LIAISON OF MONOMIAL IDEALS

CRAIG HUNEKE AND BERND ULRICH

Abstract. We give a simple algorithm to decide whether a monomial ideal of finite colength in a polynomial ring is *licci*, i.e., in the linkage class of a complete intersection. The algorithm proves that whether or not such an ideal is licci does not depend on whether we restrict the linkage by only allowing monomial regular sequences, or homogeneous regular sequences, or arbitrary regular sequences. We apply our results on monomial ideals to compare when an ideal is licci versus when its initial ideal in some term order is licci. We also apply an idea of Migliore and Nagel to prove that monomial ideals of finite colength are always *glicci*, i.e., in the Gorenstein linkage class of a complete intersection. However, our proof requires the use of non-homogeneous Gorenstein links.

1. Introduction

Let R be a commutative Noetherian ring, and let I and J be two proper ideals in R. These ideals are said to be *directly linked* if there exists a regular sequence $f_1, \ldots, f_g$ contained in $I \cap J$ such that $(f_1, \ldots, f_g) : I = J$ and $(f_1, \ldots, f_g) : J = I$. We say I and J are in the same *linkage class* (or *liaison class*) if there exists a sequence of ideals $I = I_0, \ldots, I_n = J$ such that I_j is directly linked to I_{j+1} for $0 \leq j \leq n-1$, the case $n = 2$ being referred to as *double linkage*. Such a sequence of links connecting I and J is far from unique. We call the ideal I *licci* if I is in the **li**nkage **c**lass of a **c**omplete **i**ntersection, i.e., of an ideal generated by a regular sequence.

In a similar manner, at least when R is regular, we say that I and J are *Gorenstein directly linked* if there exists an ideal $K \subset I \cap J$ such that R/K is Gorenstein, $K : I = J$, and $K : J = I$; the last equality is actually a consequence of the previous one in case I is unmixed and has the same height as K. The *Gorenstein linkage class* of I is defined by making this relation an equivalence relation as above, and I is said to be *glicci* if it is in the **G**orenstein **li**nkage **c**lass of a **c**omplete **i**ntersection. Finally, by a *Gorenstein double link* we mean a sequence of two direct Gorenstein links.

This paper studies when monomial ideals in polynomials rings are licci or glicci. Our main theorem, Theorem 2.6, gives a simple algorithm to decide whether a monomial ideal of finite colength is licci. This theorem is one of the few instances where one has not only necessary, but also sufficient conditions for an ideal to be licci. In Theorem 3.2 we compare when an

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ideal of finite colength is licci to when its initial ideal with respect to some term order is licci. In Theorem 4.2 we prove that any monomial ideal of finite colength is glicci.

For basic information on linkage we refer the reader to [6], [8], and [3].

2. Licci Monomial Ideals

We begin by establishing some notation. We will always write $S = k[x_1, \dots, x_d]$ for a polynomial ring over a field k and $\mathfrak{m}$ for its homogeneous maximal ideal $(x_1, \dots, x_d)$. By a monomial in S we mean an element of the form $x_1^{a_1} \cdots x_d^{a_d}$. We simplify this notation by using capital letters to denote d-tuples of non-negative integers, $A = (a_1, \dots, a_d)$, and writing $x^A = x_1^{a_1} \cdots x_d^{a_d}$. A *monomial ideal* is an ideal generated by monomials. Every $\mathfrak{m}$-primary monomial ideal I can be written uniquely in *standard form* $I = (x_1^{a_1}, \dots, x_d^{a_d}) + I^{\#}$, where $I^{\#}$ is generated by monomials that together with $\{x_1^{a_1}, \dots, x_d^{a_d}\}$ generate I minimally. Notice that $I^{\#} = 0$ if and only if I is a complete intersection. We will use the fact that if $x^B = x_1^{b_1} \cdots x_d^{b_d} \notin (x_1^{a_1}, \dots, x_d^{a_d})$, then $(x_1^{a_1}, \dots, x_d^{a_d}) : x^B = (x_1^{a_1 - b_1}, \dots, x_d^{a_d - b_d})$. In particular, any $\mathfrak{m}$-primary monomial almost complete intersection ideal is directly linked to a complete intersection by a monomial regular sequence.

