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Indices of normalization of ideals [☆]

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ABSTRACT

We derive numerical estimates controlling the intertwined properties of the normalization of an ideal and of the computational complexity of general processes for its construction. In [18], this goal was carried out for equimultiple ideals via the examination of Hilbert functions. Here we add to this picture, in an important case, how certain Hilbert functions provide a description of the locations of the generators of the normalization of ideals of dimension zero. We also present a rare instance of normalization of a class of homogeneous ideals by a single colon operation.

1. Introduction

Let R be a Noetherian ring and let I be an ideal. The *normalization* of I is the integral closure $\overline{A}$ in $R[t]$ of the Rees algebra $A = R[It]$ of I. The properties of $\overline{A}$ add significantly to an understanding of I and of the constructions it supports. The *index* terminology refers to the integers related to the construction of

$$\overline{A} = \sum_{n\geq 0} \overline{I^n}t^n = R[\overline{I}t, \ldots, \overline{I^{s_0}}t^{s_0}].$$

In addition to the overall task of describing the generators and relations of $\overline{A}$, we wish to understand the following quantities:

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(i) Numerical indices for module generation: find s such that

$$(\overline{A})_{n+s} = (A)_n \cdot (\overline{A})_s \quad \text{for all } n \geq 0.$$

(ii) Complexity of algorithms: estimate the number of steps that effective processes must traverse between A and $\overline{A}$,

$$A = A_0 \subset A_1 \subset \cdots \subset A_{r-1} \subset A_r = \overline{A}.$$

(iii) Generators of $\overline{A}$: number and distribution of their degrees in cases of interest.

The main goal becomes the estimation of these indices in terms of invariants of I. This paper is a sequel to [18], where some of the notions developed here originated. The focus in [18] was on deriving bounds on the coefficient $e_1(\overline{A})$ of the Hilbert polynomial associated to ideals of finite co-length in local rings, and its utilization in the estimation of the length of general normalization algorithms. Here we introduce complementary notions and use them to address some of the same goals for more general ideals, but also show how known initial knowledge about the normalization allows us to give fairly detailed descriptions of $\overline{A}$, particularly those affecting the distribution of its generators.

We now outline the organization of the paper. Section 2 gives the precise definitions of the indices mentioned above and describes some relationships amongst them (Proposition 2.3 and Theorem 2.4). These indices acquire a sharp relief when the normalization $\sum_{n\geq 0} \overline{I^n} t^n$ is Cohen–Macaulay (Theorem 2.5). This result, whose proof follows *ipsis literis* the characterization of Cohen–Macaulayness for the Rees algebras of I-adic filtrations ([1], [15], [21]), has various consequences. It is partly used to motivate the treatment in Section 3 of the Sally module of the normalization algebra as a vehicle to study the number of generators and their degrees. In case the associated graded ring of the integral closure filtration $\mathcal{F}$ of a zero-dimensional ideal I, $\mathrm{gr}_{\mathcal{F}}(R)$, is Cohen–Macaulay or has depth at least $\dim R - 1$, there are several positivity relations on the Hilbert coefficients, leading to descriptions of the distribution of the new generators (usually fewer as the degrees go up), and overall bounds for their numbers (Corollary 3.3 and Theorem 3.7).

In Section 4, we present one of the rare instances where the normalization of the Rees ring is computed using an explicit expression as a colon. Our formula applies to homogeneous ideals that are generated by forms of the same degree and satisfy some additional assumptions (Theorem 4.1).

2. Normalization of ideals

This section introduces auxiliary constructions and devices to examine the integral closure of ideals, and to study applications to normal ideals.

2.1. Indices of normalization

We begin by introducing some measures for the normalization of ideals. Suppose R is a Noetherian ring, $\mathcal{F} = \{I_n,\ n \geq 0\}$ is a weakly decreasing multiplicative filtration of ideals with $I_0 = R$, and J is an ideal contained in I_1. One says that J is a *reduction* of $\mathcal{F}$ if there exists an integer $r \geq 0$ so that $I_{n+1} = JI_n$ for all $n \geq r$. The least such r is the *reduction number* of $\mathcal{F}$ with respect to J; it is denoted $\mathrm{r}_J(\mathcal{F})$. Whenever $\{I^n,\ n \geq 0\}$ is an ideal adic filtration, we talk about a reduction J of the ideal I and its reduction number $\mathrm{r}_J(I)$ with respect to J. We will often deal with the integral closure filtration $\{\overline{I^n},\ n \geq 0\}$ of an ideal I; if this filtration is the I-adic filtration, we say that I is a *normal* ideal.

If I is an ideal of a Noetherian local ring R with infinite residue field k, then I has a *minimal reduction*. The minimal number of generators of any minimal reduction of I is the Krull dimension of the ring $k \otimes_R R[It]$;

this dimension is called the *analytic spread* of I and denoted by $\ell(I)$. For a proper ideal the analytic spread satisfies the inequality $\operatorname{ht} I \leq \ell(I) \leq \min\{\dim R, \mu(I)\}$.

Recall that if R is an analytically unramified Noetherian local ring then $\overline{A} = \sum_{n\geq 0} \overline{I^n} t^n$ is finitely generated as a module over $A = R[It]$, according to [19, Theorem 1.4]. Hence the following indices are well defined integers.

Definition 2.1. Let $(R, \mathfrak{m})$ be an analytically unramified Noetherian local ring and let I be an ideal.

(i) The *normalization index* of I is the smallest non-negative integer $s = s(I)$ such that

$$\overline{I^{n+1}} = I \cdot \overline{I^n} \quad \text{for all } n \geq s.$$

(ii) The *generation index* of I is the smallest non-negative integer $s_0 = s_0(I)$ such that

$$\sum_{n\geq 0} \overline{I^n} t^n = R[\overline{I}t, \ldots, \overline{I^{s_0}} t^{s_0}].$$

For example, if $R = k[x_1, \ldots, x_d]$ is a polynomial ring in $d \geq 1$ variables over a field and $I = (x_1^d, \ldots, x_d^d)$, then $\overline{I} = (x_1, \ldots, x_d)^d$, which is a normal ideal. It follows that $s_0(I) = 1$, while $s(I) = \mathrm{r}_I(\overline{I}) = d - 1$.

These indices have an expression in terms of the special fiber ring F of the normalization map $A \longrightarrow \overline{A}$.

