$\mathcal{D}$-MODULES ON SMOOTH TORIC VARIETIES

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ABSTRACT. Let X be a smooth toric variety. Cox introduced the homogeneous coordinate ring S of X and its irrelevant ideal $\mathfrak{b}$. Let A denote the ring of differential operators on $\mathrm{Spec}(S)$. We show that the category of $\mathcal{D}$-modules on X is equivalent to a subcategory of graded A-modules modulo $\mathfrak{b}$-torsion. Additionally, we prove that the characteristic variety of a $\mathcal{D}$-module is a geometric quotient of an open subset of the characteristic variety of the associated A-module and that holonomic $\mathcal{D}$-modules correspond to holonomic A-modules.

1. INTRODUCTION

Let X be a smooth toric variety over a field k. Cox [C] introduced the homogeneous coordinate ring S of X and the irrelevant ideal $\mathfrak{b}$. The k-algebra S is a polynomial ring, with one variable for each one-dimensional cone in the fan Δ defining X, and has a natural grading by the class group $\mathrm{Cl}(X)$. The monomial ideal $\mathfrak{b} \subset S$ encodes the combinatorial structure of Δ. The following theorem of Cox [C] indicates the significance of the pair $(S, \mathfrak{b})$. We write $\mathcal{O}$-Mod for the category of quasi-coherent sheaves on X and S-GrMod for the category of graded S-modules. A graded S-module F is called $\mathfrak{b}$-torsion if, for all $f \in F$, there exists $\ell > 0$ such that $\mathfrak{b}^\ell f = 0$. Let $\mathfrak{b}$-Tors denote the full subcategory of $\mathfrak{b}$-torsion modules.

Theorem (Cox). (1) *The category* $\mathcal{O}$-Mod *is equivalent to the quotient category* S-$\mathsf{GrMod}/\mathfrak{b}$-$\mathsf{Tors}$.

(2) *The variety X is a geometric quotient of* $\mathrm{Spec}(S) \setminus \mathrm{Var}(\mathfrak{b})$ *by a suitable torus action.*

When $X = \mathbb{P}^n$, this is Serre's description of quasi-coherent sheaves on projective space and the classical construction of projective space.

The aim of this paper is to provide the $\mathcal{D}$-module version of this theorem — $\mathcal{D}$ denotes the sheaf of differential operators on X. To state the analogue of the first part, we introduce the following notation. We

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write $\mathcal{D}$-Mod for the category of left $\mathcal{D}$-modules on X. The ring of differential operators on $\operatorname{Spec}(S)$ is the Weyl algebra A; it also has a natural $\mathrm{Cl}(X)$ grading. To each element $\bar{\mathbf{u}}$ in $\mathrm{Cl}(X)^\vee = \operatorname{Hom}_{\mathbb{Z}}\big(\mathrm{Cl}(X), \mathbb{Z}\big)$, we associate an "Euler" operator $\theta_{\bar{\mathbf{u}}} \in A$ [see (2.B) for the precise definition]. The full subcategory of graded left A-modules F satisfying $(\theta_{\bar{\mathbf{u}}} - \langle \bar{\mathbf{u}}, \bar{\mathbf{a}} \rangle) \cdot F_{\bar{\mathbf{a}}} = 0$ for all $\bar{\mathbf{a}} \in \mathrm{Cl}(X)$ and all $\bar{\mathbf{u}} \in \mathrm{Cl}(X)^\vee$ is denoted A-GrMod_θ.

Theorem 1.1. *The quotient category A-$\mathsf{GrMod}_\theta/\mathfrak{b}$-$\mathsf{Tors}$ is equivalent to the category $\mathcal{D}$-Mod.*

The special case, when X is projective space, can be found in Borel [B$^+$].

This categorical equivalence is given by two functors. The first takes an object F in A-GrMod_θ to the $\mathcal{D}$-module $\widetilde{F}$ whose sections over the affine open subset U_σ associated to $\sigma \in \Delta$ are $(F_{x^{\hat{\sigma}}})_{\bar{\mathbf{0}}}$. The second maps a $\mathcal{D}$-module $\mathcal{F}$ to $\Gamma_L(\mathcal{F}) = \bigoplus_{\bar{\mathbf{a}} \in \mathrm{Cl}(X)} H^0\big(X, \mathcal{O}(\bar{\mathbf{a}}) \otimes \mathcal{F}\big)$. In fact, if F is finitely generated then $\widetilde{F}$ is coherent and if $\mathcal{F}$ is coherent then it is of the form $\widetilde{F}$ for some finitely generated graded A-module F. Our analysis of the $\Gamma_L(-)$ extends the work of Musson [M2] and Jones [J2] on rings of twisted differential operators on toric varieties.

Our second major result is

Theorem 1.2. *If $F \in A$-GrMod_θ is finitely generated, then the characteristic variety of $\widetilde{F}$ is a geometric quotient of a suitable open subset of the characteristic variety of F.*

Moreover, given a finitely generated $F \in A$-GrMod_θ which has no $\mathfrak{b}$-torsion, we show that the dimension of $\widetilde{F}$ is equal to the dimension of F minus the rank of $\mathrm{Cl}(X)$. In particular, holonomic A-modules correspond to holonomic $\mathcal{D}$-modules.

The category of modules over the Weyl algebra is a well-studied algebraic object and we hope to study $\mathcal{D}$-modules on X by using these methods. In particular, effective algorithms have been developed for $\mathcal{D}$-modules on affine space; for example, see the work of Oaku [O], Walther [W], Saito-Sturmfels-Takayama [SST] and Oaku-Takayama [OT]. It would interesting to use our results to extend these methods to smooth toric varieties.

Many of our results are valid for a simplicial toric variety if one replaces S and A with the subrings $\bigoplus_{\bar{\mathbf{b}} \in \mathrm{Pic}(X)} S_{\bar{\mathbf{b}}}$ and $\bigoplus_{\bar{\mathbf{b}} \in \mathrm{Pic}(X)} A_{\bar{\mathbf{b}}}$; see Cox [C]. This approach allows one to recover the general results of Musson [M2] and Jones [J2]. For simplicity, we present the smooth case and leave the possible generalizations to the reader.

The contents of this paper are as follows: The second section reviews the basics about toric varieties, the Weyl algebra and $\mathcal{D}$-modules. In the third section, we determine the module associated with the sheaf $\mathcal{D} \otimes \mathcal{O}(\bar{\mathbf{b}})$. We introduce the $\mathrm{Cl}(X) \times \mathrm{Cl}(X)$-graded A-A bimodule

$$D = \bigoplus_{\bar{\mathbf{b}} \in \mathrm{Cl}(X)} \frac{A(\bar{\mathbf{b}})}{A \cdot (\theta_{\bar{\mathbf{u}}} + \langle \bar{\mathbf{u}}, \bar{\mathbf{b}} \rangle : \bar{\mathbf{u}} \in \mathrm{Cl}(X)^\vee)}$$

and construct a morphism

$$\eta \colon D \longrightarrow \bigoplus_{(\bar{\mathbf{a}}, \bar{\mathbf{b}}) \in \mathrm{Cl}(X)^2} H^0(X, \mathcal{O}(\bar{\mathbf{a}}) \otimes \mathcal{D} \otimes \mathcal{O}(\bar{\mathbf{b}}))$$

We prove that η is an isomorphism in two steps. We first show that η induces an isomorphism on the associated sheaves. Secondly, we establish that $H^0_{\mathfrak{b}}(D) = H^1_{\mathfrak{b}}(D) = 0$. The first step is a local statement and can be reduced to results of Musson [M1]. We provide a direct proof using simplifications due to Jones [J1]. The fourth section contains the proof of Theorem 1.1. We establish this result for both left and right $\mathcal{D}$-modules. In general, there is an equivalence between these and we show how this is induced at the level of A-modules. In the last section, we prove Theorem 1.2 and related dimension results.

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2. Background

We collect here a number of more or less standard definitions, results and notation. Throughout this paper, we work over an algebraically closed field k of characteristic zero.

Toric varieties. Let X be a smooth toric variety determined by the fan Δ in $N \cong \mathbb{Z}^n$. We write $\mathbf{v}_1, \dots, \mathbf{v}_d$ for the unique lattice vectors generating the one-dimensional cones in Δ and we assume that the $\mathbf{v}_i$ span $N \otimes_{\mathbb{Z}} \mathbb{R}$. Each $\mathbf{v}_i$ corresponds to an irreducible torus invariant Weil divisor in X. Since these divisors generate the torus invariant Weil divisors, we may identify the group of torus invariant Weil divisors with $\mathbb{Z}^d$. Let $\mathbf{e}_i$ denote the standard basis for $\mathbb{Z}^d$ and set $\mathbf{e} = \mathbf{e}_1 + \cdots + \mathbf{e}_d$.

There is a short exact sequence

$$0 \longrightarrow N^\vee \xrightarrow{\iota} \mathbb{Z}^d \xrightarrow{\overline{(\ )}} \mathrm{Cl}(X) \longrightarrow 0\,, \tag{2.A}$$

where $\iota(\mathbf{p}) = \langle \mathbf{p}, \mathbf{v}_1 \rangle \mathbf{e}_1 + \cdots + \langle \mathbf{p}, \mathbf{v}_d \rangle \mathbf{e}_d$ and the second map $\mathbf{a} \mapsto \bar{\mathbf{a}}$ is the projection from Weil divisors to the divisor class group. Since X is smooth, the divisor class group $\mathrm{Cl}(X)$ is isomorphic to the Picard group $\mathrm{Pic}(X)$. In particular, the invertible sheaf (line bundle) associated to $\bar{\mathbf{a}} \in \mathrm{Cl}(X)$ is denoted $\mathcal{O}(\bar{\mathbf{a}})$.

Following [C], the homogeneous coordinate ring of X is the polynomial ring $S = k[x_1, \ldots, x_d]$ with a $\mathrm{Cl}(X)$-grading induced by

$$\deg(x^{\mathbf{a}}) = \deg(x_1^{\mathbf{a}_1} \cdots x_d^{\mathbf{a}_d}) = \bar{\mathbf{a}} \in \mathrm{Cl}(X)\,.$$

For a cone $\sigma \in \Delta$, $\widehat{\sigma}$ is the set $\{i : \mathbf{v}_i \notin \sigma\}$ and $x^{\widehat{\sigma}} = \prod_{\mathbf{v}_i \notin \sigma} x_i$ is the associated monomial in S. The irrelevant ideal of X is the reduced monomial ideal $\mathfrak{b} = (x^{\widehat{\sigma}} : \sigma \in \Delta)$.

Recall that each cone $\sigma \in \Delta$ corresponds to an open affine subset of X, $U_\sigma \cong \mathrm{Spec}(S_{x^{\widehat{\sigma}}})_{\bar{\mathbf{0}}}$. Every graded S-module F gives rise to a quasi-coherent sheaf on X, denoted by $\widetilde{F}$, which corresponds to the module $(F_{x^{\widehat{\sigma}}})_{\bar{\mathbf{0}}}$ over U_σ. If F is finitely generated over S, then $\widetilde{F}$ is a coherent $\mathcal{O}$-module; $\mathcal{O}$ denotes the structure sheaf on X. Moreover, every quasi-coherent sheaf on X is of the form $\widetilde{F}$ for some graded S-module F and if the sheaf is coherent, then F can be taken finitely generated. For an S-module F, we have $\widetilde{F} = 0$ if and only if $F = H^0_{\mathfrak{b}}(F)$; in other words F is $\mathfrak{b}$-torsion.

