

# UNIQUENESS AND NON-UNIQUENESS IN INVERSE RADIATIVE TRANSFER

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ABSTRACT. We characterize the non-uniqueness in the inverse problem for the stationary transport model, in which the absorption  $a$  and the scattering coefficient  $k$  of the media are to be recovered from the albedo operator. We show that “gauge equivalent” pairs  $(a, k)$  yield the same albedo operator, and that we can recover uniquely the class of the gauge equivalent pairs. We apply this result to show unique determination of the media when the absorption  $a$  depends on the line of travel through each point while scattering  $k$  obeys a symmetry property. Previously known results concerned directional independent absorption  $a$ .

## 1. INTRODUCTION

This paper considers the problem of recovering the absorption and scattering properties of a bounded, convex medium  $\Omega \subset \mathbb{R}^n$ ,  $n \geq 3$  from the spatial-angular measurements of the density of particles at the boundary  $\partial\Omega$ . Provided that the particles interact with the medium but not with each other, the radiation transfer in the steady-state can be modeled by the transport equation

$$(1) \quad -\theta \cdot \nabla u(x, \theta) - a(x, \theta)u(x, \theta) + \int_{S^{n-1}} k(x, \theta', \theta)u(x, \theta')d\theta' = 0,$$

for  $x \in \Omega$  and  $\theta \in S^{n-1}$ ; see, e.g. [9, 22]. The function  $u(x, \theta)$  represents the density of particles at  $x$  traveling in the direction  $\theta$ ,  $a(x, \theta)$  is the absorption coefficient at  $x$  for particles traveling in the direction of  $\theta$ , and  $k(x, \theta', \theta)$  is the scattering coefficient (or the collision kernel) which accounts for particles from an arbitrary direction  $\theta'$  which scatter in the direction of travel  $\theta$ .

The medium is probed with the given radiation

$$(2) \quad u|_{\Gamma_-} = f_-$$

and the exiting radiation

$$(3) \quad u|_{\Gamma_+} =: \mathcal{A}[f_-]$$

is detected; where

$$(4) \quad \Gamma_{\pm} = \{(x, \theta) \in \partial\Omega \times S^{n-1} : \pm\theta \cdot n(x) > 0\}$$

with  $n(x)$  denoting the outer unit normal at a boundary point  $x$ . The equation (3) defines the albedo operator  $\mathcal{A}$  which takes the incoming flux  $f_-$  to the outgoing flux  $u|_{\Gamma_+}$  at the boundary.

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In general, the boundary value problem (1) and (2) may not be uniquely solvable but it has a unique solution for generic  $(a, k)$ ; see [26] and Proposition 3.1 below. Unique solvability can also be obtained under some physically relevant subcritical conditions like (6) or (7) below. We assume that  $(a, k)$  is such that the direct problem (1) and (2) is well posed.

The inverse boundary value problem of radiative transfer is to recover the absorption  $a(x, \theta)$  and the scattering kernel  $k(x, \theta', \theta)$  from knowledge of the albedo operator  $\mathcal{A}$ . One could also study the time-dependent version of (1), and then the kernel of  $\mathcal{A}$  contains one more variable that gives us more information. This problem has been solved under some restrictive assumptions (e.g.  $k$  of a special type or independent of a variable) in [1, 2, 3, 5, 16, 17, 18, 27, 28]. In three or higher dimensions, uniqueness and reconstruction results for general  $k$  and  $a = a(x)$  were established in [11]. The approach there is based on the study of the singularities of the fundamental solution of (1) (see also [8]), and the singularities of the Schwartz kernel of  $\mathcal{A}$ . Stability estimates for  $k$  of special type were established in [24, 29]; and recently, for general  $k$ , in [6]. Uniqueness and reconstruction results in a Riemannian geometry setting, including recovery of a simple metric, were established in [19]. Similar results for the time-dependent model were established in [10], and in [13] for the Riemannian case. In planar domains the work in [25] shows stable determination of the isotropic absorption and small scattering, and extensions to simple Riemannian geometry is given in [20]. Also in two dimensional domains we point out that the recovery of  $k$  is only known under smallness conditions which are more restrictive than what is needed to solve the direct problem; e.g. more restrictive than (6) or (7) below. On the other hand, in the time-dependent case, the extra variable allows us to treat the planar case without such restrictions, see [10]. We also mention here the recent works [7, 14, 15], in which the coefficients are recovered from angularly averaged measurements rather than from knowledge of the whole the albedo operator  $\mathcal{A}$ .