Proposition 2.2. *With the above assumptions let*

$$F = \overline{A}/(\mathfrak{m}, It)\overline{A} = \sum_{n\geq 0} F_n.$$

We have

$$\begin{aligned} s(I) &= \sup\{m \mid F_m \neq 0\}, \\ s_0(I) &= \inf\{m \mid F = F_0[F_1, \ldots, F_m]\} \qquad \textit{if } A \neq \overline{A}. \end{aligned}$$

Furthermore, if the index of nilpotency of F_n is r_n, then

$$s(I) \leq \binom{\lfloor \frac{s_0(I)}{2} \rfloor + 1}{2} + \sum_{n=\lfloor \frac{s_0(I)}{2} \rfloor + 1}^{s_0(I)} n(r_n - 1)\,.$$

Proof. The equalities for $s(I)$ and $s_0(I)$ are a consequence of the Nakayama Lemma.

To prove the final inequality we use the equality for $s(I)$. We write $m = s_0(I)$ so that $F = F_0[F_1, \ldots, F_m]$. Let $z = z_1 \cdots z_m$ be a non-zero element of F, where each z_n is a product of $a_n \geq 0$ factors from F_n. First let $n \leq \lfloor \frac{m}{2} \rfloor$. In this case $2n \leq m$, and if $a_n \geq 2$ we may replace 2 factors from F_n in z_n by 1 factor from F_{2n} in z_{2n}, without changing z. Repeating this procedure, we achieve that $a_n \leq 1$ for $1 \leq n \leq \lfloor \frac{m}{2} \rfloor$. Next assume that $\lfloor \frac{m}{2} \rfloor + 1 \leq n \leq m$. In this range $a_n \leq r_n - 1$, since otherwise $z_n = 0$ and then $z = 0$. Therefore

$$\deg z = \sum_{n=1}^{m} n a_n \leq \sum_{n=1}^{\lfloor \frac{m}{2} \rfloor} n + \sum_{n=\lfloor \frac{m}{2} \rfloor + 1}^{m} n(r_n - 1) = \binom{\lfloor \frac{m}{2} \rfloor + 1}{2} + \sum_{n=\lfloor \frac{m}{2} \rfloor + 1}^{m} n(r_n - 1)\,,$$

as required. $\square$

It is not clear, even when R is a regular ring, which invariants of R and of I have a bearing on the determination of $s(I)$. An affirmative case is that of a monomial ideal I of a polynomial ring over a field in $d \geq 1$ variables – when $s \leq d-1$ (according to Corollary 2.6).

2.2. Zero-dimensional ideals

For zero-dimensional ideals there are relations between the two indices of normalization.

Proposition 2.3. *Let $(R, \mathfrak{m})$ be an analytically unramified Cohen–Macaulay local ring of dimension $d \geq 1$ and assume that $\mathfrak{m}$ is a normal ideal. Let I be an $\mathfrak{m}$-primary ideal with multiplicity $e(I)$ and write $m = s_0(I)$. Then*

$$s(I) \leq e(I)\frac{d}{d+1}\Big((m+1)^{d+1} - (\lfloor \frac{m}{2} \rfloor + 1)^{d+1}\Big) + 2d\binom{\lfloor \frac{m}{2} \rfloor + 1}{2} - (2d-1)\binom{m+1}{2}.$$

Proof. Without loss of generality, we may assume that the residue field of R is infinite. Following Proposition 2.2, we estimate $s(I)$ in terms of the indices of nilpotency of the components F_n, for $n \leq s_0(I)$.

Let $J = (z_1, \ldots, z_d)$ be a minimal reduction of I. For each component $I_n = \overline{I^n}$ of $\overline{A}$, we collect the following data:

$$\begin{aligned} J_n &= (z_1^n, \ldots, z_d^n), \text{ a minimal reduction of } I_n \\ e(I_n) &= e(I)\, n^d, \text{ the multiplicity of } I_n \\ \mathrm{r}_{J_n}(I_n) &\leq \frac{e(I_n)}{n}\, d - 2d + 1, \text{ a bound on the reduction number of } I_n\,. \end{aligned}$$

The last inequality follows from [27, Theorem 2.45 and Remark 2.46], once it is observed that $I_n \subset \overline{\mathfrak{m}^n} = \mathfrak{m}^n$, by the normality of $\mathfrak{m}$.

We are now ready to estimate the index of nilpotency r_n of the component F_n. With the notation above, we have ${I_n}^{r+1} = J_n I_n^r$ for $r = \mathrm{r}_{J_n}(I_n)$, hence ${I_n}^{r+1} \subset I \cdot I_{rn+n-1}$. When this containment is read in F, it means that $r_n \leq \mathrm{r}_{J_n}(I_n) + 1$.

From Proposition 2.2 and the last inequality for the index of nilpotency we obtain

$$\begin{aligned} s(I) &\leq \binom{\lfloor \frac{m}{2} \rfloor + 1}{2} + \sum_{n=\lfloor \frac{m}{2} \rfloor + 1}^{m} n \cdot \mathrm{r}_{J_n}(I_n) \\ &= \binom{\lfloor \frac{m}{2} \rfloor + 1}{2} + \sum_{n=\lfloor \frac{m}{2} \rfloor + 1}^{m} n \cdot \left(\frac{e(I)\, n^d\, d}{n} - 2d + 1\right) \\ &= e(I)\, d \sum_{n=\lfloor \frac{m}{2} \rfloor + 1}^{m} n^d \; + 2d\binom{\lfloor \frac{m}{2} \rfloor + 1}{2} - (2d-1)\binom{m+1}{2}. \end{aligned}$$

Finally, approximating the sum with an elementary integral we get the assertion. □

We can do considerably better when I is a homogeneous ideal in a polynomial ring over a field of characteristic zero.

Theorem 2.4. *Let $R = k[x_1, \ldots, x_d]$ be a polynomial ring over a field of characteristic zero and let I be a homogeneous ideal that is $(x_1, \ldots, x_d)$-primary. One has*

$$s(I) \leq (e(I)-1)\binom{s_0(I)+1}{2} - (e(I)-2)\binom{\lfloor \frac{s_0(I)}{2} \rfloor + 1}{2}.$$

Proof. We begin by localizing R at the maximal homogeneous ideal. We are going to prove that the indices of nilpotency of the components F_n as in Proposition 2.2 are bounded above by $e(I)$. An application of the proposition and delocalizing back to the original homogeneous ideals then implies the assertion.

Let J be a minimal reduction of I. We denote the associated graded ring of the filtration of integral closures $\{I_n = \overline{I^n}\}$ by G,

$$G = \sum_{n \geq 0} I_n/I_{n+1}.$$

This ring has dimension d, it is integral over $G_0[z_1, \ldots, z_d]$, where the z_i's are the images in G_1 of a minimal generating set of J, and G_0 is a finite dimensional vector space over k. It follows that $C = k[z_1, \ldots, z_d]$ is a homogeneous Noether normalization of G.