We will need the following theorem that is a special case of a main result in [3]:

Theorem 2.1. *Let $S = k[x_1, \dots, x_d]$ be a polynomial ring over a field k and let I be an $\mathfrak{m}$-primary ideal generated by homogeneous polynomials of degrees at least δ. If $\mathfrak{m}^{(d-1)(\delta-1)} \subset I$ then $I_{\mathfrak{m}}$ is not licci in $S_{\mathfrak{m}}$.*

Proof. The maximal last shift in a minimal homogeneous free S-resolution of S/I is at most $(d-1)\delta$. Now [3, 5.13(a)] implies that $I_{\mathfrak{m}}$ cannot be licci. (The essential point in applying [3, 5.13(a)] is that the maximal last shift in a minimal homogeneous free S-resolution of S/I is at most $d-1$ times the minimal degree of a generator of I.) $\square$

Theorem 2.4 below gives a necessary condition for an $\mathfrak{m}$-primary monomial ideal to be licci. The condition is rather strong and was surprising to us. To prove this theorem we need Proposition 2.3, which basically says that an ideal J is licci if and only if $J + yS$ is licci for y a regular element on S and S/J. This result is not unexpected, but does not seem to be in the literature. Its proof requires the use of universal linkage as developed in [3]. We briefly review the definition. Let $(R, \mathfrak{m})$ be a local Gorenstein ring and let I be an unmixed ideal of height $g > 0$ in R. Fix a generating sequence $f_1, \dots, f_n$ of I. Let x_{ij} be variables for $1 \le i \le g$ and $1 \le j \le n$, and write $R(X)$ for the ring $R[\{x_{ij}\}]_{\mathfrak{m}R[\{x_{ij}\}]}$. In $R(X)$ consider the regular sequence $\alpha_1, \dots, \alpha_g$ where $\alpha_i = \sum_{j=1}^{n} x_{ij} f_j$. We define the *first universal link* $L^1(I)$ of I to be the ideal $(\alpha_1, \dots, \alpha_g)R(X) : IR(X)$ in $R(X)$. Inductively we set $L^n(I) = L^1(L^{n-1}(I))$ for $n > 1$ as long as $L^{n-1}(I)$ is not the unit ideal, and call this ideal the nth universal link. Write $L^0(I) = I$. Although these definitions apparently depend upon generating sets, it turns out that universal links are essentially unique (see [3, 2.11(b)] for a precise statement). One of the basic facts about universal linkage says that I is licci if and only if $L^n(I)$ is generated by a regular sequence for some $n \ge 0$, at least when R has an infinite residue field (see [4, 2.9]).

Lemma 2.2. *Let R be a local Gorenstein ring and let J be an ideal such that R/J is Cohen-Macaulay. Let $y \in R$ be regular on R and R/J. Then $L^1((J,y)) = (K,z)$, where K is an ideal in $R(X)$ directly linked to $JR(X)$ and $z \in R(X)$ is regular on $R(X)$ and $R(X)/K$.*

Proof. Fix a generating sequence $f_1, \dots, f_{n-1}$ of J. Set $f_n = y$ and define α_i in $R(X)$ as above. Let $z = \alpha_g$. We may write $(\alpha_1, \dots, \alpha_g)R(X) = (\beta_1, \dots, \beta_{g-1}, z)R(X)$, where $\beta_1, \dots, \beta_{g-1}$ form a regular sequence contained in $JR(X)$. Set $K = (\beta_1, \dots, \beta_{g-1})R(X) : JR(X)$. We have that $L^1((J,y)) = (\alpha_1, \dots, \alpha_g)R(X) : (J,y)R(X) = (\beta_1, \dots, \beta_{g-1}, z)R(X) : (J,z)R(X) = (K,z)$, where the last equality holds by [2, 2.12]. Finally, the element z is regular on $R(X)$ and $R(X)/K$ because $\beta_1, \dots, \beta_{g-1}, z$ form an $R(X)$-regular sequence. □

Proposition 2.3. *Let R be a local Gorenstein ring with infinite residue field. Let J be an ideal such that R/J is Cohen-Macaulay, and let $y \in R$ be regular on R and R/J. Then (J,y) is licci if and only if J is licci.*

Proof. Assume that (J,y) is licci. According to [3, 2.17(b)], $L^n((J,y))$ is generated by a regular sequence for some $n \geq 0$. By repeated use of Lemma 2.2, $L^n((J,y)) = (K,z)$ for some ideal K in the linkage class of $JR(X)$ and some $z \in R(X)$ which is regular on $R(X)$ and $R(X)/K$. Hence K is generated by a regular sequence, showing that $JR(X)$ is licci in $R(X)$. Then J is licci in R by [4, 2.12], which states that the property of being licci descends from flat local extensions of local Gorenstein rings with infinite residue fields.