The above mentioned results concern media with directionally independent absorption property; for transport with variable velocity  $v$ , which now belongs to an open subset of  $\mathbb{R}^n$ , the absorption may depend on the speed  $a = a(x, |v|)$ .

In general, in media with an anisotropic absorption, the albedo operator does not determine the coefficients uniquely. For example, if  $k \equiv 0$  the obstruction to unique determination can be readily seen. The most one can recover from the albedo operator  $\mathcal{A}$  are the integrals  $\int_{\mathbb{R}} a(x + t\theta, \theta) dt$ , see [11]. In [6], they are shown to be stably recovered independently of  $k$ . In other words, for each fixed direction  $\theta \in S^{n-1}$ , we know the integral of the map  $a(\cdot, \theta)$  over the parallel lines in the direction of  $\theta$ . This is insufficient data, since one can smoothly change the  $x$ -variable in the direction of  $\theta$  while preserving the integral; see also the theorem below.

The non-uniqueness described above and the need to assume that  $a$  is isotropic left the theory of the inverse radiative problem in a somewhat unsatisfactory state. It was not clear whether the uniqueness failed if  $k \neq 0$ , neither it was known what information about  $(a, k)$  can still be recovered. The purpose of this work is to fill this gap. We show that a certain family of “gauge transformations” of  $(a, k)$  does not change the albedo operator  $\mathcal{A}$ ; and that, given  $\mathcal{A}$ , one can recover uniquely the

class of gauge equivalent pairs  $(a, k)$ . The recovery of the gauge equivalent class is explicit, as the recovery of  $a = a(x)$  and  $k$  in [11] is explicit.

## 2. MAIN RESULTS

The pair  $(a, k)$  is assumed to satisfy the *admissibility* condition

$$(5) \quad \sup_{(x, \theta) \in \Omega \times S^{n-1}} \left( |a(x, \theta)| + \int_{S^{n-1}} |k(x, \theta, \theta')| d\theta' \right) < \infty.$$

Let  $T$  be the operator defined by the l.h.s. of (1). For  $(a, k)$  admissible, the second and the third terms of  $T$  are bounded operators on  $L^1(\Omega \times S^{n-1})$ , while the first term is unbounded. We view  $T$  as an unbounded operator on  $L^1(\Omega \times S^{n-1})$  with the domain

$$D(T) = \{f \in L^1(\Omega \times S^{n-1}); \theta \cdot \nabla f \in L^1(\Omega \times S^{n-1}), f|_{\Gamma_-} = 0\},$$

see also [11]. Since the direct problem (1) and (2) can always be reduced to a non-homogeneous problem with a homogeneous boundary condition, invertibility of  $T$  implies well-posedness. We say that the direct problem is well posed, if  $T^{-1}$  exists as a bounded operator.

As an example, we have the following two *subcritical* conditions that yield well-posedness; see, e.g., [6, 11, 12, 21, 22] and Proposition 3.1 below. Either

$$(6) \quad \sup_{(x, \theta) \in \Omega \times S^{n-1}} \left| \tau(x, \theta) \int_{S^{n-1}} k(x, \theta, \theta') d\theta' \right| < 1,$$

where  $\tau(x, \theta)$  is the total free path of a particle at  $(x, \theta)$ , see the beginning of the next section, or

$$(7) \quad a(x, \theta) - \int_{S^{n-1}} k(x, \theta, \theta') d\theta' \geq 0, \text{ a.e. } \Omega \times S^{n-1}.$$

both  $a$  and  $k$  outside  $\Omega$  to

We start with a simple observation, that seems to be new. We will show that there is non-uniqueness even if  $k \not\equiv 0$ , too. Let  $\phi(x, \theta) > 0$  be such that  $\phi = 1$  on  $\Gamma := \partial\Omega \times S^{n-1}$ . Set  $\tilde{a} = a - \theta \cdot \nabla_x \log \phi$ . Then we can rewrite (1) as

$$-\theta \cdot \nabla_x u - \phi^{-1}(\theta \cdot \nabla_x \phi)u - \tilde{a}u + \int_{S^{n-1}} k(\cdot, \theta', \theta)u(\cdot, \theta') d\theta' = 0.$$