There are two basic facts about the ring G. First, its rank as a C-module is the same as its multiplicity as a graded C-module, which is equal to $e(I)$. Second, since the Rees algebra of the integral closure filtration is a normal domain, so is the extended Rees algebra

$$D = \sum_{n \in \mathbb{Z}} I_n t^n,$$

where we set $I_n = R$ for $n \leq 0$. Consequently the algebra $G = D/(t^{-1})$ satisfies Serre's condition S_1. Since this algebra is also equidimensional, it follows that it is torsionfree as a C-module.

We now apply the theory of Cayley–Hamilton equations to the elements of the components of G (see [26, Chapter 9]): Write $r = e(I)$. Recall that $r = \operatorname{rank}_C G$ and that G is a torsionfree C-module. Thus by [26, Proposition 9.3.4], every $u \in G_n$ satisfies an equation of integrality over C of degree at most r,

$$u^r + a_1 u^{r-1} + \cdots + a_r = 0\,,$$

with $a_i \in C_{ni}$. Since k has characteristic zero, using the argument of [26, Proposition 9.3.5], we then obtain an equality

$$G_n^r = C_n G_n^{r-1}.$$

At the level of the filtration, this equality means that

$$I_n^r \subset J^n I_n^{r-1} + I_{rn+1}\,.$$

As k has characteristic zero and I is the localization of a homogeneous ideal, we have $\overline{I^t} \subset (x_1, \ldots, x_d)\,\overline{I^{t-1}}$ by [27, Corollary 7.15]. It follows that

$$I_n^r \subset I \cdot I_{rn-1} + (x_1, \ldots, x_d)\, I_{rn} \quad \text{for every } n \geq 1.$$

Finally, in F, this equation shows that the indices of nilpotency of the components F_n are bounded by $r = e(I)$, as desired. $\square$

2.3. Cohen–Macaulay normalization

Not surprisingly, normalization indices are easier to obtain when the normalization of the ideal is Cohen–Macaulay. The following is directly derived from the known characterizations of Cohen–Macaulayness of Rees algebras of ideals in terms of associated graded rings and reduction numbers ([1], [15], [21]).

Theorem 2.5. *Let R be a Cohen–Macaulay local ring and let $\{I_n,\ n \geq 0\}$ be a weakly decreasing multiplicative filtration of ideals, with $I_0 = R$, $I_1 = I$, and the property that the corresponding Rees algebra $B = \sum_{n\geq 0} I_n t^n$ is finitely generated as a module over $R[It]$. Assume that* $\operatorname{ht} I \geq 1$*, write $\ell = \ell(I)$, and let J be a reduction of I.*

If B is a Cohen–Macaulay ring, then

$$\operatorname{reg} B \leq \ell - 1\,.$$

In particular, the $R[Jt]$-module B is generated by forms of degrees at most $\ell - 1$,

$$I_{n+1} = JI_n = II_n \quad \text{for } n \geq \ell - 1,$$

and

$$B = R[I_1 t, \ldots, I_{\ell-1}t^{\ell-1}]$$

unless $\ell = 1$, in which case $B = R[It] = R[Jt]$.

By $\operatorname{reg} B$ we denote the Castelnuovo–Mumford regularity of B as a finitely generated graded module over the standard graded Noetherian R-algebra $R[It]$ or, equivalently, over $R[Jt]$. Notice that the above bound for $\operatorname{reg} B$ also shows that the ideal defining the R-algebra B can be generated by forms of degrees at most ℓ.

If R is analytically unramified, then the assumption that B is finitely generated as an $R[It]$-module simply means that the filtration $\{I_n,\ n \geq 0\}$ is a subfiltration of the integral closure filtration of the powers of I, $I_n \subset \overline{I^n}$.

The proof of Theorem 2.5 relies on substituting in any of the proofs mentioned above ([1, Theorem 5.1], [15, Theorem 2.3], [21, Corollary 3.6]) the I-adic filtration $\{I^n\}$ by the filtration $\{I_n\}$. We provide an outline here.

Proof of Theorem 2.5. We may assume that the residue field of R is infinite, that $I \neq R$, and that J is a minimal reduction of I, which is necessarily generated by ℓ elements. Let $G = \sum_{n\geq 0} I_n/I_{n+1}$ be the associated graded ring of the filtration $\{I_n,\ n \geq 0\}$. One has $\dim B = d+1$ and $\dim G = d$, where $d = \dim R$. From the Cohen–Macaulayness of B and the exact sequences (originally paired in [12]),

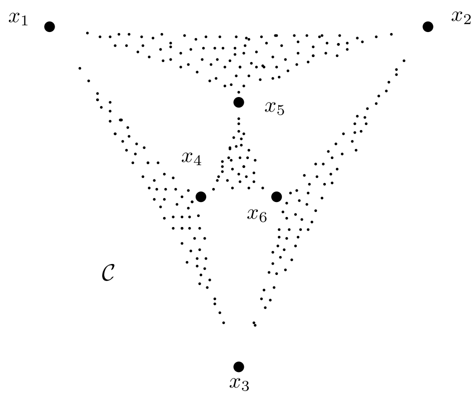
$$0 \to B_+ \longrightarrow B \longrightarrow R \to 0$$
$$0 \to B_+(1) \longrightarrow B \longrightarrow G \to 0\,,$$

one sees, as in the proof of [22, Theorem 1.1], that the local cohomology modules of G with support in the maximal homogeneous ideal of G are concentrated in negative degrees. The same is true if one replaces R by $R_\mathfrak{p}$ for any $\mathfrak{p} \in \operatorname{Spec}(R)$. Thus by [15, Proposition 2.1(i)], also the local cohomology modules of G with support in the irrelevant ideal G_+ are concentrated in negative degrees. Since G is a finitely generated graded module over $R[Jt]$ and J is generated by ℓ elements, one has $\mathrm{H}^i_{G_+}(G) = 0$ for $i > \ell$. It follows that $\operatorname{reg} G \leq \ell - 1$. Finally, $\operatorname{reg} G = \operatorname{reg} B$, as can be seen as in the proof of [14, Proposition 4.1]. $\square$

Corollary 2.6. *Let $(R, \mathfrak{m})$ be an analytically unramified Cohen–Macaulay local ring and let I be an ideal of height ≥ 1. If $\overline{R[It]}$ is Cohen–Macaulay, then both indices of normalization $s(I)$ and $s_0(I)$ are at most $\ell(I) - 1$ (unless $\ell(I) = 1$, in which case $s_0(I) = 1$). In particular, if I^n is integrally closed for $n < \ell(I)$, then I is normal.*

A case this applies to is that of monomial ideals in a polynomial ring over a field, since then $\overline{R[It]}$ is Cohen–Macaulay by Hochster's theorem ([10, Theorem 1]) (see also [20]).