Conversely, if J is licci, then $L^n(J)$ is generated by a regular sequence for some $n \geq 0$, and it is clear that y is regular modulo $L^i(J)$ for every $0 \leq i \leq n$. By [2, 2.12] it then follows that $(J,y)R(X)$ is in the same linkage class as $(L^n(J), y)R(X)$. Using [4, 2.12] again we obtain that (J,y) is licci. □

Theorem 2.4. *Let $S = k[x_1, \dots, x_d]$ be a polynomial ring over a field k and let I be an $\mathfrak{m}$-primary monomial ideal. If $I^\#$ has height at least two then $I_\mathfrak{m}$ is not licci in $S_\mathfrak{m}$. In particular, I is not licci.*

Proof. We may assume that k is infinite and we write $I = (x_1^{a_1}, \dots, x_d^{a_d}) + I^\#$. Assume that $I^\#$ has height at least two. If I contains a variable, say x_d, we may write $I = I_0S + (x_d)$, where I_0 is a $(x_1, \dots, x_{d-1})$-primary ideal in $k[x_1, \dots, x_{d-1}]$. Clearly x_d is regular on S and on S/I_0S. According to Proposition 2.3, if $I_\mathfrak{m}$ is licci, then so is $I_0S_\mathfrak{m}$. By [4, 2.12] we obtain that $I_0k[x_1, \dots, x_{d-1}]_{(x_1,\dots,x_{d-1})}$ is also licci. Since $I^\# = I_0^\# S$, $I_0^\#$ has height at least two. Inducting on the number of variables proves that $I_0k[x_1, \dots, x_{d-1}]_{(x_1,\dots,x_{d-1})}$ is not licci, and hence neither is $I_\mathfrak{m}$. Thus we may assume that $a_i \geq 2$ for $1 \leq i \leq d$.

Let T be the polynomial ring $S[y_1, \dots, y_d]$ with homogeneous maximal ideal $\mathfrak{n}$. We define a map Φ on the set of monomials in S to the set of monomials in T by first specifying its action on pure powers of the variables,

$$\Phi(x_i^n) = \begin{cases} 1 & \text{if } n = 0 \\ y_i^{n-1}x_i & \text{if } 1 \le n < a_i \\ y_i^{n-2}x_i^2 & \text{if } n \ge a_i. \end{cases}$$

We then extend Φ to a map on the set of all monomials by setting

$$\Phi(x^C) = \Phi(x_1^{c_1}) \cdots \Phi(x_d^{c_d}).$$

Notice that Φ is not multiplicative, but that it preserves divisibility. Finally, for K any monomial ideal in S, we define a monomial ideal $\widetilde{K}$ in T by applying Φ to the monomial generators of K and letting $\widetilde{K}$ be the ideal in T generated by their images.

Consider the epimorphism of S-algebras $\pi : T \to S$ mapping y_i to x_i. Notice that $\pi(\widetilde{K}) = K$ and that the kernel of this map is generated by the T-regular sequence $y_1 - x_1, \ldots, y_d - x_d$. We claim that this sequence is regular on the quotient ring $T/\widetilde{K}$ as well. The claim is equivalent to the vanishing of the first Koszul homology $H_1(y_1 - x_1, \ldots, y_d - x_d; T/\widetilde{K})$. This homology is $\operatorname{Tor}_1^T(S, T/\widetilde{K})$. Thus it suffices to prove that generating relations on the monomial minimal generators $\{x^{C_i}\}$ of K lift via π to relations on the corresponding monomial generators $\{\Phi(x^{C_i})\}$ of $\widetilde{K}$. Indeed, a set of generating syzygyies for K can be obtained as follows: let x^C and x^D be any two monomial minimal generators of K, and let x^E be the greatest common divisor of x^C and x^D. The syzygy module is generated by the syzygies given by $\frac{x^D}{x^E}x^C = \frac{x^C}{x^E}x^D$. This relation lifts to $\frac{\Phi(x^D)}{\Phi(x^E)}\Phi(x^C) = \frac{\Phi(x^C)}{\Phi(x^E)}\Phi(x^D)$. Notice that $\frac{\Phi(x^D)}{\Phi(x^E)}$ and $\frac{\Phi(x^C)}{\Phi(x^E)}$ are monomials in T because the map Φ preserves divisibility, and that we have indeed obtained a lift because the map π is multiplicative.

Thus in the language of [3, 2.2(a)], the pair $(T, \widetilde{K})$ is a deformation of (S, K). Hence according to [3, 2.16], if $I_{\mathfrak{m}}$ is licci then so is $\widetilde{I}_{\mathfrak{n}}$. In particular, the further localization $\widetilde{I}_{\mathfrak{m}T}$ would be licci as well. Set $J = I^{\#}$. Since $(T, \widetilde{J})$ is a deformation of (S, J) and $\widetilde{J}$ is homogeneous, we also obtain $\operatorname{ht} \widetilde{J} = \operatorname{ht} J \ge 2$.