Weyl algebra. By definition, the d-th Weyl algebra is

$$A = \frac{k\{x_1, \ldots, x_d, \partial_1, \ldots, \partial_d\}}{\left(\begin{smallmatrix} x_i x_j - x_j x_i & = & 0 \\ \partial_i \partial_j - \partial_j \partial_i & = & 0 \\ \partial_i x_j - x_j \partial_i & = & \delta_{ij} \end{smallmatrix}\right)}\,.$$

The canonical ring morphism $S \hookrightarrow A$ provides A with the structure of left S-module. As in the case of S, A has a $\mathrm{Cl}(X)$-grading given by

$$\deg(x^{\mathbf{a}} \partial^{\mathbf{b}}) = \deg(x_1^{\mathbf{a}_1} \cdots x_d^{\mathbf{a}_d} \partial_1^{\mathbf{b}_1} \cdots \partial_d^{\mathbf{b}_d}) = \bar{\mathbf{a}} - \bar{\mathbf{b}} \in \mathrm{Cl}(X)\,,$$

and the $\bar{\mathbf{a}}$-th graded component of A is denoted $A_{\bar{\mathbf{a}}}$. For each element $\bar{\mathbf{u}}$ of $\mathrm{Cl}(X)^\vee = \mathrm{Hom}_{\mathbb{Z}}\big(\mathrm{Cl}(X), \mathbb{Z}\big)$, we have an Euler operator

$$\theta_{\bar{\mathbf{u}}} = \langle \bar{\mathbf{u}}, \bar{\mathbf{e}}_1 \rangle \theta_1 + \cdots + \langle \bar{\mathbf{u}}, \bar{\mathbf{e}}_d \rangle \theta_d\,, \tag{2.B}$$

where $\theta_i = x_i \partial_i$. Notice that $\theta_{\bar{\mathbf{u}}}$ has degree zero.

The Weyl algebra A is isomorphic to the ring of differential operators on $\mathbb{A}^d$. The natural action of A on a polynomial $f \in S$ is $x_i \bullet f = x_i \cdot f$ and $\partial_i \bullet f = \frac{\partial f}{\partial x_i}$. Since S is also a subring of A, the symbol $\bullet$ helps distinguish this action from the product $\cdot\colon A \times A \to A$. The ring of differential operators on $S_{x^{\mathbf{a}}} = S[x^{-\mathbf{a}}]$ is denoted $A_{x^{\mathbf{a}}}$ and is equal to the localization $A[x^{-\mathbf{a}}]$.

$\mathcal{D}$-modules. The sheaf of (algebraic) differential operators on X is denoted $\mathcal{D}$. On an affine open subset $U \subseteq X$, $H^0(U, \mathcal{D}) = \bigcup_{i \geq 0} \mathcal{D}^i(U)$ where $\mathcal{D}^0(U) = H^0(U, \mathcal{O})$ and

$$\mathcal{D}^i(U) := \left\{ s \in \mathrm{End}_k\big(H^0(U, \mathcal{O})\big) : \begin{array}{l} fs - sf \in \mathcal{D}^{i-1}(U) \\ \text{for all } f \in H^0(U, \mathcal{O}) \end{array} \right\}.$$

A $\mathcal{D}$-module is a sheaf $\mathcal{F}$ on X which is quasi-coherent as an $\mathcal{O}$-module and has a structure of module over $\mathcal{D}$. A $\mathcal{D}$-module is coherent if it is locally finitely generated over $\mathcal{D}$. We write $\mathcal{D}$-Mod and Mod-$\mathcal{D}$ for the categories of left and right $\mathcal{D}$-modules respectively. The full subcategories of coherent left and right $\mathcal{D}$-modules are denoted $\mathcal{D}$-Coh and Coh-$\mathcal{D}$.

3. Sheaves of Differential Operators

The goal of this section is to describe the left S-modules corresponding to twists of the sheaf of differential operators. Recall that, for a graded A-module F and $\bar{\mathbf{b}} \in \mathrm{Cl}(X)$, $F(\bar{\mathbf{b}})$ is the shift of F by $\bar{\mathbf{b}}$: $F(\bar{\mathbf{b}})_{\bar{\mathbf{a}}} = F_{\bar{\mathbf{b}}+\bar{\mathbf{a}}}$. We define the graded left A-modules

$$D_L(\bar{\mathbf{b}}) = \frac{A(\bar{\mathbf{b}})}{A \cdot (\theta_{\bar{\mathbf{u}}} + \langle \bar{\mathbf{u}}, \bar{\mathbf{b}} \rangle : \bar{\mathbf{u}} \in \mathrm{Cl}(X)^\vee)}$$

and

$$D = \bigoplus_{\bar{\mathbf{b}} \in \mathrm{Cl}(X)} D_L(\bar{\mathbf{b}}) \, .$$

Notice that D has a $\mathrm{Cl}(X) \times \mathrm{Cl}(X)$-grading where $D_{(\bar{\mathbf{a}},\bar{\mathbf{b}})} = D_L(\bar{\mathbf{b}})_{\bar{\mathbf{a}}}$.

Lemma 3.1. *The module D is a graded A-A bimodule.*

Proof. Multiplication in the ring A yields the right action of A on D:

$$\begin{array}{ccc} A(\bar{\mathbf{b}})_{\bar{\mathbf{a}}} \otimes_k A_{\bar{\mathbf{b}}'} & \longrightarrow & A(\bar{\mathbf{b}} + \bar{\mathbf{b}}')_{\bar{\mathbf{a}}} \\ \big\downarrow & & \big\downarrow \\ D_L(\bar{\mathbf{b}})_{\bar{\mathbf{a}}} \otimes_k A_{\bar{\mathbf{b}}'} & \longrightarrow & D_L(\bar{\mathbf{b}} + \bar{\mathbf{b}}')_{\bar{\mathbf{a}}} \, . \end{array}$$

To see that the induced map is well-defined, observe that, for all elements $f \in A_{\bar{\mathbf{b}}'}$ and $\bar{\mathbf{u}} \in \mathrm{Cl}(X)^\vee$, we have

$$\begin{aligned} (\theta_{\bar{\mathbf{u}}} + \langle \bar{\mathbf{u}}, \bar{\mathbf{b}} \rangle) \cdot f &= f \cdot (\theta_{\bar{\mathbf{u}}} + \langle \bar{\mathbf{u}}, \bar{\mathbf{b}} \rangle) + \langle \bar{\mathbf{u}}, \bar{\mathbf{b}}' \rangle \cdot f \\ &= f \cdot (\theta_{\bar{\mathbf{u}}} + \langle \bar{\mathbf{u}}, \bar{\mathbf{b}} + \bar{\mathbf{b}}' \rangle) \, . \end{aligned}$$

This action is clearly compatible with the left A-module structure. It follows that D is an A-A bimodule. Since $f \in A_{\bar{\mathbf{a}}'}$ and $g \in A_{\bar{\mathbf{b}}'}$ implies $f \cdot D_{(\bar{\mathbf{a}},\bar{\mathbf{b}})} \cdot g \subseteq D_{(\bar{\mathbf{a}}+\bar{\mathbf{a}}',\bar{\mathbf{b}}+\bar{\mathbf{b}}')}$, D is bigraded. In other words, if we let A° denote the opposite algebra, then $A \otimes_k A^\circ$ is a $\mathrm{Cl}(X)^2$-graded ring and D is a graded module over $A \otimes_k A^\circ$. ☐

Analogously, we define right A-modules:

$$D_R(\bar{\mathbf{a}}) = \frac{A(\bar{\mathbf{a}})}{(\theta_{\bar{\mathbf{u}}} - \langle \bar{\mathbf{u}}, \bar{\mathbf{a}} \rangle : \bar{\mathbf{u}} \in \mathrm{Cl}(X)^\vee) \cdot A}$$

and

$$D' = \bigoplus_{\bar{\mathbf{a}} \in \mathrm{Cl}(X)} D_R(\bar{\mathbf{a}})\,.$$

Again, D' is a $\mathrm{Cl}(X) \times \mathrm{Cl}(X)$-graded A-A bimodule where the multiplication on the left is induced by the multiplication in the Weyl algebra. In fact, we obtain the same module.

Lemma 3.2. *There is a canonical identification $D = D'$ which respects the graded bimodule structure.*

Proof. Since we have

$$D_{(\bar{\mathbf{a}},\bar{\mathbf{b}})} = \frac{A_{\bar{\mathbf{a}}+\bar{\mathbf{b}}}}{A_{\bar{\mathbf{a}}+\bar{\mathbf{b}}} \cdot (\theta_{\bar{\mathbf{u}}} + \langle \bar{\mathbf{u}}, \bar{\mathbf{b}} \rangle : \bar{\mathbf{u}} \in \mathrm{Cl}(X)^\vee)}$$

and

$$D'_{(\bar{\mathbf{a}},\bar{\mathbf{b}})} = \frac{A_{\bar{\mathbf{a}}+\bar{\mathbf{b}}}}{(\theta_{\bar{\mathbf{u}}} - \langle \bar{\mathbf{u}}, \bar{\mathbf{a}} \rangle : \bar{\mathbf{u}} \in \mathrm{Cl}(X)^\vee) \cdot A_{\bar{\mathbf{a}}+\bar{\mathbf{b}}}}\,,$$

it is enough to show that $A_{\bar{\mathbf{a}}+\bar{\mathbf{b}}} \cdot (\theta_{\bar{\mathbf{u}}} + \langle \bar{\mathbf{u}}, \bar{\mathbf{b}} \rangle) = (\theta_{\bar{\mathbf{u}}} - \langle \bar{\mathbf{u}}, \bar{\mathbf{a}} \rangle) \cdot A_{\bar{\mathbf{a}}+\bar{\mathbf{b}}}$. However, for every $f \in A_{\bar{\mathbf{a}}+\bar{\mathbf{b}}}$, we have

$$\begin{aligned} f \cdot (\theta_{\bar{\mathbf{u}}} + \langle \bar{\mathbf{u}}, \bar{\mathbf{b}} \rangle) &= (\theta_{\bar{\mathbf{u}}} + \langle \bar{\mathbf{u}}, \bar{\mathbf{b}} \rangle) \cdot f - \langle \bar{\mathbf{u}}, \bar{\mathbf{a}} + \bar{\mathbf{b}} \rangle \cdot f \\ &= (\theta_{\bar{\mathbf{u}}} - \langle \bar{\mathbf{u}}, \bar{\mathbf{a}} \rangle) \cdot f\,, \end{aligned}$$

which establishes the lemma. □

Now, the cohomology of $\mathcal{D}$ has an A-A bimodule structure.

Lemma 3.3. *The direct sum*

$$\bigoplus_{(\bar{\mathbf{a}},\bar{\mathbf{b}}) \in \mathrm{Cl}(X)^2} H^0\big(X, \mathcal{O}(\bar{\mathbf{a}}) \otimes \mathcal{D} \otimes \mathcal{O}(\bar{\mathbf{b}})\big)$$

is an A-A bimodule.