Example 2.7. Let $I = I(\mathcal{C}) = (x_1x_2x_5, x_1x_3x_4, x_2x_3x_6, x_4x_5x_6)$ be the edge ideal associated to the clutter



Consider the incidence matrix M of this clutter, i.e., the matrix whose columns are the exponent vectors of the monomials of I. Since the polyhedron $Q(M) = \{x \mid xM \geq 1;\ x \geq 0\}$ is integral, we have the equality $\overline{R[It]} = R_s(I)$, where $R_s(I)$ denotes the symbolic Rees algebra of I ([7, Proposition 3.13]; see also [12, Theorem 2.1]). The ideal I is not normal because the monomial $m := x_1x_2x_3x_4x_5x_6$ is in $\overline{I^2} \setminus I^2$.

The minimal primes of I are:

$$\begin{array}{llll} \mathfrak{p}_1 = (x_1, x_6), & \mathfrak{p}_2 = (x_2, x_4), & \mathfrak{p}_3 = (x_3, x_5), & \\ \mathfrak{p}_4 = (x_1, x_2, x_5), & \mathfrak{p}_5 = (x_1, x_3, x_4), & \mathfrak{p}_6 = (x_2, x_3, x_6), & \mathfrak{p}_7 = (x_4, x_5, x_6). \end{array}$$

For any n,

$$I^{(n)} = \bigcap_{i=1}^{7} \mathfrak{p}_i^n.$$

A computation with *Macaulay* 2 ([8]) gives that $\overline{I^2} = (I^2, m)$ and $\overline{I^3} = I\overline{I^2}$. Since $\ell(I) \leq \mu(I) = 4$, Theorem 2.5 shows that $\overline{I^{n+1}} = I\overline{I^n}$ for $n \geq 3$. As a consequence,

$$R_s(I) = \overline{R[It]} = R[It, mt^2].$$

Question 2.8. Given the usefulness of Theorem 2.5, it would be worthwhile to look at the situation short of Cohen–Macaulayness. For the integral closure of a standard graded affine algebra A satisfying Serre's condition R_1, it was possible in [24, Theorem 6.5] to derive a bound for the generation degree of the A-module $\overline{A}$ assuming only that depth $\overline{A} \geq \dim \overline{A} - 1$.

3. Sally modules and normalization of ideals

In this section, we apply the notion of Sally module to obtain, among other things, upper bounds for the normalization index of an ideal I and for the number of generators of the integral closure $\overline{R[It]}$.

Let $(R, \mathfrak{m})$ be a Noetherian local ring of dimension $d \geq 1$ and I an $\mathfrak{m}$-primary ideal. Let $\mathcal{F} = \{I_n,\ n \geq 0\}$ be a weakly decreasing multiplicative filtration of ideals, with $I_0 = R$, $I_1 = I$. We will examine in detail the case when the corresponding Rees algebra

$$B = \mathcal{R}(\mathcal{F}) = \sum_{n \geq 0} I_n t^n$$

is finitely generated as a module over $R[It]$.

There are several algebraic structures attached to $\mathcal{F}$, among which we single out the associated graded ring of $\mathcal{F}$ and its Sally modules. The first is

$$\mathrm{gr}_{\mathcal{F}}(R) = \sum_{n \geq 0} I_n/I_{n+1},$$

whose properties are closely linked to $\mathcal{R}(\mathcal{F})$. It is a finitely generated graded module over the standard graded algebra $\mathrm{gr}_I(R)$.

To define the Sally module, we choose a minimal reduction J of I (if need be, we may assume that the residue field of R is infinite). Note that B is a module finite extension of the Rees algebra $A = R[Jt]$ of the ideal J. The corresponding *Sally module* S is defined by the exact sequence of finitely generated graded A-modules,

$$0 \to IA \longrightarrow B_+(1) \longrightarrow S = \bigoplus_{n=1}^{\infty} I_{n+1}/J^n I \to 0\,. \tag{1}$$

The Sally module S is annihilated by $\mathfrak{m}^t$ for some $t \geq 1$, hence it is a finitely generated graded module over $A/\mathfrak{m}^t A$. It follows that $\dim S \leq d$, with equality if $S \neq 0$ and R is Cohen–Macaulay. The Artinian A-module

$$S/JtS = \bigoplus_{n \geq 1} I_{n+1}/JI_n$$

gives some control over the number of generators of B as an A-module via the exact sequence (1). Indeed, the length of S/JtS bounds the number of generators of S as an A-module. If R and S are Cohen–Macaulay, this number is also the multiplicity of the Sally module.

The cohomological properties of B, $\mathrm{gr}_{\mathcal{F}}(R)$, and S become more entwined when R is Cohen–Macaulay. Indeed, under this condition, the exact sequence (1) and the exact sequences from the proof of Theorem 2.5,

$$0 \to B_+(1) \longrightarrow B \longrightarrow \mathrm{gr}_{\mathcal{F}}(R) \to 0 \tag{2}$$

$$0 \to B_+ \longrightarrow B \longrightarrow R \to 0\,, \tag{3}$$

together with the (inhomogeneous) isomorphism

$$B_+(1) \cong B_+\,,$$

give a fluid mechanism to pass cohomological information around.