Now write $k' = k(y_1, \ldots, y_d)$, $S' = k'[x_1, \ldots, x_d]$, $\mathfrak{m}' = (x_1, \ldots, x_d)S'$, $I' = \widetilde{I}S'$ and $J' = \widetilde{J}S'$. Notice that $T_{\mathfrak{m}T} = S'_{\mathfrak{m}'}$ and hence $\widetilde{I}_{\mathfrak{m}T} = I'_{\mathfrak{m}'}$, reducing us to prove that $I'_{\mathfrak{m}'}$ cannot be licci. By the definition of the map Φ, $I' = (x_1^2, \ldots, x_d^2) + J'$ and J' is generated by squarefree monomials of degrees at least 2. Moreover, $\operatorname{ht} J' = \operatorname{ht} \widetilde{J}S' \ge \operatorname{ht} \widetilde{J} \ge 2$ by the above. It follows that J' contains every squarefree monomial of degree $d-1$. Indeed if $\frac{x_1 \cdots x_d}{x_i}$ is not in J' then J' cannot contain a squarefree monomial not divisible by x_i. Thus x_i divides every squarefree monomial in J' and hence every monomial in J'. This forces J' to have height at most one, a contradiction. Therefore I' contains every monomial of degree $d-1$. As $\mathfrak{m}'^{d-1} \subset I'$, Theorem 2.1 now shows that $I'_{\mathfrak{m}'}$ cannot be licci. $\square$

Lemma 2.5. *Let $S = k[x_1, \ldots, x_d]$ be a polynomial ring over a field k and let I be an $\mathfrak{m}$-primary monomial ideal. If $I^{\#} = x^B K$ for some monomial $x^B = x_1^{b_1} \cdots x_d^{b_d}$ and a monomial* *ideal K with $0 \neq K \neq S$, then the ideal $I' = (x_1^{a_1-b_1}, \ldots, x_d^{a_d-b_d}) + K$ is obtained from I by a double link defined by the monomial regular sequences $x_1^{a_1}, \ldots, x_d^{a_d}$ and $x_1^{a_1-b_1}, \ldots, x_d^{a_d-b_d}$.*

Proof. It suffices to prove that $(x_1^{a_1}, \ldots, x_d^{a_d}) : I = (x_1^{a_1-b_1}, \ldots, x_d^{a_d-b_d}) : I'$. This follows from the chain of equalities $(x_1^{a_1}, \ldots, x_d^{a_d}) : I = (x_1^{a_1}, \ldots, x_d^{a_d}) : x^B K = ((x_1^{a_1}, \ldots, x_d^{a_d}) : x^B) : K = (x_1^{a_1-b_1}, \ldots, x_d^{a_d-b_d}) : K = (x_1^{a_1-b_1}, \ldots, x_d^{a_d-b_d}) : I'$. □

Let $S = k[x_1, \ldots, x_d]$ be a polynomial ring over a field k, let I be an $\mathfrak{m}$-primary monomial ideal, and consider the standard form $I = (x_1^{a_1}, \ldots, x_d^{a_d}) + I^{\#}$. We set $I^{\{0\}} = I$. If I is not a complete intersection then $I^{\#}$ can be written uniquely as $I^{\#} = x^B K$, where $x^B = x_1^{b_1} \cdots x_d^{b_d}$ is a monomial and K a monomial ideal of height at least two. We define

$$I^{\{1\}} = (x_1^{a_1-b_1}, \ldots, x_d^{a_d-b_d}) + K.$$

If on the other hand I is a complete intersection we set $I^{\{1\}} = S$. For $n > 1$ we define inductively $I^{\{n\}} = (I^{\{n-1\}})^{\{1\}}$ provided $I^{\{n-1\}} \neq S$. Observe that the representation of $I^{\{1\}} \neq S$ above may not be in standard form since K could contain a pure power of a variable; in fact this happens exactly when $\mu(I^{\{1\}}) < \mu(I)$. Also notice that according to Lemma 2.5, the ideals I and $I^{\{n\}} \neq S$ are linked by a sequence of $2n$ links defined by monomial regular sequences.

If $\mathcal{A} \subset \mathbb{R}^d_{\geq 0}$ is a finite set of points we can define a set $\mathcal{A}^{\{1\}} \subset \mathbb{R}^d_{\geq 0}$ obtained from $\mathcal{A}$ by removing the points on the coordinate axes and then translating the remaining set until each coordinate hyperplane contains a point of the set. Iterating one defines $\mathcal{A}^{\{m\}}$. The set $\mathcal{A} = \{C\}$ of exponents of the minimal monomial generators x^C of I can be reduced to the empty set by this procedure, i.e., $\mathcal{A}^{\{m\}} = \emptyset$ for some m if and only if $I^{\{n\}} = S$ for some n. It is this condition that characterizes the licci property:

Theorem 2.6. *Let $S = k[x_1, \ldots, x_d]$ be a polynomial ring over a field k and let I be an $\mathfrak{m}$-primary monomial ideal. The following conditions are equivalent :*

(1) *I can be linked to a complete intersection by a sequence of links defined by monomial regular sequences.*
(2) *I can be linked to a complete intersection by a sequence of links defined by homogeneous regular sequences.*
(3) *$I_{\mathfrak{m}}$ is licci in $S_{\mathfrak{m}}$.*
(4) *$(I^{\{n\}})^{\#}$ has height at most one whenever $I^{\{n\}} \neq S$.*
(5) *$I^{\{n\}} = S$ for some n.*

Proof. (1) $\Rightarrow$ (2) $\Rightarrow$ (3): The implications are obvious.