Multiply by  $\phi$  to get

$$-\theta \cdot \nabla_x (\phi u) - \tilde{a}\phi u + \phi \int_{S^{n-1}} k(\cdot, \theta', \cdot)u(\cdot, \theta') d\theta' = 0.$$

The function  $\tilde{u} = \phi u$  thus solves (1) with  $(a, k)$  replaced by  $(\tilde{a}, \tilde{k})$ , where those two pairs are related by the “gauge transformation”

$$(8) \quad \tilde{a} = a - \theta \cdot \nabla_x \log \phi, \quad \tilde{k}(x, \theta', \theta) = \frac{\phi(x, \theta)}{\phi(x, \theta')} k(x, \theta', \theta).$$

Since  $\phi = 1$  on  $\Gamma$ , the boundary data do not change. Therefore, if  $\tilde{\mathcal{A}}$  is the albedo operator corresponding to the pair  $(\tilde{a}, \tilde{k})$ , then  $\mathcal{A} = \tilde{\mathcal{A}}$ .

Our main result is that this is the only obstruction to non-uniqueness.

**Theorem 2.1.** *Let  $(a, k)$  and  $(\tilde{a}, \tilde{k})$  be two admissible pairs for which the direct problem is well-posed, and let  $\mathcal{A}$  and  $\tilde{\mathcal{A}}$  be the corresponding albedo operators. Then  $\mathcal{A} = \tilde{\mathcal{A}}$  if and only if there exists a positive  $\phi \in L^\infty(\Omega \times S^{n-1})$  with  $\theta \cdot \nabla_x \phi(x, \theta) \in L^\infty(\Omega \times S^{n-1})$  and  $\phi = 1$  for  $x \in \partial\Omega$ , so that (8) hold.*

The proof of the theorem is based on the analysis of the singularities of the Schwartz kernel of  $\mathcal{A}$ , as in [11].

Theorem 2.1 allows us to obtain a few new uniqueness results under additional conditions. One of them concerns the case where we have anisotropic media with absorption  $a(x, \theta)$  that depends on the line of travel through each point (but not on the direction):

$$(9) \quad a(x, \theta) = a(x, -\theta), \quad x \in \Omega, \quad \theta \in S^{n-1},$$

and a scattering coefficient  $k > 0$  satisfying the following symmetry condition

$$(10) \quad k(x, \theta, \theta') = k(x, \theta', \theta), \quad x \in \Omega, \quad \theta, \theta' \in S^{n-1}.$$

**Corollary 2.2.** *Let  $(a, k)$ ,  $(\tilde{a}, \tilde{k})$  be two admissible and subcritical pairs which yield the same albedo operator. Assume that  $k$  and  $\tilde{k} > 0$  satisfy (10).*

(i) *Then  $k = \tilde{k}$  and  $a = \tilde{a} + \theta \cdot \nabla v(x)$  for some function  $v(x)$  vanishing on  $\partial\Omega$ . In particular, one can recover the total absorption at a.e.  $x$ , i.e.,  $\int a(x, \theta) d\theta = \int \tilde{a}(x, \theta) d\theta$ .*

(ii) *If, in addition,  $a$  and  $\tilde{a}$  satisfy (9), then  $a = \tilde{a}$ .*

Note that any two pairs as in (i) yield the same albedo operator, so this is the most we can say in this case. The symmetry assumption (10) occurs naturally in some models of Optical Tomography, where the scattering of light in a tissue depends on the angle between the two directions:  $k(x, \theta, \theta') = k(x, \theta \cdot \theta')$ .

One can formulate and prove similar results in the case where the velocity belongs to an open subspace of  $\mathbb{R}^n$ , i.e., the speed can change, as in [11]. We restrict ourselves to the fixed speed case ( $|\theta| = 1$ ) for the sake of simplicity of the exposition. Also, the fixed speed model is the one that is most often discussed in the literature.

The paper is organized as follows. Section 3 recalls some results from [11] that we use later. In Section 4 we prove Theorem 2.1 and its Corollary 2.2. In section 5 we give the reconstruction formulae for continuous  $a$  and  $k$  in the symmetric case covered by Corollary 2.2. Section 6 contains concluding remarks.

### 3. PRELIMINARIES

In this section we recall some results from [11] recast to the one-speed velocities and introduce notations.