Proposition 3.1. *Let R be a Cohen–Macaulay local ring of dimension $d \geq 1$ and $\mathcal{F}$ a filtration as above. Then*

(a) $\operatorname{depth} B \leq \operatorname{depth} \mathrm{gr}_{\mathcal{F}}(R) + 1$, *with equality if* $\mathrm{gr}_{\mathcal{F}}(R)$ *is not Cohen–Macaulay.*
(b) $\operatorname{depth} B \geq d$ *if* $\mathrm{gr}_{\mathcal{F}}(R)$ *is Cohen–Macaulay.*
(c) $\operatorname{depth} S = \operatorname{depth} \mathrm{gr}_{\mathcal{F}}(R) + 1$ *if* $\mathrm{gr}_{\mathcal{F}}(R)$ *is not Cohen–Macaulay.*
(d) *S is Cohen–Macaulay if* $\mathrm{gr}_{\mathcal{F}}(R)$ *is Cohen–Macaulay.*

Proof. For (a), see the proofs of [11, Lemma 3.3 and Theorem 3.10]. Part (b) follows from the proof of [11, Proposition 3.6]. To prove (c) one uses (a), the exact sequences (2) and (1), and the fact that IA is a maximal Cohen–Macaulay A-module. Part (d) follows from (b) and the exact sequences (3) and (1). □

3.1. Hilbert functions

Again, we assume that R is a Cohen–Macaulay local ring of dimension $d \geq 1$. Another connection between $\mathcal{F}$ and S is realized via their Hilbert functions. Set

$$H_{\mathcal{F}}(n) = \lambda(R/I_n) \quad \text{and} \quad H_S(n-1) = \lambda(I_n/IJ^{n-1}).$$

The associated Poincaré-series

$$P_{\mathcal{F}}(t) = \frac{f(t)}{(1-t)^{d+1}} \quad \text{and}$$

$$P_S(t) = \frac{g(t)}{(1-t)^d}$$

are related by

$$\begin{aligned} P_{\mathcal{F}}(t) &= \frac{\lambda(R/J)\cdot t}{(1-t)^{d+1}} + \frac{\lambda(R/I)(1-t)}{(1-t)^{d+1}} - P_S(t) \\ &= \frac{\lambda(R/I) + \lambda(I/J)\cdot t}{(1-t)^{d+1}} - P_S(t). \end{aligned}$$

This fact follows as in [25, Proposition 3.1] (see also [29, Proposition 1.3.3]), replacing the I-adic filtration by the filtration $\mathcal{F}$. It implies immediately:

Proposition 3.2. *The h-polynomials $f(t)$ and $g(t)$ are related by*

$$f(t) = \lambda(R/I) + \lambda(I/J)\cdot t - (1-t)\,g(t). \tag{4}$$

In particular, if $f(t) = \sum_{i\geq 0} a_i t^i$ and $g(t) = \sum_{i\geq 1} b_i t^i$, then for $i \geq 2$

$$a_i = b_{i-1} - b_i.$$

Corollary 3.3. *If $\mathrm{gr}_{\mathcal{F}}(R)$ is Cohen–Macaulay, then the h-vector of S is non-negative and weakly decreasing,*

$$b_i \geq 0, \quad b_1 \geq b_2 \geq \cdots \geq 0.$$

Moreover, $b_k = 0$ for some $k \geq 1$ if and only if B is generated as an A-module by homogeneous elements of degree at most k.

Proof. That $b_i \geq 0$ follows because S is Cohen–Macaulay by Proposition 3.1(d). For the same reason $a_i \geq 0$. Now the difference relation in Proposition 3.2 shows that $b_{i-1} \geq b_i$.

For the other assertion, since S is Cohen–Macaulay, $b_i = \lambda(I_{i+1}/JI_i)$. The proof of this fact is a modification of [28, Theorem 1.1(iii)], using the filtration $\mathcal{F}$ instead of the I-adic filtration. Since the b_i's are weakly decreasing, it now follows that b_k vanishes if and only if $I_i = J^{i-k}I_k$ for $i \geq k$. $\square$

Remark 3.4. The equality (4) has several useful consequences, of which we remark the following. For $k \geq 2$, one has

$$f^{(k)}(1) = k\,g^{(k-1)}(1),$$

that is, the Hilbert coefficients of $\mathrm{gr}_{\mathcal{F}}(R)$ and S satisfy

$$e_{i+1}(\mathcal{F}) = e_i(S) \quad \text{for } i \geq 1.$$

Observe that when $\operatorname{depth} \mathrm{gr}_{\mathcal{F}}(R) \geq d-1$, S is Cohen–Macaulay by Proposition 3.1(c), (d), so its h-vector is non-negative, and therefore all its Hilbert coefficients along with it. As $e_0(\mathcal{F})$ and $e_1(\mathcal{F})$ are always non-negative, it follows that all $e_i(\mathcal{F})$ are non-negative. This recovers [17, Corollary 2].

Corollary 3.5. *If $\mathrm{gr}_{\mathcal{F}}(R)$ is Cohen–Macaulay and $g(t)$ is a polynomial of degree at most 4, then*

$$e_2(\mathcal{F}) \geq e_3(\mathcal{F}) \geq e_4(\mathcal{F}) \geq e_5(\mathcal{F}).$$

Proof. By our assumption

$$g(t) = b_1 t + b_2 t^2 + b_3 t^3 + b_4 t^4.$$

As

$$e_{i+1}(\mathcal{F}) = e_i(S) = \frac{g^{(i)}(1)}{i!} \qquad \text{for } i \geq 1,$$

we have the following equations:

$$\begin{aligned} e_2(\mathcal{F}) &= b_1 + 2b_2 + 3b_3 + 4b_4 \\ e_3(\mathcal{F}) &= b_2 + 3b_3 + 6b_4 \\ e_4(\mathcal{F}) &= b_3 + 4b_4 \\ e_5(\mathcal{F}) &= b_4\,. \end{aligned}$$

Now the assertion follows because $b_1 \geq b_2 \geq b_3 \geq b_4 \geq 0$ according to Corollary 3.3. $\square$

This considerably lowers the possible number of distinct Hilbert polynomials for such algebras.

Remark 3.6. The assumptions of Corollary 3.5 are satisfied for instance if $\dim R \leq 6$ and B is Cohen–Macaulay, as can be seen from Proposition 3.1(a), Theorem 2.5, and Corollary 3.3.

3.2. Number of generators

Another application of Sally modules is to obtain a bound for the number of generators (and the distribution of their degrees) of B as an A-algebra or as an A-module.

Theorem 3.7. *Let R be a Cohen–Macaulay local ring of dimension $d \geq 1$ and $\mathcal{F}$ a filtration as above.*

(a) *If $\operatorname{depth} \mathrm{gr}_{\mathcal{F}}(R) \geq d-1$, the A-module B/A can be generated by $e_1(\mathcal{F})$ elements; in particular, the A-algebra B can be generated by the same number of elements.*
(b) *If $\operatorname{depth} \mathrm{gr}_{\mathcal{F}}(R) = d$, the A-module B can be generated by $e_0(\mathcal{F})$ elements; in particular, the A-algebra B can be generated by $e_0(\mathcal{F}) - 1$ elements.*
(c) *If B is Cohen–Macaulay, the A-module B/A can be generated by*

$$\mu(I/J) + \max\{0, d-2\}\,\lambda(I_2/JI)$$

elements; in particular, the A-algebra B can be generated by the same number of elements.

