2}}{3\Gamma(n/2+3/2)},$$

and

$$\partial^4|\xi|^3/\partial\xi_i\partial\xi_j\partial\xi_k\partial\xi_l = 3|\xi|^{-1}\sigma(\varepsilon^{ij}\varepsilon^{kl}), \quad \varepsilon^{ij}(\xi) = \delta^{ij} - \xi^i\xi^j/|\xi|^2.$$

Here $\sigma(\varepsilon^{ij}\varepsilon^{kl})$ is the symmetrization of $\varepsilon^{ij}\varepsilon^{kl}$, i.e., the mean of all similar products with all possible permutation of i, j, k, l. It is easy to see that $\delta N_e f = 0$ and that $f^s_{\mathbf{R}^n}$ can be recovered from $N_e f$ by the formula

$$[\hat{f}^s_{\mathbf{R}^n}]_{ij} = a_{ijkl}\mathcal{F}(N_e f)^{kl} = a^{kl}_{ij}\mathcal{F}(N_e f)_{kl},$$

where $a_{ijkl}(\xi)$ is a rational function, homogeneous of order 1 singular only at $\xi = 0$ with explicit form

$$a_{ijkl} = |\xi|\left(c_1\delta_{ik}\delta_{jl} + c_2(\delta_{ij} - |\xi|^{-2}\xi_i\xi_j)\delta_{kl}\right). \tag{2}$$

The coefficients c_1 and c_2 depend on n only. So we get that given $N_e f$, one can recover $f^s_{\mathbf{R}^n}$ by

$$f^s_{\mathbf{R}^n} = AN_e f, \tag{3}$$

where $A = A(D)$ has the symbol in (2). In particular,

$$I_e f = 0 \Longrightarrow f^s_{\mathbf{R}^n} = 0 \Longrightarrow f = d^s v_{\mathbf{R}^n}.$$

We almost proved the following.

Theorem 1

Let $\Omega \subset \mathbf{R}^n$ be convex, let g be a Euclidean, and let (Ω, g) be simple (i.e., Ω is strictly convex). Then I is s-injective.

We claim that if $If = 0$ and $\operatorname{supp} f \subset \bar{\Omega}$, then $\operatorname{supp} v_{\mathbf{R}^n} \subset \bar{\Omega}$. Indeed, we already showed that $f = dv_{\mathbf{R}^n}$. Next, since v can be obtained from f by applying a ΨDO of order -1 with homogeneous constant (w.r.t. x) symbol, see (1), we easily get that $|v| = O(|x|^{-1})$, as $|x| \to \infty$. Now, $dv_{\mathbf{R}^n} = 0$ outside Ω. Remember,

$$\frac{d}{dt} v(x + t\xi) \cdot \xi = [dv]_{ij} \xi^i \xi^j.$$

So we get

$$v_{\mathbf{R}^n}(x) \cdot \xi = v_{\mathbf{R}^n}(x + s\xi) \cdot \xi, \quad \forall (x, \xi) \in \partial_+ S\Omega,\ s > 0. \tag{4}$$

Take the limit $s \to \infty$ to conclude that $v_{\mathbf{R}^n}(x) \cdot \xi = 0$. Varying ξ, we get $v_{\mathbf{R}^n} = 0$ on $\partial\Omega$. This also holds if we enlarge $\partial\Omega$, then we get that $\operatorname{supp} v \subset \bar{\Omega}$. So we get that $v_{\mathbf{R}^n}$, restricted to Ω, coincides with v in the decomposition $f = f^s + dv$! Moreover, that restriction commutes with taking the symmetric differential d because $v = 0$ on $\partial\Omega$. So we get $f = dv$, i.e., f is potential field.

In general, given f and v, $v_{\mathbf{R}^n}$ related to f, we have $v_{\mathbf{R}^n} \neq v$ in Ω, and in particular, $v_{\mathbf{R}^n} \neq 0$ on $\partial\Omega$. We got an equality only under the assumption $If = 0$!

If our goal is not a proof of s-injectivity of I_e but a recovery of f^s from $N_e f$, then we can proceed as above. Namely, since $f^s_{\mathbf{R}^n} = f - dv_{\mathbf{R}^n}$ in Ω, d commutes with the extension as zero, and $f = 0$ outside Ω, similarly to (4), we can write

$$v_{\mathbf{R}^n}(x)\cdot\xi - v_{\mathbf{R}^n}(x+s\xi)\cdot\xi = \int_0^s (AN_e f)(x+tv)\,dt, \quad \forall (x,\xi)\in\partial_+ S\Omega,\ s>0. \tag{5}$$

Take the limit $s\to\infty$, to get

$$v_{\mathbf{R}^n}(x)\cdot\xi = \int_0^\infty (AN_e f)(x+tv)\,dt, \quad \forall (x,\xi)\in\partial_+ S\Omega,\ s>0. \tag{6}$$