Let  $\tau_\pm(x, \theta)$  be the travel time it takes a particle at  $x$  to reach the boundary while moving in the direction of  $\pm\theta$  and define  $\tau(x, \theta) = \tau_-(x, \theta) + \tau_+(x, \theta)$ . Since we work with unit-speed velocities, note that  $\tau(x, \theta) \leq \text{diam}(\Omega)$ . Let  $d\xi(x, \theta) = |n(x) \cdot \theta| d\mu(x) d\theta$  where  $d\mu(x)$  is the induced Lebesgue measure on the boundary and  $d\theta$  is the normalized measure on the sphere. Also let  $\delta_{\{x\}}(x')$  represent the delta

distribution with respect to the boundary measure  $d\mu(x')$  supported at  $x \in \partial\Omega$  and  $\delta_{\{\theta\}}(\theta')$  represents the delta distribution with respect to  $d\theta$  centered at  $\theta \in S^{n-1}$ .

Also, for  $(x, \theta) \in \Omega \times S^{n-1}$ , let  $x_\theta^+$  denote the exiting point if traveling from  $x$  in the  $\theta$ -direction and  $x_{\theta'}^-$  be the entrance point at the boundary to reach the inside point  $x$  by traveling in the  $\theta'$ -direction, i.e.

$$(11) \quad x_\theta^+ := x + \tau_+(x, \theta) \text{ and } x_{\theta'}^- := x - \tau_-(x, \theta')\theta'.$$

For the proposition below, we introduce the class of *regular* scattering kernels  $k \in C(\bar{\Omega}, L^\infty(S^{n-1}, L^1(S^{n-1})))$ . Then the map  $\int k(x, \theta', \theta)\phi(\theta')d\theta'$  is bounded on  $L^1(S^{n-1})$  continuously depending on  $x$ . Our notion of regular  $k$  is stronger than that in [21], and, in particular, it allows us to use the results in there.

**Proposition 3.1.** *The direct problem is well-posed, i.e.,  $T$  has a bounded inverse on  $L^1(\Omega \times S^{n-1})$ , if either (6) or (7) holds. Moreover, the direct problem is well-posed for an open dense set of  $(a, k) \in L^\infty(\Omega \times S^{n-1}) \times C(\bar{\Omega}, L^\infty(S^{n-1}, L^1(S^{n-1})))$ .*

*Proof.* We will first discuss the well-posedness under the conditions (6) or (7). Assume (6) first. By [11, Proposition 2.3], and since  $\tau$  is bounded, we get that  $T^{-1}$  is bounded. The subcritical case (7) is covered in [6, Section 2].

The generic statement is proven in  $L^2(\Omega \times S^{n-1})$  for  $C^2$  coefficients in [26]. In the  $L^1$  spaces under consideration, we proceed in a similar way. Let  $K$  be the integral operator in (1) and  $T_1 = T - K$ . Then  $KT_1^{-1}K$  is weakly compact in  $L^1(\Omega \times S^{n-1})$ , see [21]. Therefore,  $(KT_1^{-1})^2$  is weakly compact, and its square is compact. For a fixed  $a$ , consider the family  $\lambda k$ , where  $\lambda$  is a real parameter. By the analytic Fredholm alternative in Banach spaces [23],  $\lambda \mapsto (I - (\lambda KT_1^{-1})^4)^{-1}$  is a meromorphic family. This implies that  $\lambda \mapsto (I - \lambda KT_1^{-1})^{-1}$  is also meromorphic, and thus  $T^{-1} = (I - \lambda KT_1^{-1})^{-1}T_1^{-1}$  exists for all but a discrete set of  $\lambda$ 's. This shows that there is a dense set of pairs yielding a well-posed problem. The fact that this set is also open follows from a perturbation argument around each  $(a, k)$ , for which  $T^{-1}$  is bounded; thus  $(I - \lambda KT_1^{-1})^{-1}$  corresponding to nearby pairs exists.  $\square$

Note that one can set  $a(x, \theta) = a_0(x, \theta) + \int k(x, \theta, \theta')d\theta'$ , where the integral represents the attenuation due to the change of direction, while  $a_0$  is the absorption. Then one can prove in the same way that the direct problem is well posed for generic  $(a_0, k)$ , and moreover, for any  $a_0$ , this is true for generic  $k$ 's.