Proof. (a) From the relation (4), we have

$$e_1(\mathcal{F}) = f'(1) = \lambda(I/J) + g(1).$$

The sequence (1) defining S shows that

$$\mu_A(B/A) \le \mu_R(I/J) + \mu_A(S) \le \lambda(I/J) + \mu_A(S).$$

By Proposition 3.1(c), (d), S is a Cohen–Macaulay module over the standard graded ring $A/\mathfrak{m}^t A$ for some $t \ge 1$. We factor out a system of parameters of this module consisting of linear forms to see that

$$\mu_A(S) \le e_0(S) = g(1).$$

These facts combined imply the assertion.

(b) Since $\mathrm{gr}_{\mathcal{F}}(R)$ is a Cohen–Macaulay module over $A/\mathfrak{m}^s A$ for some $s \ge 1$, it is generated by $e := e_0(\mathcal{F})$ homogeneous elements. Let $x_1, \ldots, x_e$ be homogeneous lifts of these elements to B. For every i and every $j \ge 1$ one has

$$B_i \subset Ax_1 + \ldots + Ax_e + B_{i+j}\,.$$

Since B is finitely generated as an A-module, there exists j so that $B_{i+j} = JB_{i+j-1} \subset JB_i$. Now Nakayama's Lemma shows that

$$B_i \subset Ax_1 + \ldots + Ax_e \quad \text{for every } i.$$

(c) Since B is Cohen–Macaulay, the reduction number of $\mathcal{F}$ with respect to J is at most $d-1$ by Theorem 2.5. Recall that $\mathrm{gr}_{\mathcal{F}}(R)$ is Cohen–Macaulay by Proposition 3.1(a). Thus the h-polynomial of $\mathrm{gr}_{\mathcal{F}}(R)$ has degree $\le d-1$, and consequently the h-polynomial of the Sally module has degree at most $d-2$ by Proposition 3.2. The Sally module S is Cohen–Macaulay by Proposition 3.1(d), and its h-vector $(b_1, b_2, \ldots)$ is weakly decreasing with $b_1 = \lambda(I_2/JI)$ by Corollary 3.3 and its proof. It follows that its multiplicity satisfies $e_0(S) \le \max\{0, d-2\}\, \lambda(I_2/JI)$, and we conclude as in the proof of (a). $\square$

Remark 3.8. A typical application is to the case $d = 2$ with $\mathcal{F} = \{\overline{I^n}, n \ge 0\}$.

Remark 3.9. An issue is to get bounds for $e_1(\mathcal{F})$. This is addressed in [18]. For instance, when R is a regular local ring, $e_1(\mathcal{F}) \le \dfrac{(d-1)e_0}{2}$ by [18, Corollary 3.4].

4. One-step normalization of Rees algebras

In this section we present one of the rare instances where the normalization of the Rees ring can be computed in a single step using an explicit expression as a colon. Our formula applies to homogeneous ideals that are generated by forms of the same degree and satisfy some additional assumptions.

Let I be an ideal of a Noetherian ring R. The G_d assumption in the next theorem means that the minimal number of generators $\mu(I_\mathfrak{p})$ is at most $\dim R_\mathfrak{p}$ for every prime ideal $\mathfrak{p}$ containing I with $\dim R_\mathfrak{p} \le d-1$. In the proof of the theorem we use the theory of residual intersections. Let $s \ge 0$ be an integer (for convenience, we drop the usual requirement that $s \ge \mathrm{ht}\, I$); the ideal $\mathfrak{a} : I$ is an *s-residual intersection* of I if $\mathfrak{a}$ is an s-generated R-ideal properly contained in I and $\mathrm{ht}\, \mathfrak{a} : I \ge s$.

An ideal is said to be *of linear type* if the natural epimorphism from the symmetric algebra onto the Rees algebra is an isomorphism. We will frequently use the fact that an ideal of linear type has no proper reductions.

Theorem 4.1. *Let k be an infinite field, $R = k[x_1, \ldots, x_d]$ a positively graded polynomial ring in $d \geq 1$ variables, and I an ideal of height g generated by forms of degree $\delta \geq 1$. Assume I satisfies G_d,* $\operatorname{depth} R/I^j \geq \dim R/I - j + 1$ *for $1 \leq j \leq d-g$, and I is normal locally on the punctured spectrum. Let J be a homogeneous minimal reduction of I and write $\sigma = \sum_{i=1}^{d} \deg x_i$. Then*

$$\overline{R[It]} = R[Jt] :_{R[t]} R_{\geq g\delta - \delta - \sigma + 1}\,;$$

in particular, $\overline{I} = J :_R R_{\geq g\delta-\delta-\sigma+1}$. Furthermore, I is a normal ideal of linear type if and only if $\delta \leq \frac{\sigma-1}{g-1}$ $(:= \infty$ if $g = 1)$ or $\mu(I) \leq d-1$.

Proof. We may assume $g \geq 1$ and $d \geq 2$. By ℓ we denote the analytic spread $\ell(I) = \ell(J)$. As before we write $A = R[Jt]$. Notice that $\overline{R[It]} = \overline{A}$.

Locally on the punctured spectrum of R, the ideal I is of linear type by [23, Theorem 2.9(a) and Proposition 1.11], hence I and J coincide locally, and so J is locally normal. The ideal J is of linear type and A is Cohen–Macaulay by [23, Theorem 2.9(a), Remark 1.12, and Proposition 1.11] and [9, Theorem 6.1]. Write $\mathfrak{m}$ for the maximal homogeneous ideal of R. Since J is of linear type, $\mathfrak{m}A$ is a prime ideal in A, necessarily of height $d + 1 - \ell$, and every other prime ideal of smaller or equal height contracts to a prime ideal of R properly contained in $\mathfrak{m}$. We write $\mathfrak{b} = R_{\geq g\delta-\delta-\sigma+1}$.

First assume that $\ell < d$. Since A is Cohen–Macaulay, since J is normal locally on the punctured spectrum of R, and since $\operatorname{ht} \mathfrak{m}A = d + 1 - \ell \geq 2$, it follows that $\overline{A} = A$. On the other hand $A :_{R[t]} \mathfrak{b} = A$, because $\operatorname{ht} \mathfrak{b}A \geq \operatorname{ht} \mathfrak{m}A \geq 2$ and A is Cohen–Macaulay. Therefore $\overline{A} = A :_{R[t]} \mathfrak{b}$ in this case. Also notice that $I = J$ is a normal ideal of linear type and $\mu(I) = \ell \leq d-1$. Thus we may from now on assume that $\ell = d$, or equivalently, $\operatorname{ht} \mathfrak{m}A = 1$.