Choose $n-1$ linearly independent ξ's above, and we have recovered $h := v_{\mathbf{R}^n}(x)|_{\partial\Omega}$ in terms of $N_e f$. Now,

$$f^s_{\mathbf{R}^n} = f - dv_{\mathbf{R}^n} \quad \text{but } v_{\mathbf{R}^n}|_{\partial M} \neq 0.$$

On the other hand,

$$f^s = f - dv \quad \text{with } v|_{\partial M} = 0.$$

Set $w = v_{\mathbf{R}^n} - v$. Then w is the solution w of the BVP

$$\delta dw = 0 \quad \text{in } \Omega, \quad w_{\mathbf{R}^n}|_{\partial\Omega} = h, \tag{7}$$

that we can solve (h is known). Then in Ω,

$$f^s = f^s_{\mathbf{R}^n} + dw = AN_e f + dw, \tag{8}$$

and w is expressible in terms of $N_e f$.

One can show that the principal symbol of N is the same as in the Euclidean case (!) with the proper interpretation of raised and lowered indices. However, that is not even needed. We do not need the formulas, we just need to show ellipticity on solenoidal tensors.

Recall that the principal symbol of N (in case of the weighted ray transform of functions)

$$p(x,\xi) = 2\pi \int_{S_xM} |\alpha(x,\theta)|^2 \delta(\xi\cdot\theta)\, d\sigma(\theta).$$

For tensors, the "weight" is $\theta^i\theta^j$. We get that the principal symbol in that case is

$$\sigma_p(N)^{ijkl}(x,\xi) = 2\pi \int_{S_xM} \theta^i\theta^j\theta^k\theta^l \delta(\xi\cdot\theta)\, d\sigma(\theta).$$

(OK, it is not exactly what we have for functions. The reason is that the meaning of adjoint is somewhat different for tensors).

Let us check the ellipticity of $\sigma_p(N)$.

$$\begin{aligned}
\sigma_p(N)^{ijkl} h_j \bar{h}_{kl} &= 2\pi \int_{S_xM} \left(h_{ij}\theta^i\theta^j\right)\left(\bar{h}_{kl}\theta^k\theta^l\right)\delta(\xi\cdot\theta)\, d\sigma(\theta)\\
&= 2\pi \int_{S_xM} \left|h_{ij}\theta^i\theta^j\right|^2 \delta(\xi\cdot\theta)\, d\sigma(\theta)
\end{aligned}$$

For this to be elliptic, we need to show that $h_{ij}\theta^i\theta^j = 0$ for $\theta \perp \xi$ implies $h = 0$. Of course, this is not true. But if in addition, we know that

$$\xi^i h_{ij} = 0, \tag{9}$$

then this would be true.

Since $h = \hat{f}(\xi)$, the l.h.s. of (9) is the principal part of δf. So we get that

N is elliptic on solenoidal tensors.

Therefore, one can apply a parametrix to recover the solenoidal projection of f. Well, not exactly because we can only work in the interior of M.

What we actually do is to work in a neighborhood M_1 of M, and recover $f^s_{M_1}$ in the interior of M_1. Then we compare f^s and $f^s_{M_1}$ to recover (up to a smoothing operator) f^s in M. This allows us to use the Fredholm theory arguments.

Main results for the linear problem

- I_g has a finitely dimensional smooth kernel in the subspace of solenoidal tensors.
- If g is analytic, the kernel is trivial, i.e., I_g is s-injective (requires analytic microlocal analysis)
- If I_g is s-injective, then there is an a priori stability estimate

$$\|f^s\|_{L^2(M)} \leq C\|Nf\|_{H^1(M_1)}$$

with $C > 0$ that is locally constant under small C^k perturbations of g, $k \gg 1$.
- Therefore, I_g is s-injective for a dense open set of simple metrics.

Main results for the boundary rigidity problem for simple metrics

- Near any g with an s-injective I_g, there is local uniqueness.
- Moreover, there is a Hölder stability estimate

Theorem 2

For any $\mu < 1$, there exits $k \gg 1$ such that for any simple $g_0 \in C^k$ with an s-injective I_{g_0}, there are $\varepsilon_0 > 0$ and $C > 0$ with the property that that for any two metrics g_1, g_2 with $\|g_m - g_0\|_{C^2(M)} \le \varepsilon_0$, and $\|g_m\|_{C^k(M)} \le A$, $m = 1, 2$, with some $A > 0$, we have the following stability estimate

$$\|g_2 - \psi_* g_1\|_{C^2(M)} \le C(A) \|\rho_{g_1} - \rho_{g_2}\|^{\mu}_{C(\partial M \times \partial M)}$$

with some diffeomorphism $\psi : M \to M$ fixing the boundary pointwise.

Sharafutdinov and Pestov have shows with different methods s-injectivity under an explicit upper bound of the curvature.