Proof. It suffices to give k-linear maps:

$$\begin{aligned} \text{(3.A)} \quad \mu\colon A_{\bar{\mathbf{a}}'} \otimes_k H^0\big(X, \mathcal{O}(\bar{\mathbf{a}}) \otimes \mathcal{D} \otimes \mathcal{O}(\bar{\mathbf{b}})\big) \otimes_k A_{\bar{\mathbf{b}}'} & \\ \longrightarrow H^0\big(X, \mathcal{O}(\bar{\mathbf{a}} + \bar{\mathbf{a}}') \otimes \mathcal{D} \otimes \mathcal{O}(\bar{\mathbf{b}} + \bar{\mathbf{b}}')\big)\,. & \end{aligned}$$

Locally, a section $s \in H^0\big(U_\sigma, \mathcal{O}(\bar{\mathbf{a}}) \otimes \mathcal{D} \otimes \mathcal{O}(\bar{\mathbf{b}})\big)$ can be identified with an element of $\mathrm{Hom}_k\big((S_{x^{\hat{\sigma}}})_{-\bar{\mathbf{b}}}, (S_{x^{\hat{\sigma}}})_{\bar{\mathbf{a}}}\big)$, where σ is a cone in Δ. Moreover, the action of $f \in A_{\bar{\mathbf{a}}}$ on S descends to action on $S_{x^{\hat{\sigma}}}$ which increases degrees by $\bar{\mathbf{a}}$. Thus, we may define $(\mu|_{U_\sigma})(f \otimes s \otimes g) = f \circ s \circ g$. One verifies that $\mu|_{U_\sigma}$ maps into $H^0\big(U_\sigma, \mathcal{O}(\bar{\mathbf{a}} + \bar{\mathbf{a}}') \otimes \mathcal{D} \otimes \mathcal{O}(\bar{\mathbf{b}} + \bar{\mathbf{b}}')\big)$ and that these local definitions glue to give the required map. □

We next construct a morphism of graded A-A bimodules

$$\eta\colon D \longrightarrow \bigoplus_{(\bar{\mathbf{a}},\bar{\mathbf{b}})} H^0\big(X, \mathcal{O}(\bar{\mathbf{a}}) \otimes \mathcal{D} \otimes \mathcal{O}(\bar{\mathbf{b}})\big)\,. \tag{3.B}$$

Since η is a graded k-linear morphism, it is enough to define $\eta(f)$ for $f \in D_{(\bar{\mathbf{a}},\bar{\mathbf{b}})}$. For $f \in D_{(\bar{\mathbf{a}},\bar{\mathbf{b}})}$, let $\eta(f)$ be the section of $H^0\big(X, \mathcal{O}(\bar{\mathbf{a}}) \otimes \mathcal{D} \otimes \mathcal{O}(\bar{\mathbf{b}})\big)$ whose restriction over each U_σ corresponds to the map induced by the action of $f\colon (S_{x^{\hat{\sigma}}})_{-\bar{\mathbf{b}}} \longrightarrow (S_{x^{\hat{\sigma}}})_{\bar{\mathbf{a}}}$. To see that $\eta(f)$ is well-defined, consider $f \in A_{\bar{\mathbf{a}}+\bar{\mathbf{b}}} \cdot (\theta_{\bar{\mathbf{u}}} + \langle \bar{\mathbf{u}}, \bar{\mathbf{b}} \rangle)$. It follows that, for $g \in (S_{x^{\hat{\sigma}}})_{-\bar{\mathbf{b}}}$, we have

$$(\theta_{\bar{\mathbf{u}}} + \langle \bar{\mathbf{u}}, \bar{\mathbf{b}} \rangle) \bullet g = -\langle \bar{\mathbf{u}}, \bar{\mathbf{b}} \rangle \cdot g + \langle \bar{\mathbf{u}}, \bar{\mathbf{b}} \rangle \cdot g = 0$$

and therefore $\eta(f) = 0$. It is clear that η is a morphism of graded A-A bimodules. The main result of this section is the following.

Theorem 3.4. *The morphism η [see equation (3.B)] is an isomorphism of graded A-A bimodules.*

Before proving Theorem 3.4, we collect some local results. We first consider a local version of η. By composing η with the restriction to U_σ where $\sigma \in \Delta$, we obtain a morphism of left S-modules

$$D \longrightarrow \bigoplus_{(\bar{\mathbf{a}},\bar{\mathbf{b}}) \in \mathrm{Cl}(X)^2} H^0\big(U_\sigma, \mathcal{O}(\bar{\mathbf{a}}) \otimes \mathcal{D} \otimes \mathcal{O}(\bar{\mathbf{b}})\big)\,,$$

which induces a morphism

$$\eta^\sigma\colon D_{x^{\hat{\sigma}}} \longrightarrow \bigoplus_{(\bar{\mathbf{a}},\bar{\mathbf{b}}) \in \mathrm{Cl}(X)^2} H^0\big(U_\sigma, \mathcal{O}(\bar{\mathbf{a}}) \otimes \mathcal{D} \otimes \mathcal{O}(\bar{\mathbf{b}})\big)\,.$$

Taking the degree zero component yields the ring morphism

$$\bar{\varphi}_\sigma = \eta^\sigma_{(\bar{\mathbf{0}},\bar{\mathbf{0}})}\colon \frac{(A_{x^{\hat{\sigma}}})_{\bar{\mathbf{0}}}}{(A_{x^{\hat{\sigma}}})_{\bar{\mathbf{0}}} \cdot (\theta_{\bar{\mathbf{u}}} : \bar{\mathbf{u}} \in \mathrm{Cl}(X)^\vee)} \longrightarrow H^0(U_\sigma, \mathcal{D})\,. \tag{3.C}$$

Understanding $\bar{\varphi}_\sigma$ is an important ingredient in establishing Theorem 3.4. We begin by studying the map $\varphi_\sigma\colon (A_{x^{\hat{\sigma}}})_{\bar{\mathbf{0}}} \longrightarrow H^0(U_\sigma, \mathcal{D})$ which induces $\bar{\varphi}_\sigma$.

By definition, $H^0(U_\sigma, \mathcal{D})$ is the ring of differential operators on the ring $(S_{x^{\hat{\sigma}}})_{\bar{\mathbf{0}}}$. However, the inclusion $\iota_\sigma\colon \sigma^\vee \cap N^\vee \hookrightarrow \mathbb{Z}^d$ induces a ring isomorphism (denoted by the same name) $\iota_\sigma\colon k[\sigma^\vee \cap N^\vee] \xrightarrow{\cong} (S_{x^{\hat{\sigma}}})_{\bar{\mathbf{0}}}$. To see this, observe that $x^{\iota(\mathbf{p})} \in S_{x^{\hat{\sigma}}}$ if and only if $\langle \mathbf{p}, \mathbf{v}_i \rangle \geq 0$, for all $\mathbf{v}_i \in \sigma$, which is equivalent to $\mathbf{p} \in \sigma^\vee$. It follows that the isomorphism ι_σ induces an isomorphism, called ψ_σ, from the differential operators on $(S_{x^{\hat{\sigma}}})_{\bar{\mathbf{0}}}$ to the differential operators R_σ on $k[\sigma^\vee \cap N^\vee]$. More explicitly, we have $\psi_\sigma(f) = \iota_\sigma^{-1} \circ f \circ \iota_\sigma$. We will actually focus on the morphism $\psi_\sigma \circ \varphi_\sigma\colon (A_{x^{\hat{\sigma}}})_{\bar{\mathbf{0}}} \longrightarrow R_\sigma$.

Following [M1] and [J1], we decompose $(A_{x^{\hat{\sigma}}})_{\bar{\mathbf{0}}}$ and R_σ under the appropriate torus actions and express $\psi_\sigma \circ \varphi_\sigma$ in terms of these decompositions. To be more concrete, we identify $N^\vee$ with $\mathbb{Z}^n$ by fixing a basis. Let $\boldsymbol{\varepsilon}_1, \ldots, \boldsymbol{\varepsilon}_n$ denote the standard basis of $\mathbb{Z}^n$ and let $\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}_1 + \cdots + \boldsymbol{\varepsilon}_n$. We continue to call the natural embedding $\iota \colon N^\vee = \mathbb{Z}^n \longrightarrow \mathbb{Z}^d$.

The torus $(k^*)^d$ acts on $k[x_1^{\pm 1}, \ldots, x_d^{\pm 1}]$ by $\lambda * (x^{\mathbf{a}}) = \lambda^{\mathbf{a}} x^{\mathbf{a}}$ which produces an action on $A_{x^{\mathbf{e}}}$ given by $\lambda * (x^{\mathbf{a}} \partial^{\mathbf{b}}) = \lambda^{\mathbf{a}-\mathbf{b}} x^{\mathbf{a}} \partial^{\mathbf{b}}$ for $\lambda \in (k^*)^d$. The corresponding eigenspace decomposition is $A_{x^{\mathbf{e}}} = \bigoplus_{\mathbf{a} \in \mathbb{Z}^d} x^{\mathbf{a}} \cdot W$, where W is the polynomial ring $k[\theta_1, \ldots, \theta_d]$. Taking the degree zero part, we have $(A_{x^{\mathbf{e}}})_{\bar{\mathbf{0}}} = \bigoplus_{\mathbf{p} \in \mathbb{Z}^n} x^{\iota(\mathbf{p})} \cdot W$. Since $(A_{x^{\hat{\sigma}}})_{\bar{\mathbf{0}}}$ is invariant under the action of $(k^*)^d$, the decomposition of $(A_{x^{\mathbf{e}}})_{\bar{\mathbf{0}}}$ yields

$$(A_{x^{\hat{\sigma}}})_{\bar{\mathbf{0}}} = \bigoplus_{\mathbf{p} \in \mathbb{Z}^n} x^{\iota(\mathbf{p})} \cdot J(\mathbf{p}) \,,$$

where $J(\mathbf{p})$ is an ideal in W. To describe $J(\mathbf{p})$, recall that $A_{x^{\hat{\sigma}}}$ is the ring of differential operators on $S_{x^{\hat{\sigma}}}$. Thus, if $g \in W$, then $x^{\iota(\mathbf{p})} g$ belongs to $J(\mathbf{p})$ if and only if $(x^{\iota(\mathbf{p})} g) \bullet S_{x^{\hat{\sigma}}} \subseteq S_{x^{\hat{\sigma}}}$. Equivalently, for every $x^{\mathbf{a}}$ satisfying $\mathbf{a}_i \geq 0$ when $\mathbf{v}_i \in \sigma$, we have $g(\mathbf{a}) x^{\iota(\mathbf{p}) + \mathbf{a}} \in S_{x^{\hat{\sigma}}}$. We conclude that $J(\mathbf{p})$ is the ideal of polynomials vanishing on

$$Z(\mathbf{p}) = \left\{ \mathbf{a} \in \mathbb{Z}^d : \begin{array}{l} \mathbf{a}_i \geq 0 \text{ if } \mathbf{v}_i \in \sigma \text{ and} \\ \iota(\mathbf{p})_j + \mathbf{a}_j < 0 \text{ for some } \mathbf{v}_j \in \sigma \end{array} \right\} .$$

Analogously, the affine space $\mathbb{A}_k^n$ has as its associated Weyl algebra,

$$A' = \frac{k\{y_1, \ldots, y_n, \eth_1, \ldots, \eth_n\}}{\left(\begin{array}{rcl} y_i y_j - y_j y_i & = & 0 \\ \eth_i \eth_j - \eth_j \eth_i & = & 0 \\ \eth_i y_j - y_j \eth_i & = & \delta_{ij} \end{array} \right)}$$

and the torus $(k^*)^n$ acts on the Laurent ring $k[y_1^{\pm 1}, \ldots, y_n^{\pm 1}] = k[N^\vee]$ inducing an action on $A'_{y^{\boldsymbol{\varepsilon}}}$. The corresponding eigenspace decomposition is $A'_{y^{\boldsymbol{\varepsilon}}} = \bigoplus_{\mathbf{p} \in \mathbb{Z}^n} y^{\mathbf{p}} \cdot W'$, where W' is the polynomial ring $k[\vartheta_1, \ldots, \vartheta_n]$ and $\vartheta_i = y_i \eth_i$. The inclusion $R_\sigma \hookrightarrow A'_{y^{\boldsymbol{\varepsilon}}}$ identifies R_σ with

$$\left\{ f \in A'_{y^{\boldsymbol{\varepsilon}}} : f \bullet (k[\sigma^\vee \cap N^\vee]) \subseteq k[\sigma^\vee \cap N^\vee] \right\} .$$

Since this condition is torus invariant, R_σ is also torus invariant and we obtain $R_\sigma = \bigoplus_{\mathbf{p} \in \mathbb{Z}^n} y^{\mathbf{p}} \cdot I(\mathbf{p})$, where

$$I(\mathbf{p}) = \{ f \in W' : (y^{\mathbf{p}} f) \bullet (k[\sigma^\vee \cap N^\vee]) \subseteq k[\sigma^\vee \cap N^\vee] \} .$$

Identifying W' with the coordinate ring of $\mathbb{A}^n$, we have $f \bullet y^{\mathbf{q}} = f(\mathbf{q}) y^{\mathbf{q}}$ for every $\mathbf{q} \in N^\vee$. Hence, $I(\mathbf{p})$ is the ideal of polynomials vanishing on

$$Y(\mathbf{p}) = \{ \mathbf{q} \in \sigma^\vee \cap N^\vee : \mathbf{q} + \mathbf{p} \notin \sigma^\vee \cap N^\vee \} .$$

Finally, we define a map $\rho\colon W \longrightarrow W'$. The inclusion $\iota\colon \mathbb{Z}^n \to \mathbb{Z}^d$ induces, by tensoring with k, a linear embedding (denoted by the same name) $\iota : \mathbb{A}^n \to \mathbb{A}^d$ of the corresponding affine spaces. Identifying W and W' with the coordinate rings of $\mathbb{A}^d$ and $\mathbb{A}^n$ respectively, we obtain the ring homomorphism $\rho = \iota^*\colon W \to W'$. Clearly, ρ is surjective and $\operatorname{Ker}(\rho) = (\theta_{\bar{\mathbf{u}}} : \bar{\mathbf{u}} \in \operatorname{Cl}(X)^\vee)$.