**Proposition 3.2.** *Assume that the direct problem is well-posed. Then the albedo operator  $\mathcal{A} : L^1(\Gamma_-, d\xi) \rightarrow L^1(\Gamma_+, d\xi)$  is bounded and its Schwartz kernel  $\alpha$  is given by  $\alpha = \alpha_1 + \alpha_2 + \alpha_3$ , where*

$$(12) \quad \alpha_1(x, \theta, x', \theta') = e^{\int_0^{\tau_-(x, \theta)} a_\theta(x-t\theta)dt} \delta_{\{x_\theta^-\}}(x') \delta_{\{\theta\}}(\theta')$$

$$(13) \quad \alpha_2(x, \theta, x', \theta') = \int_0^{\tau_-(x, \theta)} e^{-\int_0^s a(x-t\theta, \theta)dt} e^{-\int_0^{\tau_-(x-s\theta, \theta')} a(x-s\theta-t\theta', \theta')dt} \\ \times k(x-s\theta, \theta', \theta) \delta_{\{x-s\theta-\tau_-(x-s\theta, \theta')\theta'\}}(x') ds$$

$$(14) \quad |n(x') \cdot \theta'|^{-1} \alpha_3 \in L^\infty(\Gamma_-; L^1(\Gamma_+, d\xi)).$$

This proposition is formulated in [11] under the assumption that the system is subcritical, i.e., either (6) or (7) holds. We remark that those conditions are only used in the proof in the analysis of  $\alpha_3$ , to guarantee that  $T^{-1}$  exists in  $L^1$ ; something that we assume here.

Let  $\phi \in C_0^\infty(B(0;1))$  with  $0 \leq \phi \leq 1$  and  $\phi \equiv 1$  near the origin be a cut-off function. Given  $\epsilon > 0$  we define for  $x, x' \in \mathbb{R}^n$  and  $\theta, \theta' \in S^{n-1}$

$$(15) \quad \phi_\epsilon(x, \theta, x', \theta') = \phi\left(\frac{x_\theta^- - x'}{\epsilon}\right) \phi\left(\frac{\theta - \theta'}{\epsilon}\right).$$

**Proposition 3.3.** *Assume that the direct problem is well posed. Then the limit below holds in  $L^1(\Gamma_+, d\xi)$*

$$(16) \quad \lim_{\epsilon \rightarrow 0} \int_{\Gamma_-} \alpha(x, \theta, x', \theta') \phi_\epsilon(x, \theta, x', \theta') d\mu(x') d\theta = e^{-\int_{-\infty}^{\infty} a(x+t\theta, \theta) dt} > 0 \text{ a.e. } \Gamma_+;$$

For linearly independent  $\theta, \theta' \in S^{n-1}$ , let  $\pi_{\theta, \theta'}(x)$  denote the projection of  $x$  onto the plane through the origin spanned by  $\theta$  and  $\theta'$ . Let  $\theta'_\perp$  be a unit vector in  $\text{span}\{\theta, \theta'\}$  orthogonal to  $\theta'$ :  $\theta'_\perp \cdot \theta' = 0$ . Let  $\varphi \in C_0^\infty(-1, 1)$  with  $0 \leq \varphi \leq 1$  and  $\int_{\mathbb{R}} \varphi(t) dt = 1$  be a cut-off function in  $\mathbb{R}$  and  $\phi$  be the cut-off function in  $\mathbb{R}^n$  introduced earlier. Define the test function

$$(17) \quad \phi_{\epsilon, \delta}(x; \theta, \theta') = \frac{1}{\epsilon} \varphi\left(\frac{x \cdot \theta'_\perp}{\epsilon \theta \cdot \theta'_\perp}\right) \phi\left(\frac{x - \pi_{\theta, \theta'}(x)}{\delta}\right).$$

With  $x_\theta^+$  and  $x_\theta^-$  given by (11) and  $\phi_{\epsilon, \delta}$  given by (17), we define in  $\Omega \times S^{n-1} \times S^{n-1}$

$$(18) \quad I_{\epsilon, \delta}(x, \theta', \theta) := \int_{\partial\Omega} \alpha(x_\theta^+, \theta, x', \theta') \phi_{\epsilon, \delta}(x' - x_\theta^-, \theta, \theta') d\mu(x').$$

**Proposition 3.4.** *Assume that the direct problem is well posed. Then the following limit holds in  $L^1(\Omega; L_{loc}^1(I^n))$*

$$(19) \quad I_{0,0}(x, \theta', \theta) := \lim_{\epsilon \rightarrow 0} \lim_{\delta \rightarrow 0} I_{\epsilon, \delta}(x, \theta', \theta) = e^{-\int_{-\infty}^0 a(x+t\theta', \theta') dt - \int_0^{\infty} a(x+t\theta, \theta) dt} k(x, \theta', \theta),$$

where  $I^n := \{(\theta, \theta') \in S^{n-1} \times S^{n-1} : \theta' \neq \pm\theta\}$ .