(3) $\Rightarrow$ (4): According to Lemma 2.5 the ideals $(I^{\{n\}})_{\mathfrak{m}}$ are licci as well. Now apply Theorem 2.4.

(4) $\Rightarrow$ (5): Write σ for the sum of the degrees of homogeneous minimal generators of a homogeneous ideal. We use induction on $\sigma(I)$. We may assume that $I^{\{1\}} \neq S$. Since $I^{\#}$ has height at most one it follows that $x^B \neq 1$ in the definition of $I^{\{1\}}$. Therefore $\sigma(I^{\{1\}}) < \sigma(I)$ and we may apply the induction hypothesis to $I^{\{1\}}$.

(5) $\Rightarrow$ (1): By Lemma 2.5 we may replace I by $I^{\{n-1\}}$ to assume that $I^{\{1\}} = S$. But then I is an almost complete intersection and hence linked to a complete intersection by a monomial regular sequence. $\square$

Notice that Theorem 2.6 immediately implies the well known fact that I is licci if $d \leq 2$, as can be seen from condition (4).

Corollary 2.7. *Let $S = k[x_1, \dots, x_d]$ be a polynomial ring over a field k and let I and L be $\mathfrak{m}$-primary monomial ideals with $I^\# = L^\#$. Then I is licci if and only if L is licci.*

Proof. The ideals $I^{\{1\}}$ and $L^{\{1\}}$ have the same monomial minimal generators except possibly for the pure powers. Hence $(I^{\{1\}})^\# = (L^{\{1\}})^\#$, and inductively $(I^{\{n\}})^\# = (L^{\{n\}})^\#$ whenever either ideal is defined. Now use condition (4) of Theorem 2.6. $\square$

Remark 2.8. Let $S = k[x_1, \dots, x_d]$ be a polynomial ring over a field k and let I be an $\mathfrak{m}$-primary ideal generated by at most 5 monomials. Then I is licci. Indeed since I is licci if it is an almost complete intersection or if $d \leq 2$, we may assume that $d = 3$. Hence $I^\#$ is generated by at most two monomials, and the same is true for $(I^{\{n\}})^\#$ as long as $I^{\{n\}} \neq S$. The two monomials generating $(I^{\{n\}})^\#$ must have a common factor $\neq 1$. Therefore condition (4) of Theorem 2.6 implies that I is licci.

Discussion 2.9. Let $S = k[x_1, \dots, x_d]$ be a polynomial ring over a field k and let I be an $\mathfrak{m}$-primary monomial ideal. If I is licci then Theorem 2.6 and Lemma 2.5 show that the sequence of double links $I, I^{\{1\}}, \dots, I^{\{n\}}, \dots$ leads to a monomial almost complete intersection $I^{\{n\}}$, which is either a complete intersection or directly linked to a complete intersection. Part (4) of Theorem 2.6 provides an algorithm for deciding when an $\mathfrak{m}$-primary monomial is licci. Furthermore, the following statements hold:

(1) $I \subset I^{\{1\}} \subset \dots \subset I^{\{n\}} \subset \dots$.
(2) The chain of ideals in (1) stabilizes at S if and only if I is licci.
(3) The sequence of double links $I, I^{\{1\}}, \dots, I^{\{n\}}, \dots$ is the unique sequence of double links defined by using monomial regular sequences of smallest possible degrees at every step.

Part (1) is obvious from the definition and (2) follows from Theorem 2.6. To see (3) we may assume that I is not a complete intersection. Consider the standard form $I = (x_1^{a_1}, \dots, x_d^{a_d}) + I^\#$, and write $I^\# = x^B K$ with $x^B = x_1^{b_1} \cdots x_d^{b_d}$ a monomial and K a monomial ideal of height at least two. According to Lemma 2.5 it suffices to show that $x_1^{a_1 - b_1}, \dots, x_d^{a_d - b_d}$ is the monomial regular sequence of minimal degrees in $(x_1^{a_1}, \dots, x_d^{a_d}) : I$. Suppose without loss of generality that $x_1^n \in (x_1^{a_1}, \dots, x_d^{a_d}) : I$ for some $n \leq a_1$. It follows that $x^B K \subset (x_1^{a_1 - n}, x_2^{a_2}, \dots, x_d^{a_d})$. However, no minimal monomial generator of $x^B K$ lies in $(x_2^{a_2}, \dots, x_d^{a_d})$ by definition of $I^\# = x^B K$. We conclude that $x^B K \subset (x_1^{a_1 - n})$, hence $x^B \in (x_1^{a_1 - n})$ as K has height at least two. Therefore $b_1 \geq a_1 - n$, which gives $n \geq a_1 - b_1$. This proves part (3).