With this notation, we have

Lemma 3.5. *The ring homomorphism $\psi_\sigma \circ \varphi_\sigma : (A_{x^\sigma})_{\bar{\mathbf{0}}} \longrightarrow R_\sigma$ is given by $x^{\iota(\mathbf{p})} \cdot g \mapsto y^{\mathbf{p}} \cdot \rho(g)$ where $\mathbf{p} \in \mathbb{Z}^n$ and $g \in J(\mathbf{p})$.*

Proof. It suffices to show that $(\psi_\sigma \circ \varphi_\sigma)(x^{\iota(\mathbf{p})}g)$ and $y^{\mathbf{p}}\rho(g)$ have the same action on $y^{\mathbf{q}} \in k[\sigma^\vee \cap N^\vee]$. On one hand, we have

$$\begin{aligned}
(\psi_\sigma \circ \varphi_\sigma)(x^{\iota(\mathbf{p})}g) \bullet y^{\mathbf{q}} &= \iota^{-1}\left(\varphi_\sigma(x^{\iota(\mathbf{p})}g \bullet \iota(y^{\mathbf{q}})\right) \\
&= \iota^{-1}\left(\varphi_\sigma(x^{\iota(\mathbf{p})}g \bullet x^{\iota(\mathbf{q})}\right) \\
&= \iota^{-1}\left(g\big(\iota(\mathbf{q})\big)x^{\iota(\mathbf{p}+\mathbf{q})}\right) \\
&= g\big(\iota(\mathbf{q})\big)y^{\mathbf{p}+\mathbf{q}}\,.
\end{aligned}$$

On the other hand, we also have

$$\big(y^{\mathbf{p}}\rho(g)\big) \bullet y^{\mathbf{q}} = \big(\rho(g)(\mathbf{q})y^{\mathbf{p}+\mathbf{q}}\big) = g\big(\iota(\mathbf{q})\big)y^{\mathbf{p}+\mathbf{q}}\,,$$

which establishes the claim. $\square$

Before returning our attention to $\bar{\varphi}_\sigma$, we need one more lemma.

Lemma 3.6. *If the elements θ_i and θ_j in $W = k[\theta_1, \dots, \theta_d]$ are distinct and correspond to rays $\mathbf{v}_i$ and $\mathbf{v}_j$ in the same cone σ, then, for every pair of integers m_i and m_j, the elements $\rho(\theta_i) + m_i$ and $\rho(\theta_j) + m_j$ are linearly independent over the field k.*

Proof. Suppose otherwise: for some $c \in k$, we have $\rho(\theta_i) + m_i = c \cdot (\rho(\theta_j) + m_j)$. It follows that $\rho(\theta_i) = c \cdot \rho(\theta_j)$ and hence

$$\sum_{\ell=1}^{n} \langle \boldsymbol{\varepsilon}_\ell, \mathbf{v}_i \rangle \vartheta_\ell = c \cdot \left(\sum_{\ell=1}^{n} \langle \boldsymbol{\varepsilon}_\ell, \mathbf{v}_j \rangle \vartheta_\ell \right)\,.$$

We deduce that $\langle \boldsymbol{\varepsilon}_\ell, \mathbf{v}_i \rangle = \langle \boldsymbol{\varepsilon}_\ell, c \cdot \mathbf{v}_j \rangle$ for all ℓ and thus $\mathbf{v}_i = c \cdot \mathbf{v}_j$. However, the $\mathbf{v}_i$ in any cone σ are linearly independent. $\square$

We are now in a position to understand $\bar{\varphi}_\sigma$. In particular, we obtain the following proposition which is a special case of Musson's results on rings of differential operators; see [M1].

Proposition 3.7 (Musson)**.** *For every $\sigma \in \Delta$, the map $\bar{\varphi}_\sigma$ [see equation (3.C)] is an isomorphism of rings.*

Proof. The fact that $\bar{\varphi}_\sigma$ is a ring homomorphism follows directly from the definition. Thus, the assertion reduces to showing that $\psi_\sigma \circ \varphi_\sigma$ is surjective and to describe its kernel. To achieve this, we determine the Zariski closures of $Y(\mathbf{p})$ and $Z(\mathbf{p})$. Since the Zariski closure of the set of integer points inside a rational polyhedral cone is the linear space spanned by that cone, it is easy to check that

$$\text{(3.D)} \qquad \begin{cases} \overline{Z(\mathbf{p})} & = \displaystyle\bigcup_{(i,m)\in\Lambda'} \big\{\mathbf{b} \in k^d : b_i = m\big\}, \\ \overline{Y(\mathbf{p})} & = \displaystyle\bigcup_{(i,m)\in\Lambda'} \big\{\mathbf{q} \in k^n : \iota(\mathbf{q})_i = m\big\} = \iota^{-1}\big(\overline{Z(\mathbf{p})}\big)\,, \end{cases}$$

where $\Lambda' = \{(i,m) : \mathbf{v}_i \in \sigma \text{ and } 0 \le m \le -\iota(\mathbf{p})_i\}$. We claim that $\rho(J(\mathbf{p})) = I(\mathbf{p})$. Indeed, equation (3.D) implies that $I(\mathbf{p}) = \sqrt{\rho(J(\mathbf{p}))}$, so it is enough to check that $\rho(J(\mathbf{p}))$ is a radical ideal. However, $J(\mathbf{p})$ is the principal ideal generated by

$$\text{(3.E)} \qquad h_{\mathbf{p}} := \prod_{(i,m)\in\Lambda'} (\theta_i - m)\,.$$

From the equation (3.E) and Lemma 3.6, we see that $\rho(J(\mathbf{p}))$ is reduced and hence $I(\mathbf{p}) = \rho(J(\mathbf{p}))$. Applying Lemma 3.5, it follows that $\psi_\sigma \circ \varphi_\sigma$ and $\bar{\varphi}_\sigma$ are surjective. To prove that $\psi_\sigma \circ \bar{\varphi}_\sigma$ is injective, recall that $\operatorname{Ker}(\rho) = (\theta_{\bar{\mathbf{u}}} : \bar{\mathbf{u}} \in \operatorname{Cl}(X)^\vee)$. Thus, it is enough to show that, for every $\mathbf{p} \in \mathbb{Z}^n$, we have $J(\mathbf{p}) \cap \operatorname{Ker}(\rho) = J(\mathbf{p}) \cdot \operatorname{Ker}(\rho)$. To see this, observe that $f \in J(\mathbf{p}) \cap \operatorname{Ker}(\rho)$ implies $f = h_{\mathbf{p}} f_1$. Therefore, it suffices to notice that $\rho(h_{\mathbf{p}}) \neq 0$, which follows from Lemma 3.6. $\square$

We now prove the main result in this section.

Proof of Theorem 3.4. We must show that, for every $\bar{\mathbf{b}} \in \operatorname{Cl}(X)$, the homomorphism of graded left A-modules

$$\eta_{(*,\bar{\mathbf{b}})} \colon D_L(\bar{\mathbf{b}}) \longrightarrow \bigoplus_{\bar{\mathbf{a}}\in\operatorname{Cl}(X)} H^0\big(X, \mathcal{O}(\bar{\mathbf{a}}) \otimes \mathcal{D} \otimes \mathcal{O}(\bar{\mathbf{b}})\big)$$

is an isomorphism.

The first step is to prove that $\eta_{(*,\bar{\mathbf{b}})}$ induces an isomorphism of the associated sheaves. Fix $\sigma \in \Delta$ and choose $\mathbf{b} \in \mathbb{Z}^d$, mapping to $\bar{\mathbf{b}}$ in $\operatorname{Cl}(X)$, such that $x^{\mathbf{b}}$ is an invertible element in $S_{x^{\hat{\sigma}}}$. Since the restriction of $\mathcal{O}(\bar{\mathbf{b}})$ to U_σ is trivial, one may always find such a $\mathbf{b}$. We then have a commutative diagram:

$$\begin{array}{ccc}
\frac{(A_{x^{\widehat{\sigma}}})_{\bar{\mathbf{0}}}}{(A_{x^{\widehat{\sigma}}})_{\bar{\mathbf{0}}}\cdot(\theta_{\bar{\mathbf{u}}}:\bar{\mathbf{u}}\in\mathrm{Cl}(X)^\vee)} & \xrightarrow{\bar{\varphi}_\sigma} & H^0(U_\sigma,\mathcal{D}) \\
{\scriptstyle \cdot x^{\mathbf{b}}}\Big\downarrow & & \Big\downarrow{\scriptstyle \otimes x^{\mathbf{b}}} \\
\frac{(A_{x^{\widehat{\sigma}}})_{\bar{\mathbf{b}}}}{(A_{x^{\widehat{\sigma}}})_{\bar{\mathbf{b}}}\cdot(\theta_{\bar{\mathbf{u}}}+\langle\bar{\mathbf{u}},\bar{\mathbf{b}}\rangle:\bar{\mathbf{u}}\in\mathrm{Cl}(X)^\vee)} & \xrightarrow{\bar{\varphi}'_\sigma} & H^0\big(U_\sigma,\mathcal{D}\otimes\mathcal{O}(\bar{\mathbf{b}})\big)
\end{array}$$

where $\bar{\varphi}_\sigma$ is the morphism in Proposition 3.7 and $\bar{\varphi}'_\sigma$ is the analogous morphism induced by $\eta_{(\bar{\mathbf{0}},\bar{\mathbf{b}})}$. Now, Proposition 3.7 implies that $\bar{\varphi}_\sigma$ is an isomorphism and the vertical arrows are clearly isomorphisms. It follows that $\bar{\varphi}'_\sigma$ is an isomorphism and therefore $\eta_{(*,\bar{\mathbf{b}})}$ induces an isomorphism of the associated sheaves.