#### 4. PROOF OF THEOREM 2.1 AND COROLLARY 2.2

*Proof of Theorem 2.1.* From Proposition 3.3, we can recover

$$(20) \quad \int_{-\infty}^{\infty} a(x + s\theta, \theta) ds, \quad (x, \theta) \in \Omega \times S^{n-1}.$$

In particular, the integrals above for  $f = a - \tilde{a}$  vanish:

$$(21) \quad \int_{-\infty}^{\infty} f(x + s\theta, \theta) ds = 0, \quad \forall (x, \theta) \in \Omega \times S^{n-1}.$$

The kernel of the linear transform (20) is easy to describe. Set

$$(22) \quad v(x, \theta) = \int_{-\infty}^0 f(x + s\theta, \theta) ds, \quad (x, \theta) \in \Omega \times S^{n-1}.$$

Then  $v \in L^\infty(\Omega \times S^{n-1})$  with  $\theta \cdot \nabla_x v = f \in L^\infty(\Omega \times S^{n-1})$  and  $v = 0$  on  $\Gamma_-$ . From (21) we get also that  $v = 0$  on  $\Gamma_+$ , therefore  $v(x, \theta) = 0$  for  $x \in \partial\Omega$  and a.e.  $\theta \in S^{n-1}$ .

Set

$$(23) \quad \phi(x, \theta) = e^{v(x, \theta)}$$

to get  $a - \tilde{a} = \theta \cdot \nabla_x \log \phi$  as claimed. This shows the first part of (8).

Now, from Proposition 3.4 we get

$$e^{-\int_{-\infty}^0 a(x+t\theta', \theta') - \int_0^\infty a(x+t\theta, \theta)} k(x, \theta', \theta) = e^{-\int_{-\infty}^0 \tilde{a}(x+t\theta', \theta') - \int_0^\infty \tilde{a}(x+t\theta, \theta)} \tilde{k}(x, \theta', \theta).$$

Using the first equality in (8) that we already proved, after a simple calculation, we derive

$$(24) \quad \tilde{k}(x, \theta', \theta) = \frac{\phi(x, \theta)}{\phi(x, \theta')} k(x, \theta', \theta).$$

□

*Proof of Corollary 2.2.* Next we use the characterization above to prove uniqueness in the symmetric case. By swapping  $\theta$  and  $\theta'$  in (24) we get

$$\begin{aligned} \tilde{k}(x, \theta, \theta') &= \frac{\phi(x, \theta')}{\phi(x, \theta)} k(x, \theta, \theta') = \frac{\phi(x, \theta')}{\phi(x, \theta)} k(x, \theta', \theta) \\ &= \frac{\phi(x, \theta')^2}{\phi(x, \theta)^2} \tilde{k}(x, \theta', \theta) = \frac{\phi(x, \theta')^2}{\phi(x, \theta)^2} \tilde{k}(x, \theta, \theta'). \end{aligned}$$

For the second and fourth equality we used (10), while for the third one we used (24). Since  $\tilde{k}$  does not vanish, we conclude that  $\phi(x, \theta') = \phi(x, \theta)$  for all  $\theta, \theta' \in S^{n-1}$ . Therefore  $\phi = \phi(x)$  is independent of  $\theta$  and applying (24) again we conclude that  $k = \tilde{k}$ .