Since Theorem 2.6 gives a complete characterization of when an $\mathfrak{m}$-primary monomial ideal is licci, a natural question becomes the following: when is an arbitrary $\mathfrak{m}$-primary ideal in the same linkage class as a monomial ideal? Obviously this is a broader class than licci ideals, but we do not know any criterion for an ideal to be in the linkage class of a monomial ideal. However, there is an obstruction: if $S = k[x_1, \dots, x_d]$ is a polynomial ring over a field k and I a Cohen-Macaulay ideal in the linkage class of a monomial ideal, then $(S/I)_{\mathfrak{m}} \cong T/(x)$, where T is a reduced local ring and x is regular on T.

3. Strong Liaison

In this section we consider the consequences of Theorem 2.6 for non-monomial ideals. In particular we study the relationship between an ideal being licci and its initial ideal in some term order being licci. Fix a polynomial ring $S = k[x_1, \dots, x_d]$ and a term order $>$. When I is an ideal in S, we let $\mathrm{in}(I)$ denote the initial ideal of I with respect to $>$. We call a sequence of elements $f_1, \dots, f_n$ *super regular* if $\mathrm{in}(f_1), \dots, \mathrm{in}(f_n)$ form a regular sequence. When $f_1, \dots, f_n$ generate a zero-dimensional ideal, this condition means that after possibly reordering $f_1, ..., f_n$, one has $\mathrm{in}(f_i) = x_i^{a_i}$ for some positive integers $a_1, \dots, a_n$. We say I is *strongly licci* if I can be linked to an ideal generated by a super regular sequence in such a way that all the regular sequences used in the chain of links are super regular. Notice that in this language Theorem 2.6 implies that an $\mathfrak{m}$-primary licci monomial ideal is strongly licci. Finally, observe that a super regular sequence is a Gröbner basis for the ideal it generates [1, 15.15].

Lemma 3.1. *Let $S = k[x_1, \dots, x_d]$ with a fixed term order $>$, and let I and J be two zero-dimensional ideals linked via the super regular sequence $f_1, \dots, f_d$. Then $\mathrm{in}(I)$ and $\mathrm{in}(J)$ are linked via the regular sequence $\mathrm{in}(f_1), \dots, \mathrm{in}(f_d)$.*

Proof. We first prove that $\mathrm{in}(I) \cdot \mathrm{in}(J) \subset (\mathrm{in}(f_1), \dots, \mathrm{in}(f_d))$. Let $\mathrm{in}(f) \in \mathrm{in}(I)$ and $\mathrm{in}(g) \in \mathrm{in}(J)$ for elements $f \in I$ and $g \in J$. Then $\mathrm{in}(f) \cdot \mathrm{in}(g) = \mathrm{in}(fg) \in \mathrm{in}((f_1, \dots, f_d)) = (\mathrm{in}(f_1), \dots, \mathrm{in}(f_d))$, where the last equality follows from the fact that $f_1, \dots, f_d$ are a Gröbner basis for $(f_1, \dots, f_d)$.

Hence $\mathrm{in}(J) \subset (\mathrm{in}(f_1), \dots, \mathrm{in}(f_d)) : \mathrm{in}(I)$. To prove equality, it suffices to show that $\dim_k(S/\mathrm{in}(J)) = \dim_k S/((\mathrm{in}(f_1), \dots, \mathrm{in}(f_d)) : \mathrm{in}(I))$. But $\dim_k(S/\mathrm{in}(J)) = \dim_k(S/J) = \dim_k(S/(f_1, \dots, f_d)) - \dim_k(S/I) = \dim_k(S/\mathrm{in}((f_1, \dots, f_d))) - \dim_k(S/\mathrm{in}(I)) = \dim_k(S/((\mathrm{in}(f_1), \dots, \mathrm{in}(f_d)) : \mathrm{in}(I)))$. □

Theorem 3.2. *Let $S = k[x_1, \dots, x_d]$ with a fixed term order $>$, and let I be a zero-dimensional ideal. Then $\mathrm{in}(I)$ is licci if and only if I is strongly licci.*