We prove the displayed equality $\overline{A} = A :_{R[t]} \mathfrak{b}$. Notice that $A :_{R[t]} \mathfrak{b} = A :_{\mathrm{Quot}(R[t])} \mathfrak{b}$, as $\operatorname{ht} \mathfrak{b} \geq d \geq 2$ and R is Cohen–Macaulay. Since the two A-modules $\overline{A}$ and $A :_{\mathrm{Quot}(R[t])} \mathfrak{b}$ satisfy S_2 and since locally on the punctured spectrum of R, J is normal, it suffices to prove the equality

$$\overline{A}_{\mathfrak{m}A} = (A :_{\mathrm{Quot}(R[t])} \mathfrak{b})_{\mathfrak{m}A}\,.$$

Let $f_1, \ldots, f_d$ be a generating set of J consisting of forms of degree δ and let φ be a minimal homogeneous presentation matrix of $f_1, \ldots, f_d$. Notice that the entries along any column of φ are forms of the same degree. As J is of linear type, one has $A \cong R[T_1, \ldots, T_d]/I_1(\underline{T} \cdot \varphi)$. Let $K = k(T_1, \ldots, T_d)$ and $B = K[x_1, \ldots, x_d]/I_1(\underline{T} \cdot \varphi)$. Notice that

$$B_{\mathfrak{m}B} \cong A_{\mathfrak{m}A}\,.$$

Since B is a positively graded K-algebra with maximal homogeneous ideal $\mathfrak{m}B$, we conclude that B is a domain of dimension one because $A_{\mathfrak{m}A}$ is.

The elements $T_1, \ldots, T_d$ are non-zero in K; hence in the ring $R' = K[x_1, \ldots, x_d]$, the ideal $I_1(\underline{T} \cdot \varphi)$ can be written as $\mathfrak{a} : J$, where $\mathfrak{a}$ is generated by $d-1$ forms in JR' of degree δ. Since this ideal has height $d-1$, it is a $(d-1)$-residual intersection of JR'. On the other hand,

$$(\mathfrak{a} : J)(JR' : I) \subset \mathfrak{a} : I \subset \mathfrak{a} : J\,,$$

where $\operatorname{ht} JR' : I \geq d$. Hence $\mathfrak{a} : I$ is also a $(d-1)$-residual intersection of IR'. Thus this ideal is unmixed of height $d-1$ by [23, Theorem 2.9(a) and Proposition 1.7(a)], and applying the above containments once more, we deduce that $\mathfrak{a} : I = \mathfrak{a} : J$. From this we conclude that $I_1(\underline{T} \cdot \varphi)$ is a $(d-1)$-residual intersection of IR'. Now [23, Theorem 2.9(b)] and [16, the proof of Proposition 2.1] imply $\omega_B \cong \left(I^{d-g}R'/I^{d-g-1}\mathfrak{a}\right)((d-1)\delta - \sigma)$ (where $I^n = R$ if $n \leq 0$). Thus the a-invariant of B satisfies

$$a(B) = (g-1)\delta - \sigma\,.$$

We are now going to apply this information to compute the conductor of B.

Since B is a positively graded K-domain, it follows that $\overline{B}$ is a positively graded L-domain for some finite field extension L of K. As $\dim B = 1$, $\overline{B}$ is a principal ideal domain, hence $\overline{B} = L[t]$ for some homogeneous element t of degree $\alpha > 0$. Since the conductor of B is a homogeneous $\overline{B}$-ideal, it is of the form $B :_B \overline{B} = B_{\geq \varepsilon}$ for some ε, where $\varepsilon = \max\{i \,|\, [\overline{B}/B]_i \neq 0\} + 1$.

The sequence

$$0 \to B \longrightarrow \overline{B} \longrightarrow \overline{B}/B \to 0$$

yields an exact sequence

$$0 \to \overline{B}/B \longrightarrow H^1_{\mathfrak{m}B}(B) \longrightarrow H^1_{\mathfrak{m}B}(\overline{B}) \to 0\,.$$

If $\overline{B}/B \neq 0$, then $a(B) \geq 0$ since $\overline{B}/B$ is concentrated in non-negative degrees. On the other hand $a(\overline{B}) = -\alpha < 0$. Thus $\varepsilon = a(B) + 1 = g\delta - \delta - \sigma + 1$, and we obtain $B :_B \overline{B} = B_{\geq g\delta-\delta-\sigma+1}$. If on the other hand $\overline{B}/B = 0$, then $a(B) = a(\overline{B})$, hence $g\delta - \delta - \sigma = -\alpha < 0$. Thus $B_{\geq g\delta-\delta-\sigma+1} = B = B :_B \overline{B}$. Therefore in either case $B :_B \overline{B} = B_{\geq g\delta-\delta-\sigma+1} = \mathfrak{b}B$, or equivalently,

$$\overline{B} = B :_{\mathrm{Quot}(B)} \mathfrak{b}\,.$$

Localizing at $\mathfrak{m}B$ and using the equalities $\overline{B}_{\mathfrak{m}B} = \overline{B_{\mathfrak{m}B}} = \overline{A_{\mathfrak{m}A}} = \overline{A}_{\mathfrak{m}A}$, we conclude that

$$\overline{A}_{\mathfrak{m}A} = (A :_{\mathrm{Quot}(R[t])} \mathfrak{b})_{\mathfrak{m}A}\,,$$

as required.

As to the additional assertion of the theorem, recall that $\mu(I) \geq \ell = d$ by our standing assumption. By the equality $\overline{A} = A :_{R[t]} \mathfrak{b}$ and since J is of linear type, the ideal I is normal and of linear type if and only if $A = A :_{R[t]} \mathfrak{b}$. This equality obviously holds if $\delta \leq \frac{\sigma-1}{g-1}$, since then $\mathfrak{b} = R$. Conversely, assume that $\delta > \frac{\sigma-1}{g-1}$. Since $\ell = d$, we have depth $R/J^n = 0$ for some $n \geq 1$ by [4, Corollary, p. 373] (even for all $n \gg 0$ by [3, Theorem, p. 36]). It follows that $A \subsetneq A :_{R[t]} \mathfrak{m}$. This colon is contained in $A :_{R[t]} \mathfrak{b}$, because $\delta > \frac{\sigma-1}{g-1}$ and so $\mathfrak{b} \subset \mathfrak{m}$. Thus $A \neq A :_{R[t]} \mathfrak{b}$. $\square$

Remark 4.2. Notice that if in the previous theorem R is standard graded then $R_{\geq g\delta-\delta-\sigma+1} = \mathfrak{m}^{g\delta-\delta-\sigma+1}$.