So far we showed that  $a(x, \theta) - \tilde{a}(x, \theta) = \theta \cdot \nabla_x \phi(x)$ . If the symmetry relation (9) holds then  $a(x, \theta) - \tilde{a}(x, \theta) = -\theta \cdot \nabla_x \phi(x)$ . Therefore  $a - \tilde{a} = 0$ . □

## 5. RECONSTRUCTION FORMULAS IN THE SYMMETRIC CASE

We showed in the previous section that under the symmetry hypotheses (10) and (9) there is uniqueness for  $a, k$ . The proof is constructive but it still leads to the problem of recovering  $a(x, \theta)$  first, up to  $\theta \cdot \nabla \log \phi$ , from its underdetermined X-ray transform, see the l.h.s. of (21). In this section, we show that under the same assumptions, there is an explicit reconstruction of a different, simpler type. Namely, we recover the integrals in (21) first, that measure the total absorption along straight lines, but we are not trying to determine  $a$  from them. Then by (13) we recover

$$e^{-\int_{-\infty}^0 a(x+t\theta', \theta') dt - \int_0^\infty a(x+t\theta, \theta) dt} k(x, \theta', \theta),$$

see (28) below. We do not know the attenuation term above because we know the attenuation along straight but not broken lines. We can however swap  $\theta$  and  $\theta'$  and multiply the results. Then we get  $k^2$  multiplied by the attenuation along the two lines through  $x$  parallel to  $\theta$ , and  $\theta'$ , respectively, and we know that attenuation. This recovers  $k$ , and then we recover  $a$ .

We need to strengthen the regularity assumptions on the coefficients to

$$(25) \quad a \in C(\Omega \times S^{n-1}), \quad k \in C(\Omega \times S^{n-1} \times S^{n-1}).$$

Firstly, we extend the limit in Proposition 3.3 valid for maps in  $\Gamma_+$  to maps in  $\Omega \times S^{n-1}$ . For  $(x, \theta) \in \Omega \times S^{n-1}$  and  $\epsilon > 0$ , let us denote

$$(26) \quad J_\epsilon(x, \theta) := \int_{\Gamma_-} \alpha(x_\theta^+, \theta, x', \theta') \phi_\epsilon(x, \theta, x', \theta') d\mu(x') d\theta',$$

where  $x_\theta^+$  is given in (11). We remark here that  $J_\epsilon$  is constant with  $x$  varying in the direction of  $\theta$ , i.e.  $J_\epsilon(x + t\theta, \theta) = J_\epsilon(x, \theta)$  for all  $t$ .

**Lemma 5.1.** *The limit below holds in  $L^1(\Omega \times S^{n-1}; \tau(x, \theta)^{-1} dx d\theta)$*

$$(27) \quad J_0(x, \theta) := \lim_{\epsilon \rightarrow 0} J_\epsilon(x, \theta) = \exp\left(-\int_{-\infty}^{\infty} a(x + t\theta, \theta) dt\right) > 0 \text{ a.e. } \Omega \times S^{n-1}.$$

*Proof.* We use the change of variable formula

$$\int_{\Omega \times S^{n-1}} f(x, \theta) dx d\theta = \int_{\Gamma_+} \int_0^{\tau_-(x^+, \theta)} f(x^+ - s\theta) ds d\xi(x^+, \theta)$$

for  $f(x, \theta) = \left| J_\epsilon(x, \theta) - \exp\left(\int_{-\infty}^{\infty} a(x + t\theta, \theta) dt\right) \right| \tau(x, \theta)^{-1}$  to estimate

$$\begin{aligned} & \int_{\Omega \times S^{n-1}} \left| J_\epsilon(x, \theta) - \exp\left(\int_{-\infty}^{\infty} a(x + t\theta, \theta) dt\right) \right| \tau(x, \theta)^{-1} dx d\theta \\ &= \int_{\Gamma_+} \int_0^{\tau_-(x^+, \theta)} \frac{\left| J_\epsilon(x^+ - s\theta, \theta) - e^{\left(\int_{-\infty}^{\infty} a(x^+ - s\theta + t\theta, \theta) dt\right)} \right|}{\tau(x^+ - s\theta, \theta)} ds d\xi(x^+, \theta) \\ &= \int_{\Gamma_+} \left| J_\epsilon(x^+, \theta) - \exp\left(\int_{-\infty}^{\infty} a(x^+ + t\theta, \theta) dt\right) \right| d\xi(x^+, \theta). \end{aligned}$$

The last identity holds due to the fact that  $\tau(x^+ - s\theta, \theta) = \tau(x^+, \theta) = \tau_-(x^+, \theta)$  and that the integrand is constant in  $s$ . An application of Proposition 3.3 finishes the proof.  $\square$