If F is a graded S-module, we write

$$\Gamma_L(\widetilde{F}) = \bigoplus_{\bar{\mathbf{a}}\in\mathrm{Cl}(X)} H^0\big(X,\mathcal{O}(\bar{\mathbf{a}})\otimes\widetilde{F}\big)\,.$$

For every such F, there is an exact sequence

$$0 \longrightarrow H^0_{\mathfrak{b}}(F) \longrightarrow F \longrightarrow \Gamma_L(\widetilde{F}) \longrightarrow H^1_{\mathfrak{b}}(F) \longrightarrow 0\,, \tag{3.F}$$

where $\mathfrak{b}$ is the irrelevant ideal; see [EMS]. Hence, if $H^0_{\mathfrak{b}}\big(D_L(\bar{\mathbf{b}})\big)$ and $H^1_{\mathfrak{b}}\big(D_L(\bar{\mathbf{b}})\big)$ both vanish, then $\eta_{(*,\bar{\mathbf{b}})}$ is an isomorphism. We relegated these vanishing results to Propositions 3.8 and 3.9 below. $\square$

Our first vanishing result is

Proposition 3.8. *The element $x^{\mathbf{e}}\in\mathfrak{b}$ is not a zero divisor on $D_L(\bar{\mathbf{b}})$ and $H^0_{\mathfrak{b}}(D_L(\bar{\mathbf{b}}))=0$.*

Proof. The first assertion implies the second, so it suffices to show that $x^{\mathbf{e}}$ is not a zero divisor. Every $\mathbf{a}\in\mathbb{Z}^d$ can be written uniquely as $\mathbf{a}=\mathbf{a}^+-\mathbf{a}^-$ where $\mathbf{a}^+$ and $\mathbf{a}^-$ are non-negative and have disjoint support. Consider the action of the torus $(k^*)^d$ on the Weyl algebra A; the corresponding eigenspace decomposition is $A=\bigoplus_{\mathbf{a}\in\mathbb{Z}^d} x^{\mathbf{a}^+}\partial^{\mathbf{a}^-}\cdot W$, where $W=k[\theta_1,\dots,\theta_d]$. Let $L_0\subset W$ be the ideal generated by $\theta_{\bar{\mathbf{u}}}+\langle\bar{\mathbf{u}},\bar{\mathbf{b}}\rangle$ for all $\bar{\mathbf{u}}\in\mathrm{Cl}(X)^\vee$. Since L_0 is generated by linear forms, it is a prime ideal. Let L denote the left A-ideal $(\theta_{\bar{\mathbf{u}}}+\langle\bar{\mathbf{u}},\bar{\mathbf{b}}\rangle:\bar{\mathbf{u}}\in\mathrm{Cl}(X)^\vee)$. With this notation, we have $L=\bigoplus_{\mathbf{a}\in\mathbb{Z}^d} x^{\mathbf{a}^+}\partial^{\mathbf{a}^-}\cdot L_0$.

For $x^{\mathbf{e}}$ to be a non-zero divisor on $D_L(\bar{\mathbf{b}})$, it suffices to prove that, for $g\in W$ and $\mathbf{a}\in\mathbb{Z}^d$, the relation $x^{\mathbf{e}}x^{\mathbf{a}^+}\partial^{\mathbf{a}^-}g\in L$ implies $g\in L_0$. To accomplish this, we first note

$$\begin{aligned} x^{\mathbf{e}} \cdot x^{\mathbf{a}^+}\partial^{\mathbf{a}^-} &= \left(\prod_{\mathbf{a}_i \geq 0} x_i^{\mathbf{a}_i+1}\right)\left(\prod_{\mathbf{a}_i<0} x_i\partial^{-\mathbf{a}_i}\right) \\ &= \left(\prod_{\mathbf{a}_i \geq 0} x_i^{\mathbf{a}_i+1}\right)\left(\prod_{\mathbf{a}_i<0} \partial^{-\mathbf{a}_i-1}(\theta_i+\mathbf{a}_i+1)\right) \\ &= x^{(\mathbf{a}+\mathbf{e})^+}\partial^{(\mathbf{a}+\mathbf{e})^-}\prod_{\mathbf{a}_i<0}(\theta_i+\mathbf{a}_i+1)\,. \end{aligned}$$

Hence, $x^{\mathbf{e}} \cdot x^{\mathbf{a}^+}\partial^{\mathbf{a}^-} g \in L$ implies $\left(\prod_{\mathbf{a}_i<0}(\theta_i+\mathbf{a}_i+1)\right)\cdot g \in L_0$. Suppose $g \notin L_0$. Since L_0 is a prime ideal, there exists an index i such that $\mathbf{a}_i < 0$ and $\theta_i + \mathbf{a}_i + 1 \in L_0$. Expressing $\theta_i + \mathbf{a}_i + 1$ in terms of the linear generators of L_0, it follows that there is $\bar{\mathbf{u}} \in \mathrm{Cl}(X)^\vee$ such that $\theta_i = \theta_{\bar{\mathbf{u}}}$. However, for every $\mathbf{w} \in N^\vee$, equation (2.A) implies

$$\big\langle \bar{\mathbf{u}}, \langle \mathbf{w}, \mathbf{v}_1\rangle \bar{\mathbf{e}}_1 + \cdots + \langle \mathbf{w}, \mathbf{v}_d\rangle \bar{\mathbf{e}}_d \big\rangle = 0\,,$$

from which we deduce that $\langle \mathbf{w}, \mathbf{v}_i\rangle = 0$ and $\mathbf{v}_i = 0$ giving a contradiction. □

We end this section with

Proposition 3.9. *The local cohomology module $H^1_{\mathfrak{b}}(D_L(\bar{\mathbf{b}}))$ vanishes.*

Proof. Since $x^{\mathbf{e}}$ is not a zero divisor on $D_L(\bar{\mathbf{b}})$, there is a short exact sequence of S-modules

$$0 \longrightarrow D_L(\bar{\mathbf{b}}) \xrightarrow{x^{\mathbf{e}}\cdot} D_L(\bar{\mathbf{b}}) \longrightarrow Q = \frac{D_L(\bar{\mathbf{b}})}{x^{\mathbf{e}} \cdot D_L(\bar{\mathbf{b}})} \longrightarrow 0\,,$$

and the long exact sequence of local cohomology gives

$$0 \longrightarrow H^0_{\mathfrak{b}}(Q) \longrightarrow H^1_{\mathfrak{b}}(D_L(\bar{\mathbf{b}})) \xrightarrow{x^{\mathbf{e}}\cdot} H^1_{\mathfrak{b}}(D_L(\bar{\mathbf{b}}))\,. \tag{3.G}$$

Because every element in $H^1_{\mathfrak{b}}\big(D_L(\bar{\mathbf{b}})\big)$ is annihilated by a power of $\mathfrak{b}$, the injectivity of $x^{\mathbf{e}}\cdot$ in equation (3.G) implies that $H^1_{\mathfrak{b}}\big(D_L(\bar{\mathbf{b}})\big) = 0$. Thus, it suffices to prove that $H^0_{\mathfrak{b}}(Q) = 0$

Let K be the left A-ideal satisfying $Q = A(\bar{\mathbf{b}})/K$. To prove that $H^0_{\mathfrak{b}}(Q) = 0$, we must show that if $f \in A$ satisfies $(x^{\widehat{\sigma}})^m f \in K$ for some $m \geq 1$ and all $\sigma \in \Delta$, then $f \in K$. Using the notation from the proof of Proposition 3.8, we have $K = L + x^{\mathbf{e}} \cdot A$. From the decomposition of A, we have

$$\begin{aligned} x^{\mathbf{e}} \cdot A &= \bigoplus_{\mathbf{a} \in \mathbb{Z}^d} x^{\mathbf{e}} \cdot x^{\mathbf{a}^+} \partial^{\mathbf{a}^-} \cdot W \\ &= \bigoplus_{\mathbf{a} \in \mathbb{Z}^d} \left(x^{(\mathbf{a}+\mathbf{e})^+} \partial^{(\mathbf{a}+\mathbf{e})^-} \left(\prod_{\mathbf{a}_i < 0} (\theta_i + \mathbf{a}_i + 1) \right) \cdot W \right), \end{aligned}$$

and we deduce $K = \bigoplus_{\mathbf{a} \in \mathbb{Z}^d} x^{\mathbf{a}^+} \partial^{\mathbf{a}^-} \cdot K(\mathbf{a})$, where $K(\mathbf{a})$ is defined to be $L_0 + \big(\prod_{\mathbf{a}_i \leq 0} (\theta_i + \mathbf{a}_i) \big) \cdot W$. Thus, it is enough to consider elements $f \in A$ of the form $x^{\mathbf{a}^+} \partial^{\mathbf{a}^-} g$ with $g \in W$ and prove that $g \in K(\mathbf{a})$. Moreover, we may assume that $m + \mathbf{a}_i > 0$ for $1 \leq i \leq d$.

By induction on r, we see that $x_i^r \partial_i^r = \prod_{j=1}^{r} (\theta_i - j + 1)$ for $r \geq 1$. Hence, for $\mathbf{a}_i < 0$, we have

$$x_i^m \partial^{-\mathbf{a}_i} = x_i^{m+\mathbf{a}_i} x_i^{-\mathbf{a}_i} \partial_i^{-\mathbf{a}_i} = x_i^{m+\mathbf{a}_i} \cdot \prod_{j=1}^{-\mathbf{a}_i} (\theta_i - j + 1)$$

and, for $\sigma \in \Delta$, we obtain

$$(x^{\widehat{\sigma}})^m x^{\mathbf{a}^+} \partial^{\mathbf{a}^-} = \left(x^{(\mathbf{a}+m\widehat{\sigma})^+} \partial^{(\mathbf{a}+m\widehat{\sigma})^-} \right) \cdot \prod_{i \in \widehat{\Lambda}} \prod_{j=1}^{-\mathbf{a}_i} (\theta_i - j + 1),$$

where $\widehat{\Lambda} = \{ i : \mathbf{v}_i \notin \sigma \text{ and } \mathbf{a}_i < 0 \}$. We deduce

$$\left(\prod_{i \in \widehat{\Lambda}} \prod_{j=1}^{-\mathbf{a}_i} (\theta_i - j + 1) \right) \cdot g \in L_0 + \left(\prod_{i \in \Lambda} (\theta_i + \mathbf{a}_i) \right) \cdot W, \tag{3.H}$$

where $\Lambda = \{ i : \mathbf{v}_i \in \sigma \text{ and } \mathbf{a}_i \leq 0 \}$.

For each $\mathbf{b} \in \mathbb{Z}^d$, we define an automorphism $\alpha_{\mathbf{b}} \colon W \longrightarrow W$ given by $\alpha_{\mathbf{b}}(\theta_i) = \theta_i - \mathbf{b}_i$ and we define $\rho_{\mathbf{b}} \colon W \longrightarrow W'$ to be the composition $\rho_{\mathbf{b}} = \rho \circ \alpha_{\mathbf{b}}$ (the map ρ is defined in the paragraph before Lemma 3.5). It is clear that $\rho_{\mathbf{b}}$ is surjective and $\operatorname{Ker}(\rho_{\mathbf{b}}) = L_0$. Applying $\rho_{\mathbf{b}}$ to equation (3.H) gives

$$\left(\prod_{i \in \widehat{\Lambda}} \prod_{j=1}^{-\mathbf{a}_i} \rho_{\mathbf{b}}(\theta_i - j + 1) \right) \cdot \rho_{\mathbf{b}}(g) \in \left(\prod_{i \in \Lambda} \rho_{\mathbf{b}}(\theta_i + \mathbf{a}_i) \right) \cdot W'.$$

Lemma 3.6 obviously extends to $\rho_{\mathbf{b}}$ and implies

$$\rho_{\mathbf{b}}(g) \in \left(\prod_{i \in \Lambda} \rho_{\mathbf{b}}(\theta_i + \mathbf{a}_i) \right) \cdot W'.$$

Since this relation holds for every $\sigma \in \Delta$, a second application of Lemma 3.6 shows that $\rho_{\mathbf{b}}(g) \in \left(\prod_{\mathbf{a}_i \leq 0} \rho_{\mathbf{b}}(\theta_i + \mathbf{a}_i)\right) \cdot W'$ and therefore $g \in K(\mathbf{a}) = L_0 + \left(\prod_{a_i \leq 0}(\theta_i + \mathbf{a}_i)\right) \cdot W$. □

4. $\mathcal{D}$-modules

We now use Theorem 3.7 to describe the relation between A-modules and $\mathcal{D}$-modules on X. We begin by showing that the S-module associated to a $\mathcal{D}$-module has a graded A-module structure.

Proposition 4.1. *If $\mathcal{F}$ is a left $\mathcal{D}$-module, then the graded S-module*

$$\Gamma_L(\mathcal{F}) = \bigoplus_{\bar{\mathbf{a}} \in \mathrm{Cl}(X)} H^0\big(X, \mathcal{O}(\bar{\mathbf{a}}) \otimes \mathcal{F}\big)\,,$$

has a graded left A-module structure, extending the left S-module structure. Similarly, if $\mathcal{G}$ is a right $\mathcal{D}$-module, then the graded S-module

$$\Gamma_R(\mathcal{G}) = \bigoplus_{\bar{\mathbf{b}} \in \mathrm{Cl}(X)} H^0\big(X, \mathcal{G} \otimes \mathcal{O}(\bar{\mathbf{b}})\big)\,,$$

has a graded right A-module structure, extending the right S-module structure.