Proof. Lemma 3.1 immediately implies that if I is strongly licci then $\mathrm{in}(I)$ is licci. For the proof of the converse suppose that $\mathrm{in}(I)$ is licci. Theorem 2.6 shows that in this case $\mathrm{in}(I)$ can be linked to a complete intersection by a sequence of links only using monomial regular sequences. Let $x_1^{a_1}, \dots, x_d^{a_d}$ be the first such regular sequence contained in $\mathrm{in}(I)$. Choose $f_i \in I$ such that $\mathrm{in}(f_i) = x_i^{a_i}$. By construction $f_1, \dots, f_d$ is a super regular sequence in I. Setting $J = (f_1, \dots, f_d) : I$, we use Lemma 3.1 to conclude that $\mathrm{in}(J)$ is the link of $\mathrm{in}(I)$

with respect to $(x_1^{a_1}, \ldots, x_d^{a_d})$. Inducting on the least number of links needed to link in(I) to a complete intersection via monomial linkage completes the proof. $\square$

It is worth remarking that the conclusion that I is strongly licci if in(I) is licci relies on Theorem 2.6 and does not follow directly from standard techniques of "Gröbner deformation".

Corollary 3.3. *Let $S = k[x, y]$ and let I be a zero-dimensional ideal. Then I is strongly licci with respect to every term order.*

Proof. It suffices to prove in(I) is licci, which is well-known (see also the remark after Theorem 2.6). $\square$

Remark 3.4. Notice that Theorems 2.6 and 3.2 give a complete characterization for when a zero-dimensional ideal is strongly licci with respect to a given term order. However, one might want to change either the variables or the term order as the next example shows.

Example 3.5. Let $S = k[x, y, z]$ and $I = (x^2 + y^2, y^2 + z^2, xy, xz, yz)$. Use the term order revlex with the variables ordered $x > y > z$. The initial ideal is in$(I) = (x^2, y^2, xy, xz, yz, z^3)$ which is not licci since $(\text{in}(I))^{\#}$ has height two. Hence I is not strongly licci with respect to this order. However, I is licci as it is a height three Gorenstein ideal.

On the other hand, one has $I = ((x-y)^2, (y-z)^2, xy, (x-y)z, yz)$, and changing variables to $x' = x - y$, $y' = y - z$ and $z' = z$ allows us to rewrite the ideal $I = ((x')^2, (y')^2, (z')^2 - x'y', x'z', x'y' + y'z')$. In revlex order with $z' > y' > x'$ these generators form a Gröbner basis and the initial ideal is $((x')^2, (y')^2, (z')^2, x'z', y'z')$. This is a licci monomial ideal.

The above theorem and example raise the question of whether or not zero-dimensional licci ideals have licci initial ideals with respect to some term order if in addition we allow a change of variables. Equivalently, are zero-dimensional licci ideals strongly licci after a suitable change of variables and choice of term order? This seems unlikely. A weaker question is whether there exists a monomial licci ideal with the same Hilbert function as any given $\mathfrak{m}$-primary homogeneous ideal I linked to a complete intersection by a sequence of links defined by homogeneous regular sequences. There is an "obvious" way to try to construct such a monomial ideal: starting with a sequence of links from I to $(x_1, \ldots, x_d)$, simply go backwards by always using monomial regular sequences of the same degrees as the homogeneous regular sequences in the original linking sequence. The problem with this idea is that there may not be the appropriate pure powers in the monomial ideals obtained via this algorithm. It is an interesting question whether or not the appropriate pure powers would actually exist.

4. Glicci Mononial Ideals

The next proposition can be found in [5, 5.10] in the graded case; the local case given below follows easily from the same proof. We give an easy proof for the benefit of the reader.

Proposition 4.1. *Let $(R, \mathfrak{m})$ be a local Gorenstein ring, J and K proper ideals of R, and $x \in \mathfrak{m}$. Assume that the ring R/J is Cohen-Macaulay and generically Gorenstein and that the ideal $J + xK$ is unmixed of height greater than* $\operatorname{ht} J$. *Then $J + xK$ and $J + K$ are Gorenstein doubly linked.*

Proof. We write $^-$ for images in $\overline{R} = R/J$. Notice that $\overline{x}$ is a regular element on $\overline{R}$ and that $\overline{K}$ is an unmixed ideal of height one. Since the ring $\overline{R}$ is generically Gorenstein it has a canonical ideal, meaning an ideal ω of positive height that is a canonical module of $\overline{R}$. Multiplying with an $\overline{R}$-regular element we may assume that $\omega \subset \overline{K}$. Let H be an ideal in S with $J \subset H \subset J + K$ and $\overline{H} = \omega$. Since ω and $\overline{x}\omega$ are canonical ideals, it follows that both $R/H \cong \overline{R}/\omega$ and $R/J + xH \cong \overline{R}/\overline{x}\omega$ are Gorenstein rings. The element $\overline{x}$ being $\overline{R}$-regular one also has $\overline{x}\omega : \overline{x}\overline{K} = \omega : \overline{K}$ in $\overline{R}$. Therefore back in R we obtain $(J + xH) : (J + xK) = H : (J + K)$. As $J + xH \subset J + xK$ and $H \subset J + K$ and all four ideals are unmixed of the same height, we conclude that $J + xK$ and $J + K$ are Gorenstein doubly linked. □