Remark 4.3. In the presence of the G_d assumption the depth conditions in Theorem 4.1 are satisfied, for example, if I is strongly Cohen–Macaulay, which means that the Koszul homology modules of I are Cohen–Macaulay [23, Remark 2.10]. The ideal I is strongly Cohen–Macaulay if I is perfect and generated by $g+2$ elements [2, p. 259] or if I is in the linkage class of a complete intersection [13, Theorem 1.11]. Examples of ideals in the linkage class of a complete intersection are perfect ideals of height 2 [6] and Gorenstein ideals of height 3 [30, the proof of Thm.].

In the presence of the depth assumptions in Theorem 4.1, the ideal I is normal locally on the punctured spectrum of R if I is reduced and the G_d assumption is replaced by the condition $\ell(I_\mathfrak{p}) \leq \dim R_\mathfrak{p} - 1$ for every prime ideal $\mathfrak{p}$ containing I with $g+1 \leq \dim R_\mathfrak{p} \leq d-1$ (see [23, Theorem 2.9(a) and Proposition 1.11], [9, Theorem 6.1], and [5, Proposition 3.3]).

Another class of ideals satisfying the assumptions of the theorem are one-dimensional ideals:

Example 4.4. Let k be an infinite field, $R = k[x_1, \dots, x_d]$ a standard graded polynomial ring with maximal homogeneous ideal $\mathfrak{m}$, I a one-dimensional reduced ideal generated by forms of degree δ, and J an ideal generated by d general forms of degree δ in I. Then for every n,

$$\overline{I^n} = J^n : \mathfrak{m}^{(d-2)(\delta-1)-1}.$$

References

[1] I.M. Aberbach, C. Huneke, N.V. Trung, Reduction numbers, Briançon–Skoda theorems and depth of Rees algebras, Compos. Math. 97 (1995) 403–434.

[2] L. Avramov, J. Herzog, The Koszul algebra of a codimension 2 embedding, Math. Z. 175 (1980) 249–260.

[3] M. Brodmann, The asymptotic nature of the analytic spread, Math. Proc. Camb. Philos. Soc. 86 (1979) 35–39.

[4] L. Burch, Codimension and analytic spread, Proc. Camb. Philos. Soc. 72 (1972) 369–373.

[5] D. Eisenbud, C. Huneke, Cohen–Macaulay Rees algebras and their specialization, J. Algebra 81 (1983) 202–224.

[6] F. Gaeta, Ricerche intorno alle varietá matriciali ed ai loro ideali, in: Atti del Quarto Congresso dell'Unione Matematica Italiana, Taormina, 1951, vol. II, Casa Editrice Perrella, Roma, 1953, pp. 326–328.

[7] I. Gitler, E. Reyes, R.H. Villarreal, Blowup algebras of square-free monomial ideals and some links to combinatorial optimization problems, Rocky Mt. J. Math. 39 (2009) 71–102.

[8] D. Grayson, M. Stillman, Macaulay 2, a software system for research in algebraic geometry, available at http://www.math.uiuc.edu/Macaulay2/.

[9] J. Herzog, A. Simis, W.V. Vasconcelos, Approximation complexes of blowing-up rings. II, J. Algebra 82 (1983) 53–83.

[10] M. Hochster, Rings of invariants of tori, Cohen–Macaulay rings generated by monomials, and polytopes, Ann. Math. 96 (1972) 318–337.

[11] S. Huckaba, T. Marley, Depth formulas for certain graded rings associated to an ideal, Nagoya Math. J. 133 (1994) 57–69.

[12] C. Huneke, On the associated graded ring of an ideal, Ill. J. Math. 26 (1982) 121–137.

[13] C. Huneke, Linkage and the Koszul homology of ideals, Am. J. Math. 104 (1982) 1043–1062.

[14] M. Johnson, B. Ulrich, Artin–Nagata properties and Cohen–Macaulay associated graded rings, Compos. Math. 103 (1996) 7–29.

[15] B. Johnston, D. Katz, Castelnuovo regularity and graded rings associated to an ideal, Proc. Am. Math. Soc. 123 (1995) 727–734.

[16] A. Kustin, B. Ulrich, If the socle fits, J. Algebra 147 (1992) 63–80.

[17] T. Marley, The coefficients of the Hilbert polynomial and the reduction number of an ideal, J. Lond. Math. Soc. 40 (1989) 1–8.

[18] C. Polini, B. Ulrich, W.V. Vasconcelos, Normalization of ideals and Briançon–Skoda numbers, Math. Res. Lett. 12 (2005) 827–842.

[19] D. Rees, A note on analytically unramified local rings, J. Lond. Math. Soc. 36 (1961) 24–28.

[20] L. Reid, L.G. Roberts, M.A. Vitulli, Some results on normal homogeneous ideals, Commun. Algebra 31 (2003) 4485–4506.

[21] A. Simis, B. Ulrich, W.V. Vasconcelos, Cohen–Macaulay Rees algebras and degrees of polynomial relations, Math. Ann. 301 (1995) 421–444.

[22] N.V. Trung, S. Ikeda, When is the Rees algebra Cohen–Macaulay?, Commun. Algebra 17 (1989) 2893–2922.

[23] B. Ulrich, Artin–Nagata properties and reductions of ideals, Contemp. Math. 159 (1994) 373–400.

[24] B. Ulrich, W.V. Vasconcelos, On the complexity of the integral closure, Trans. Am. Math. Soc. 357 (2005) 425–442.

[25] W.V. Vasconcelos, Hilbert functions, analytic spread and Koszul homology, Contemp. Math. 159 (1994) 401–422.

[26] W.V. Vasconcelos, Computational Methods in Commutative Algebra and Algebraic Geometry, Springer, Berlin, 1998.

[27] W.V. Vasconcelos, Integral Closure, Springer Monographs in Mathematics, Springer, Berlin, 2005.

[28] M. Vaz Pinto, Structure of Sally Modules and Hilbert Functions, Ph.D. dissertation, Rutgers University, 1995.

[29] M. Vaz Pinto, Hilbert functions and Sally modules, J. Algebra 192 (1997) 504–523.

[30] J. Watanabe, A note on Gorenstein rings of embedding codimension three, Nagoya Math. J. 50 (1973) 227–232.