Proof. We present the left $\mathcal{D}$-modules case here — the proof for right $\mathcal{D}$-modules is completely analogous. For the first assertion, it is enough as well to construct k-linear maps

$$\mu^{\mathcal{F}}_{\bar{\mathbf{a}}',\bar{\mathbf{a}}} \colon A_{\bar{\mathbf{a}}'} \otimes_k H^0\big(X, \mathcal{O}(\bar{\mathbf{a}}) \otimes \mathcal{F}\big) \longrightarrow H^0\big(X, \mathcal{O}(\bar{\mathbf{a}} + \bar{\mathbf{a}}') \otimes \mathcal{F}\big)\,,$$

for all $\bar{\mathbf{a}}, \bar{\mathbf{a}}' \in \mathrm{Cl}(X)$, satisfying the obvious axioms. To accomplish this, we consider a local version of the left multiplication map (3.A) when $\bar{\mathbf{b}} = 0$. More explicitly, for each $\sigma \in \Delta$, this morphism is

$$\mu_{\bar{\mathbf{a}}',\bar{\mathbf{a}}}|_{U_\sigma} \colon A_{\bar{\mathbf{a}}'} \otimes_k H^0\big(U_\sigma, \mathcal{O}(\bar{\mathbf{a}}) \otimes \mathcal{D}\big) \longrightarrow H^0\big(U_\sigma, \mathcal{O}(\bar{\mathbf{a}} + \bar{\mathbf{a}}') \otimes \mathcal{D}\big)\,,$$

given by $\mu_{\bar{\mathbf{a}}',\bar{\mathbf{a}}}|_{U_\sigma}(f \otimes s) = f \circ s$. We claim that $\mu_{\bar{\mathbf{a}}',\bar{\mathbf{a}}}|_{U_\sigma}$ is a morphism of right $H^0(U_\sigma, \mathcal{D})$-modules. To see this, recall that sections $s \in H^0(U_\sigma, \mathcal{O}(\bar{\mathbf{b}}) \otimes \mathcal{D})$ and $g \in H^0(U_\sigma, \mathcal{D})$ can be identified with elements in $\mathrm{Hom}_k\big((S_{x^{\hat\sigma}})_0, (S_{x^{\hat\sigma}})_{\bar{\mathbf{b}}}\big)$ and $\mathrm{Hom}_k\big((S_{x^{\hat\sigma}})_0, (S_{x^{\hat\sigma}})_0\big)$ respectively. In particular, we have

$$\big(\mu_{\bar{\mathbf{a}}',\bar{\mathbf{a}}}|_{U_\sigma}(f \otimes s)\big) \cdot g = f \circ s \circ g = \mu_{\bar{\mathbf{a}}',\bar{\mathbf{a}}}|_{U_\sigma}(f \otimes s \cdot g)\,,$$

for all $f \in A_{\bar{\mathbf{a}}'}$. It follows that, by tensoring the map $\mu_{\bar{\mathbf{a}}',\bar{\mathbf{a}}}|_{U_\sigma}$ on the right with $H^0(U_\sigma, \mathcal{F})$ over $H^0(U_\sigma, \mathcal{D})$, we obtain a k-linear map

$$\mu^{\mathcal{F}}_{\bar{\mathbf{a}}',\bar{\mathbf{a}}}|_{U_\sigma} \colon A_{\bar{\mathbf{a}}'} \otimes_k H^0\big(U_\sigma, \mathcal{O}(\bar{\mathbf{a}}) \otimes \mathcal{F}\big) \longrightarrow H^0\big(U_\sigma, \mathcal{O}(\bar{\mathbf{a}}' + \bar{\mathbf{a}}) \otimes \mathcal{F}\big)\,.$$

These maps glue together to give $\mu^{\mathcal{F}}_{\bar{\mathbf{a}}',\bar{\mathbf{a}}}$ which makes F into a graded left A-module. $\square$

Let $A\text{-}\mathsf{GrMod}_\theta$ be the category of graded left A-modules F such that

$$(\theta_{\bar{\mathbf{u}}} - \langle \bar{\mathbf{u}}, \bar{\mathbf{a}} \rangle) \cdot F_{\bar{\mathbf{a}}} = 0 \text{ for all } \bar{\mathbf{a}} \in \mathrm{Cl}(X) \text{ and all } \bar{\mathbf{u}} \in \mathrm{Cl}(X)^\vee. \tag{4.A}$$

A graded A-module F is called $\mathfrak{b}$-torsion if, for every $f \in F$, there exists $\ell > 0$ such that $\mathfrak{b}^\ell f = 0$. Let $\mathfrak{b}\text{-}\mathsf{Tors}$ denote the full subcategory of $\mathfrak{b}$-torsion modules. Similarly, $\mathsf{GrMod}_\theta\text{-}A$ is the category of graded right A-modules G satisfying $G_{\bar{\mathbf{b}}} \cdot (\theta_{\bar{\mathbf{u}}} + \langle \bar{\mathbf{u}}, \bar{\mathbf{b}} \rangle) = 0$ for all $\bar{\mathbf{b}} \in \mathrm{Cl}(X)$ and all $\bar{\mathbf{u}} \in \mathrm{Cl}(X)^\vee$. Let $\mathsf{Tors}\text{-}\mathfrak{b}$ denote the full subcategory of $\mathfrak{b}$-torsion modules in $\mathsf{GrMod}_\theta\text{-}A$. It is clear that $A\text{-}\mathsf{GrMod}_\theta$ and $\mathsf{GrMod}_\theta\text{-}A$ are both abelian categories closed under taking graded subquotients. The main result in this section is:

Theorem 4.2. *The map $F \mapsto \widetilde{F}$ is an exact functor from $A\text{-}\mathsf{GrMod}_\theta$ to $\mathcal{D}\text{-}\mathsf{Mod}$, $\mathcal{F} \mapsto \Gamma_L(\mathcal{F})$ is a left exact functor from $\mathcal{D}\text{-}\mathsf{Mod}$ to $A\text{-}\mathsf{GrMod}_\theta$ and there are natural transformations of functors*

$$\mathrm{id}_{A\text{-}\mathsf{GrMod}_\theta/\mathfrak{b}\text{-}\mathsf{Tors}} \xrightarrow{\cong} \Gamma_L \circ \widetilde{} \quad \textit{and} \quad \widetilde{} \circ \Gamma_L \xrightarrow{\cong} \mathrm{id}_{\mathcal{D}\text{-}\mathsf{Mod}}\,.$$

Similarly, the map $G \mapsto \widetilde{G}$ is an exact functor from $\mathsf{GrMod}_\theta\text{-}A$ to $\mathsf{Mod}\text{-}\mathcal{D}$, $\mathcal{G} \mapsto \Gamma_R(\mathcal{G})$ is a left exact functor from $\mathsf{Mod}\text{-}\mathcal{D}$ to $\mathsf{GrMod}_\theta\text{-}A$ and there are natural transformations of functors

$$\mathrm{id}_{\mathsf{GrMod}_\theta\text{-}A/\mathsf{Tors}\text{-}\mathfrak{b}} \xrightarrow{\cong} \Gamma_R \circ \widetilde{} \quad \textit{and} \quad \widetilde{} \circ \Gamma_R \xrightarrow{\cong} \mathrm{id}_{\mathsf{Mod}\text{-}\mathcal{D}}\,.$$

In particular, every left $\mathcal{D}$-module is of the form $\widetilde{F}$ for some graded left A-module F and every right $\mathcal{D}$-module is of the form $\widetilde{G}$ for some graded right A-module G.

Proof of Theorem 4.2. Again, we give the proof only for left modules. For the first part, we consider an object F in $A\text{-}\mathsf{GrMod}_\theta$. By definition, we have $H^0(U_\sigma, \widetilde{F}) = (F_{x^{\widehat{\sigma}}})_{\bar{\mathbf{0}}}$, where $(F_{x^{\widehat{\sigma}}})_{\bar{\mathbf{0}}}$ is a left $(A_{x^{\widehat{\sigma}}})_{\bar{\mathbf{0}}}$-module and $\sigma \in \Delta$. In light of Theorem 3.7, we must show that, for every $\bar{\mathbf{u}} \in \mathrm{Cl}(X)^\vee$, we have $\theta_{\bar{\mathbf{u}}} \cdot (F_{x^{\widehat{\sigma}}})_{\bar{\mathbf{0}}} = 0$. Now, if $\frac{f}{(x^{\widehat{\sigma}})^m} \in (F_{x^{\widehat{\sigma}}})_{\bar{\mathbf{0}}}$, then $f \in F_{m\bar{\mathbf{e}}_\sigma}$ where $\mathbf{e}_\sigma = \sum_{i \in \widehat{\sigma}} \mathbf{e}_i$ and the hypotheses on F imply

$$\begin{aligned} \theta_{\bar{\mathbf{u}}} \cdot \tfrac{f}{(x^{\widehat{\sigma}})^m} &= \left(\theta_{\bar{\mathbf{u}}} \tfrac{1}{(x^{\widehat{\sigma}})^m}\right) \cdot f \\ &= \tfrac{1}{(x^{\widehat{\sigma}})^m} \theta_{\bar{\mathbf{u}}} \cdot f - \sum_{i \in \widehat{\sigma}} m \langle \bar{\mathbf{u}}, \bar{\mathbf{e}}_i \rangle \tfrac{1}{(x^{\widehat{\sigma}})^m} \cdot f \\ &= \tfrac{1}{(x^{\widehat{\sigma}})^m} \big(\theta_{\bar{\mathbf{u}}} - \langle \bar{\mathbf{u}}, m\bar{\mathbf{e}}_\sigma \rangle\big) \cdot f = 0\,. \end{aligned}$$

Therefore $H^0(U_\sigma, \widetilde{F})$ has a structure of left module over $H^0(U_\sigma, \mathcal{D})$ for every $\sigma \in \Delta$. It is straightforward to verify that these structures glue together to give a $\mathcal{D}$-module structure on $\widetilde{F}$. By construction, we see that $\widetilde{F}$ is quasi-coherent sheaf over $\mathcal{D}$.

Conversely, let $\mathcal{F}$ be an object of $\mathcal{D}$-Mod. Applying Proposition 4.1, we know that $F = \Gamma_L(\mathcal{F})$ is a graded left A-module, so it is enough to prove that F satisfies (4.A). Fixing $\bar{\mathbf{u}} \in \mathrm{Cl}(X)^\vee$ and $\bar{\mathbf{a}} \in \mathrm{Cl}(X)$, it suffices to show that

$$\mu^{\mathcal{F}}_{\bar{\mathbf{0}},\bar{\mathbf{a}}}|_{U_\sigma}\big((\theta_{\bar{\mathbf{u}}} - \langle \bar{\mathbf{u}}, \bar{\mathbf{a}}\rangle) \otimes s'\big) = 0\,,$$

for every $\sigma \in \Delta$ and all sections $s' \in H^0\big(U_\sigma, \mathcal{O}(\bar{\mathbf{a}}) \otimes \mathcal{F}\big)$. As explained in Proposition 4.1, we have

$$\begin{aligned} H^0\big(U_\sigma, \mathcal{O}(\bar{\mathbf{a}}) \otimes \mathcal{F}\big) &= H^0\big(U_\sigma, \mathcal{O}(\bar{\mathbf{a}}) \otimes \mathcal{D} \otimes_{\mathcal{D}} \mathcal{F}\big) \\ &= H^0\big(U_\sigma, \mathcal{O}(\bar{\mathbf{a}}) \otimes \mathcal{D}\big) \otimes_{H^0(U_\sigma, \mathcal{D})} H^0(U_\sigma, \mathcal{F})\,, \end{aligned}$$

so that we may identify s' with a linear combination of elements of the form $s \otimes f$. By definition, we have

$$\mu^{\mathcal{F}}_{\bar{\mathbf{0}},\bar{\mathbf{a}}}|_{U_\sigma}\big((\theta_{\bar{\mathbf{u}}} - \langle \bar{\mathbf{u}}, \bar{\mathbf{a}}\rangle) \otimes s \otimes f\big) = (\theta_{\bar{\mathbf{u}}} - \langle \bar{\mathbf{u}}, \bar{\mathbf{a}}\rangle) \circ s \otimes f\,,$$

and we claim that $(\theta_{\bar{\mathbf{u}}} - \langle \bar{\mathbf{u}}, \bar{\mathbf{a}}\rangle) \circ s$ is zero. Indeed, for every $x^{\mathbf{c}} \in (S_{x^{\hat{\sigma}}})_{\bar{\mathbf{a}}}$, we have $\theta_{\bar{\mathbf{u}}}(x^{\mathbf{c}}) = \langle \bar{\mathbf{u}}, \bar{\mathbf{e}}_i\rangle c_1 x^{\mathbf{c}} + \cdots + \langle \bar{\mathbf{u}}, \bar{\mathbf{e}}_i\rangle c_d x^{\mathbf{c}} = \langle \bar{\mathbf{u}}, \bar{\mathbf{a}}\rangle x^{\mathbf{c}}$. Therefore, if $g \in H^0(U_\sigma, \mathcal{O}) = (S_{x^{\hat{\sigma}}})_0$, then $s(g) \in (S_{x^{\hat{\sigma}}})_{\bar{\mathbf{a}}}$ and

$$(\theta_{\bar{\mathbf{u}}} - \langle \bar{\mathbf{u}}, \bar{\mathbf{a}}\rangle)(s(g)) = \langle \bar{\mathbf{u}}, \bar{\mathbf{a}}\rangle s(g) - \langle \bar{\mathbf{u}}, \bar{\mathbf{a}}\rangle s(g) = 0\,.$$