Following from Proposition 3.4 for a.e.  $(x, \theta, \theta') \in \Omega \times I^n$  we get that

$$(28) \quad e^{-\int_{-\infty}^0 a(x+t\theta', \theta') dt - \int_0^{\infty} a(x+t\theta, \theta) dt} k(x, \theta', \theta) = I_{0,0}(x, \theta', \theta),$$

$$(29) \quad e^{-\int_{-\infty}^0 a(x+t\theta, \theta) dt - \int_0^{\infty} a(x+t\theta', \theta') dt} k(x, \theta, \theta') = I_{0,0}(x, \theta, \theta').$$

Using the symmetry assumption (10) on  $k$ , by multiplication of (28) and (29) we obtain

$$(30) \quad e^{-\int_{-\infty}^0 a(x+t\theta', \theta') dt - \int_{-\infty}^{\infty} a(x+t\theta, \theta) dt} k^2(x, \theta', \theta) = I_{0,0}(x, \theta', \theta) I_{0,0}(x, \theta, \theta').$$

Now using Lemma 5.1 we recover

$$(31) \quad k(x, \theta', \theta) = \left( \frac{I_{0,0}(x, \theta', \theta) I_{0,0}(x, \theta, \theta')}{J_0(x, \theta) J_0(x, \theta')} \right)^{\frac{1}{2}}.$$



Using the formula (31) in (29) we get for a.e.  $(x, \theta, \theta') \in \Omega \times I^n$  that

$$(32) \quad \int_{-\infty}^0 a(x + t\theta, \theta) dt + \int_0^{\infty} a(x + t\theta', \theta') dt = \frac{1}{2} \log \left( \frac{I_{0,0}(x, \theta', \theta)}{I_{0,0}(x, \theta, \theta') J_0(x, \theta) J_0(x, \theta')} \right).$$

The continuity assumption (25) imply that the identity above extends pointwise in  $\Omega \times S^{n-1} \times S^{n-1}$ . By looking at backscattering  $\theta' = -\theta$  and making one change of variables  $t \leftrightarrow -t$  we obtain

$$(33) \quad \int_{-\infty}^0 a(x + t\theta, \theta) dt + \int_{-\infty}^0 a(x + t\theta, -\theta) dt = \frac{1}{2} \log \left( \frac{I_{0,0}(x, -\theta, \theta)}{I_{0,0}(x, \theta, -\theta) J_0(x, \theta) J_0(x, -\theta)} \right).$$

The left hand side is differentiable in  $x$  in the direction of  $\theta$ , hence so is the right hand side. By differentiating (33) in the direction of  $\theta$  we obtain

$$(34) \quad a(x, \theta) + a(x, -\theta) = \frac{1}{2} \theta \cdot \nabla_x \log \left( \frac{I_{0,0}(x, -\theta, \theta)}{I_{0,0}(x, \theta, -\theta) J_0(x, \theta) J_0(x, -\theta)} \right).$$

Since the absorption depends on the line through  $x$ , we have  $a(x, \theta) = a(x, -\theta)$  and thus it can be recovered from the formula:

$$(35) \quad a(x, \theta) = \frac{1}{4} \theta \cdot \nabla_x \log \left( \frac{I_{0,0}(x, -\theta, \theta)}{I_{0,0}(x, \theta, -\theta) J_0(x, \theta) J_0(x, -\theta)} \right).$$

## 6. REMARKS

**6.1. The isotropic absorption case.** The previously known uniqueness result [11] for isotropic absorption follow from Theorem 2.1 combined with the injectivity of the X-ray transform. If  $f = a - \tilde{a}$  depends on the position only, then (21) implies  $f = 0$ . From the definition (23) we get  $\phi \equiv 1$ , which by (8) yields  $k = \tilde{k}$ .

**6.2. Other conditions that imply uniqueness.** Assuming (10), we can only recover  $a$  up to  $\theta \cdot \nabla v(x)$  as above. For unique recovery, the condition (9) suffices but it can be replaced by something weaker. For example, we may require that  $a$  is orthogonal to all such functions w.r.t. some measure  $d\mu(\theta)$ . This is equivalent to

$$\int_{S^{n-1}} \theta \cdot \nabla_x a(x, \theta) d\mu(\theta) = 0.$$

Note that the symmetry assumption (9) implies such a condition if  $d\mu(\theta)$  is even.

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