The proof of the next theorem was inspired by the work of Migliore and Nagel in [7, 3.5], where they prove that strongly stable Cohen-Macaulay monomial ideals are glicci in the graded sense, i.e., using only homogeneous Gorenstein ideals in the links. The problem for such monomial ideals reduces at once to the $\mathfrak{m}$-primary case. Their proof can be generalized as follows:

Theorem 4.2. *Let $S = k[x_1, \ldots, x_d]$ be a polynomial ring over an infinite field k and let I be an $\mathfrak{m}$-primary monomial ideal. Then $I_\mathfrak{m}$ is glicci in $S_\mathfrak{m}$.*

Proof. Write $x = x_d$. We use induction on a, the smallest integer so that $x^a \in I$. If $a = 1$ we are done by induction on d. Otherwise we may write $I = I_0S + xK$ with $I_0 = I \cap k[x_1, \ldots, x_{d-1}]$ and K a proper monomial ideal in S. By induction on a it will suffice to prove that $I_\mathfrak{m}$ and $(I_0S + K)_\mathfrak{m}$ are Gorenstein doubly linked.

To this end we wish to apply Proposition 4.1. Thus write $y = x^a$ and let $\{\alpha_j | j \in \mathbb{N}\}$ be a sequence of pairwise distinct elements in k. Similar to the proof of Theorem 2.4 we define a map Ψ from the set of monomials in $k[x_1, \ldots, x_{d-1}]$ to S. For $1 \le i \le d - 1$ we set

$$\Psi(x_i^n) = \prod_{j=1}^{n} (x_i + \alpha_j y),$$

and we extend this definition multiplicatively,

$$\Psi(x_1^{c_1} \cdots x_{d-1}^{c_{d-1}}) = \Psi(x_1^{c_1}) \cdots \Psi(x_{d-1}^{c_{d-1}}).$$

Finally, we define J to be the ideal in S generated by the images $\Psi(x^C)$ of the monomials x^C in I_0.

Obviously $I_0S + yS = J + yS$. Since $y \in xK$ it then follows that $I = I_0S + xK = J + xK$ and $I_0S + K = J + K$. Hence it suffices to prove that $(J + xK)_\mathfrak{m}$ and $(J + K)_\mathfrak{m}$ are Gorenstein doubly linked. As in the proof of Theorem 2.4 one sees that the element y is regular on S/J. In particular $S_\mathfrak{m}/J_\mathfrak{m}$ is a one-dimensional Cohen-Macaulay ring. In view of Proposition 4.1

is remains to show that this ring is reduced, hence generically Gorenstein. Thus let $\mathfrak{p}$ be a minimal prime of J. Let n be an integer so that $x_i^n \in I_0$ for $1 \leq i \leq d-1$, and consider the products contained in J,

$$J \ni \Psi(x_i^n) = \prod_{j=1}^{n}(x_i + \alpha_j y).$$

For every i, the ideal $\mathfrak{p}$ contains at most one factor $x_i+\alpha_{j(i)}y$, because $\alpha_1, \ldots, \alpha_n$ are pairwise distinct elements of k and $\mathfrak{p}$ does not contain y. Therefore in the ring $S_\mathfrak{p}$ we obtain

$$J_\mathfrak{p} \supset (\Psi(x_1^n), \ldots, \Psi(x_{d-1}^n))_\mathfrak{p} = (x_1 + \alpha_{j(1)}y, \ldots, x_{d-1} + \alpha_{j(d-1)}y)_\mathfrak{p}.$$

This shows that indeed $J_\mathfrak{p} = \mathfrak{p}_\mathfrak{p}$. □

The special type of Gorenstein double linkage that arises in Proposition 4.1 has been dubbed *Gorenstein biliaison*. In this language the proof of Theorem 4.2 gives the stronger statement that $I_\mathfrak{m}$ is linked to a complete intersection in $S_\mathfrak{m}$ by a sequence of Gorenstein biliaisons.

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Department of Mathematics, University of Kansas, Lawrence, KS 66045
E-mail address: huneke@math.ku.edu
URL: http://www.math.ku.edu/~huneke

Department of Mathematics, Purdue University, West Lafayette, IN 47907
E-mail address: ulrich@math.purdue.edu
URL: http://www.math.purdue.edu/~ulrich