Finally, the exact sequence (3.F) provides the first natural transformation, once we observe that $H^i_{\mathfrak{b}}(F)$ is $\mathfrak{b}$-torsion. It follows from Cox [C] that the sheaf associated to $\Gamma_L(\mathcal{F})$ is isomorphic to $\mathcal{F}$. □

As a corollary, we obtain

Proof of Theorem 1.1. Follows immediately from Theorem 4.2. □

We next turn our attention to coherent $\mathcal{D}$-modules and finitely generated graded A-modules. We write A-GrMod^f_θ and GrMod^f_θ-A for the full subcategories of A-GrMod_θ and GrMod_θ-A consisting of finitely generated A-modules.

Proposition 4.3. *If F is an object in A-GrMod^f_θ then $\widetilde{F}$ is a coherent left $\mathcal{D}$-module. Moreover, every coherent left $\mathcal{D}$-module is of the form $\widetilde{F}$ for some $F \in A$-GrMod^f_θ. Similarly, $G \in \mathsf{GrMod}^f_\theta$-$A$ implies $\widetilde{G}$ belongs to Coh-$\mathcal{D}$ and every coherent right $\mathcal{D}$-module is isomorphic to $\widetilde{G}$ for some $G \in \mathsf{GrMod}^f_\theta$-$A$.*

Our proof is analogous to the $\mathcal{O}$-module case found in Cox [C].

Proof. Once again, we present the proof only for left modules. Suppose that F belongs to $A\text{-}\mathsf{GrMod}^f_\theta$. To establish that $\widetilde{F} \in \mathcal{D}\text{-}\mathsf{Coh}$, we have to check that, for every $\sigma \in \Delta$, $(F_{x^{\hat{\sigma}}})_{\bar{\mathbf{0}}}$ is finitely generated over $(A_{x^{\hat{\sigma}}})_{\bar{\mathbf{0}}}$. Thus, it suffices to show that, for every element $f \in F$, there is an invertible element $g \in S_{x^{\hat{\sigma}}}$ such that $g \cdot f$ has degree zero. Consider f in $F_{\bar{\mathbf{a}}}$. Since X is smooth, there is a divisor corresponding to $\mathbf{a} \in \mathbb{Z}^d$ supported outside σ and whose class is $\bar{\mathbf{a}}$. It follows that $x^{-\mathbf{a}}$ is an invertible element in $S_{x^{\hat{\sigma}}}$ and $x^{-\mathbf{a}} \cdot f$ has degree zero.

For the second assertion, we must show that given a coherent left $\mathcal{D}$-module $\mathcal{F}$ there exists a finitely generated A-submodule F of $\Gamma_L(\mathcal{F})$ such that $\widetilde{F} \cong \mathcal{F}$. Since $\mathcal{F}$ is coherent, $H^0(U_\sigma, \mathcal{F})$ is finitely generated over $H^0(U_\sigma, \mathcal{D})$, for every $\sigma \in \Delta$. Choose, for each $\sigma \in \Delta$, a finite set of homogeneous elements in $\Gamma_L(\mathcal{F})$ which are the numerators for a corresponding set of generators of $H^0(U_\sigma, \mathcal{F})$. Setting F to be the A-submodule of $\Gamma_L(\mathcal{F})$ generated by the union of these sets, we have $\widetilde{F} \cong \mathcal{F}$ and F is finitely generated over A. □

Remark 4.4. As a consequence of Theorem 4.2, we have $\widetilde{F} = 0$ if and only if the graded S-module F satisfies $F = H^0_{\mathfrak{b}}(F)$; this is equivalent to saying that F is a $\mathfrak{b}$-torsion module. At the other extreme, F has no $\mathfrak{b}$-torsion when $H^0_{\mathfrak{b}}(F) = 0$ and we say that F is $\mathfrak{b}$-saturated when $H^0_{\mathfrak{b}}(F) = H^1_{\mathfrak{b}}(F) = 0$. Now, every left $\mathcal{D}$-module $\mathcal{F}$ can be represented by a unique saturated A-module, namely $\Gamma_L(\mathcal{F})$. Unfortunately, this may not be finitely generated, even if $\mathcal{F}$ is coherent. However, by replacing F with a suitable submodule of $F/H^0_{\mathfrak{b}}(F)$, we may assume that F has no $\mathfrak{b}$-torsion and is finitely generated, whenever $\mathcal{F}$ is coherent.

Example 4.5. The A-module corresponding to the structure sheaf $\mathcal{O}$ (which is a left $\mathcal{D}$-module) is $\Gamma_L(\mathcal{O}) = \frac{A}{A\cdot(\partial_1,\dots,\partial_d)}$. In particular, $\Gamma_L(\mathcal{O})$ is isomorphic to S, where S has the standard A-module structure.

Corollary 4.6. *For every $\bar{\mathbf{b}} \in \mathrm{Cl}(X)$, there is an isomorphism of graded left A-modules $\Gamma_L\big(\mathcal{D} \otimes \mathcal{O}(\bar{\mathbf{b}})\big) \cong D_L(\bar{\mathbf{b}})$. Similarly, we have $\Gamma_R\big(\mathcal{O}(\bar{\mathbf{a}}) \otimes \mathcal{D}\big) \cong D_R(\bar{\mathbf{a}})$.*

Proof. Theorem 3.4 provides $\eta_{(*,\bar{\mathbf{0}})} \colon D_L(\bar{\mathbf{b}}) \xrightarrow{\cong} \Gamma_L\big(\mathcal{D} \otimes \mathcal{O}(\bar{\mathbf{b}})\big)$ and $\eta_{(\bar{\mathbf{0}},*)} \colon D_R(\bar{\mathbf{a}}) \xrightarrow{\cong} \Gamma_R\big(\mathcal{O}(\bar{\mathbf{a}}) \otimes \mathcal{D}\big)$. □

Corollary 4.7. *For $\bar{\mathbf{a}}, \bar{\mathbf{b}} \in \mathrm{Cl}(X)$, we have $D_L(\bar{\mathbf{a}}) \in A\text{-}\mathsf{GrMod}_\theta$ and $D_R(\bar{\mathbf{b}}) \in \mathsf{GrMod}_\theta\text{-}A$.*

Proof. This follows from Corollary 4.6 and Theorem 4.2. □

Corollary 4.8. *Let F be a graded left A-module generated by homogeneous elements $\{f_i\}_i$ with $\deg f_i = \bar{\mathbf{b}}_i$. For F to belong to $A\text{-}\mathsf{GrMod}_\theta$,* *it is necessary and sufficient that* $(\theta_{\bar{\mathbf{u}}} - \langle \bar{\mathbf{u}}, \bar{\mathbf{b}}_i \rangle) \cdot f_i = 0$ *for all* i *and all* $\bar{\mathbf{u}} \in \mathrm{Cl}(X)^\vee$. *A similar assertion holds for graded right* A*-modules.*

Proof. This condition is clearly necessary. To see the other direction, consider the surjective graded morphism defined by the given generators: $\bigoplus_i A(-\bar{\mathbf{b}}_i) \longrightarrow F$. By hypothesis, this factors to an epimorphism $\bigoplus_i D_L(-\bar{\mathbf{b}}_i) \longrightarrow F$ and Corollary 4.7 implies that $F \in A\text{-}\mathsf{GrMod}_\theta$. □

Corollary 4.9. *For each* $\mathcal{F} \in \mathcal{D}\text{-}\mathsf{Mod}$, *there exist a family* $\{\bar{\mathbf{b}}_i\}_i$ *of elements in* $\mathrm{Cl}(X)$ *and an epimorphism* $\bigoplus_i \mathcal{D} \otimes \mathcal{O}(-\bar{\mathbf{b}}_i) \longrightarrow \mathcal{F}$. *The analogous result also holds for* $\mathcal{G} \in \mathsf{Mod}\text{-}\mathcal{D}$.

Proof. Theorem 4.2 implies that there exists $F \in A\text{-}\mathsf{GrMod}_\theta$ such that $\mathcal{F} \cong \widetilde{F}$. Now, Corollary 4.8 gives an epimorphism $\bigoplus_i D_L(-\bar{\mathbf{b}}_i) \longrightarrow F$. Taking the corresponding morphism of sheaves and applying Corollary 4.6 establishes the claim. □

Remark 4.10. Applying Corollary 4.9, one can construct a resolution of a $\mathcal{D}$-module using twisted modules $\mathcal{D} \otimes \mathcal{O}(\bar{\mathbf{b}}_i)$. More concretely, given a graded module F satisfying $\widetilde{F} \cong \mathcal{F}$, Gröbner basis techniques can be used to construct a resolution of F by modules $D_L(\bar{\mathbf{b}})$ which will then lift to a resolution of $\mathcal{F}$. It would be interesting to investigate the relationship between these two types of resolutions.

The last part of this section is devoted to the categorical equivalence between right and left $\mathcal{D}$-modules on the toric variety X. Recall that, for a smooth variety X of dimension n, the sheaf of differential forms of top degree Ω^n has a natural structure of right $\mathcal{D}$-module extending the usual $\mathcal{O}$-module structure. Locally, right multiplication of an n-form ω with a vector field ν is defined by $\omega \cdot \nu = -\operatorname{Lie}_\nu(\omega)$, where $\operatorname{Lie}_\nu(\omega)$ is the Lie derivative of ω along ν (see [B+] Chapter VI.3.4 for details).

The equivalence of categories $\tau_{LR} \colon \mathcal{D}\text{-}\mathsf{Mod} \longrightarrow \mathsf{Mod}\text{-}\mathcal{D}$ with inverse $\tau_{RL} \colon \mathsf{Mod}\text{-}\mathcal{D} \longrightarrow \mathcal{D}\text{-}\mathsf{Mod}$ is defined as follows: For a left $\mathcal{D}$-module $\mathcal{F}$, we have $\tau_{LR}(\mathcal{F}) = \mathcal{F} \otimes \Omega^n$ where the right multiplication with a vector field ν is $(f \otimes \omega) \cdot \nu = -\nu(f) \otimes \omega + f \otimes \omega \cdot \nu$. Similarly, if $\mathcal{G}$ is a right $\mathcal{D}$-module, then $\tau_{RL}(\mathcal{G}) = \mathcal{H}om_{\mathcal{O}}(\Omega^n, \mathcal{G})$ and left multiplication with a vector field ν is given by $\nu \cdot \psi(\omega) = \psi(\omega \cdot \nu) - \psi(\omega) \cdot \nu$.

In particular, the left–right equivalence of A-modules is given by the algebra involution $\tau \colon A \longrightarrow A$, where $x^{\mathbf{a}}\partial^{\mathbf{b}} \mapsto (-\partial)^{\mathbf{b}} x^{\mathbf{a}}$. Specifically, given a graded left A-module F, we obtain a graded right A-module F^τ which has the same underlying additive structure and has multiplication defined by $g \cdot f = \tau(f) \cdot g$ for $g \in A$ and $f \in F$. Similarly, if G is a graded right A-module, then an analogous procedure yields the left

A-module G^τ. It is clear that $(F^\tau)^\tau = F$ and $(G^\tau)^\tau = G$. Furthermore, for graded A-modules, we have

Proposition 4.11. *There are inverse equivalences of categories*

$$\tau_{LR}^{\mathsf{mod}} \colon A\text{-}\mathsf{GrMod}_\theta \longrightarrow \mathsf{GrMod}_\theta\text{-}A\,, \quad \tau_{RL}^{\mathsf{mod}} \colon \mathsf{GrMod}_\theta\text{-}A \longrightarrow A\text{-}\mathsf{GrMod}_\theta$$

given by $\tau_{LR}^{\mathsf{mod}}(F) = F^\tau(-\bar{\mathbf{e}})$ *and* $\tau_{RL}^{\mathsf{mod}}(G) = G^\tau(\bar{\mathbf{e}})$ *where* $\bar{\mathbf{e}} \in \mathrm{Cl}(X)$ *is the class of* $\mathbf{e} = \mathbf{e}_1 + \cdots + \mathbf{e}_d \in \mathbb{Z}^d$.

Proof. We only need to show that the graded components of $\tau_{LR}^{\mathsf{mod}}(F)$ and $\tau_{RL}^{\mathsf{mod}}(G)$ are annihilated by suitable Euler operators. However, this follows from the fact that $\tau(\theta_{\bar{\mathbf{u}}}) = -\theta_{\bar{\mathbf{u}}} - \langle \bar{\mathbf{u}}, \bar{\mathbf{e}} \rangle$, where $\bar{\mathbf{u}} \in \mathrm{Cl}(X)$. $\square$

Example 4.12. Since $\tau(\theta_{\bar{\mathbf{u}}} + \langle \bar{\mathbf{u}}, \bar{\mathbf{a}} \rangle) = -(\theta_{\bar{\mathbf{u}}} - \langle \bar{\mathbf{u}}, \bar{\mathbf{a}} - \bar{\mathbf{e}} \rangle)$, there is an isomorphism $\tau_{LR}^{\mathsf{mod}}\big(D_L(\bar{\mathbf{a}})\big) \cong D_R(\bar{\mathbf{a}} - \bar{\mathbf{e}})$.

We next show that these equivalences of categories are compatible with the functors in Theorem 4.2. We will use the fact that, for a smooth toric variety X, there is a natural isomorphism $\Omega^n \cong \mathcal{O}(-\bar{\mathbf{e}})$. In fact, if 0 denotes the unique zero dimensional cone in Δ, then this isomorphism identifies the section $\frac{dy_1}{y_1} \wedge \cdots \wedge \frac{dy_n}{y_n}$ with $\frac{1}{x_1 \cdots x_d}$ on the open subset U_0 (see Section 4.3 in [F]).

Proposition 4.13. *For the pair of functors* $(\tau_{LR}^{\mathsf{mod}}, \tau_{LR})$, *the diagrams*

$$\begin{tikzcd} A\text{-}\mathsf{GrMod}_\theta \arrow[r, "\tau_{LR}^{\mathsf{mod}}"] \arrow[d, "\sim"'] & \mathsf{GrMod}_\theta\text{-}A \arrow[d, "\sim"'] \\ \mathcal{D}\text{-}\mathsf{Mod} \arrow[r, "\tau_{LR}"] & \mathsf{Mod}\text{-}\mathcal{D} \end{tikzcd} \qquad \begin{tikzcd} \mathcal{D}\text{-}\mathsf{Mod} \arrow[r, "\tau_{LR}"] \arrow[d, "\Gamma_L"'] & \mathsf{Mod}\text{-}\mathcal{D} \arrow[d, "\Gamma_R"'] \\ A\text{-}\mathsf{GrMod}_\theta \arrow[r, "\tau_{LR}^{\mathsf{mod}}"] & \mathsf{GrMod}_\theta\text{-}A \end{tikzcd}$$

are commutative, up to natural isomorphisms. A similar statement holds for the pair $(\tau_{RL}^{\mathsf{mod}}, \tau_{RL})$.

Proof. Since $\tau_{RL} = (\tau_{LR})^{-1}$ and $\tau_{RL}^{\mathsf{mod}} = (\tau_{LR}^{\mathsf{mod}})^{-1}$, the second assertion is a consequence of the first. Because $F^\tau \cong F$ as S-modules and $\Omega^n \cong \mathcal{O}(-\bar{\mathbf{e}})$ as $\mathcal{O}$-modules, there is a natural isomorphism of $\mathcal{O}$-modules $\beta_F \colon \tau_{LR}(\widetilde{F}) = \widetilde{F} \otimes \Omega^n \longrightarrow \widetilde{\tau_{LR}^{\mathsf{mod}}(F)} = \widetilde{F^\tau(-\bar{\mathbf{e}})}$. Thus, it suffices to prove that β_F is compatible with the right $\mathcal{D}$-module structures.

By taking a presentation $\bigoplus_i D_L(\bar{\mathbf{b}}_i) \longrightarrow F$, we see that it suffices to establish the claim for $F = D_L(\bar{\mathbf{b}}_i)$. In this case, the restriction map

$$H^0\big(U, \widetilde{F^\tau(-\bar{\mathbf{e}})}\big) \longrightarrow H^0\big(U', \widetilde{F^\tau(-\bar{\mathbf{e}})}\big)\,,$$

is injective for open subsets $U' \subseteq U \subseteq X$. Thus, the claim reduces to showing that β_F is compatible with the right $\mathcal{D}$-module structure on U_0. Over U_0, the map

$$\beta_F|_{U_0} \colon (F_{x_1 \cdots x_d})_{\bar{\mathbf{0}}} \otimes_{(S_{x_1 \cdots x_d})_{\bar{\mathbf{0}}}} H^0(U_0, \Omega^n) \longrightarrow (F_{x_1 \cdots x_d})^\tau_{-\bar{\mathbf{e}}}$$

is given by $f \otimes \omega \mapsto \frac{f}{x_1 \cdots x_d}$, where $\omega = \frac{dy_1}{y_1} \wedge \cdots \wedge \frac{dy_n}{y_n}$. Now, it is enough to check that $\beta_F|_{U_0}$ is compatible with right multiplication with a vector field ν over U_0. Using the notation from Lemma 3.5, we may assume that $\nu = y^{\mathbf{p}} \rho(\theta_i)$, for some $\mathbf{p} \in \mathbb{Z}^n$. We first compute

$$(f \otimes \omega) \cdot y^{\mathbf{p}} \rho(\theta_i) = -\big(y^{\mathbf{p}} \rho(\theta_i) \cdot f\big) \otimes \omega + f \otimes \big(\omega \cdot y^{\mathbf{p}} \rho(\theta_i)\big) .$$

By definition, we have

$$\begin{aligned}
(\omega \cdot y^{\mathbf{p}} \vartheta_j)(\eth_1, \ldots, \eth_n) &= -\big(\operatorname{Lie}_{y^{\mathbf{p}}\vartheta_j}(\omega)\big)(\eth_1, \ldots, \eth_n) \\
&= \sum_{i=1}^{n} \omega(\eth_1, \ldots, [y^{\mathbf{p}}\vartheta_j, \eth_i], \ldots, \eth_n) - \operatorname{Lie}_{y^{\mathbf{p}}\vartheta_j} \big(\omega(\eth_1, \ldots, \eth_n)\big) \\
&= -\sum_{i=1}^{n} \Big(\delta_{ij} \tfrac{(\mathbf{p}_j+1) y^{\mathbf{p}}}{y_1 \cdots y_n}\Big) + \tfrac{y^{\mathbf{p}}}{y_1 \cdots y_n} \\
&= -\tfrac{\mathbf{p}_j y^{\mathbf{p}}}{y_1 \cdots y_n} ,
\end{aligned}$$

from which we deduce $\omega \cdot y^{\mathbf{p}} \rho(\theta_i) = -\iota(\mathbf{p})_i y^{\mathbf{p}} \cdot \omega$. Identifying the action of $y^{\mathbf{p}} \rho(\theta_i)$ on F with the action of $x^{\iota(\mathbf{p})}\theta_i$, we obtain

$$(f \otimes \omega) \cdot y^{\mathbf{p}} \rho(\theta_i) = \big((-x^{\iota(\mathbf{p})}\theta_i - \iota(\mathbf{p})_i x^{\iota(\mathbf{p})}) \cdot f\big) \otimes \omega .$$

On the other hand, we have

$$\begin{aligned}
\tfrac{f}{x_1 \cdots x_d} \cdot x^{\iota(\mathbf{p})}\theta_i &= \tau\big(x^{\iota(\mathbf{p})}\theta_i\big) \cdot \tfrac{f}{x_1 \cdots x_d} \\
&= \big(-x^{\iota(\mathbf{p})}\theta_i - (\iota(\mathbf{p})_i + 1) x^{\iota(\mathbf{p})}\big) \tfrac{f}{x_1 \cdots x_d} \\
&= \tfrac{1}{x_1 \cdots x_d} (-x^{\iota(\mathbf{p})}\theta_i - \iota(\mathbf{p})_i x^{\iota(\mathbf{p})}) f ,
\end{aligned}$$

and we conclude that $\beta_F\big((f \otimes \omega) \cdot \nu\big) = \beta_F(f \otimes \omega) \cdot \nu$.

In the second part, we consider $\mathcal{F} \in \mathcal{D}\text{-}\mathsf{Mod}$. For $F = \Gamma_L(\mathcal{F})$, we construct the natural map $\beta'_{\mathcal{F}} \colon \tau_{LR}^{\mathsf{mod}}\big(\Gamma_L(\mathcal{F})\big) \longrightarrow \Gamma_R\big(\tau_{LR}(\mathcal{F})\big)$, by composing the morphisms:

$$\tau_{LR}^{\mathsf{mod}}\big(\Gamma_L(\mathcal{F})\big) = F^{\tau}(-\bar{\mathbf{e}}) \longrightarrow \Gamma_R\big(\widetilde{F^{\tau}(-\bar{\mathbf{e}})}\big) , \tag{4.B}$$

$$\beta_{F^{\tau}(-\bar{\mathbf{e}})} \colon \Gamma_R\big(\widetilde{F^{\tau}(-\bar{\mathbf{e}})}\big) \longrightarrow \Gamma_R(\mathcal{F} \otimes \Omega^n) = \Gamma_R\big(\tau_{LR}(\mathcal{F})\big) . \tag{4.C}$$

Since F is $\mathfrak{b}$-saturated, it follows that $F^{\tau}(-\bar{\mathbf{e}})$ is also $\mathfrak{b}$-saturated and hence (4.B) is an isomorphism. Moreover, Γ_R is a functor and $\beta_{F^{\tau}(-\bar{\mathbf{e}})}$ is an isomorphism which implies that (4.C) is also an isomorphism. $\square$

Example 4.14. The isomorphism of $\mathcal{O}$-modules $\Omega^n \cong \mathcal{O}(-\bar{\mathbf{e}})$ yields an isomorphism of S-modules $\Gamma_R(\Omega^n) \cong S(-\bar{\mathbf{e}})$. Proposition 4.13 shows that this is an isomorphism of right A-modules if $S(-\bar{\mathbf{e}})$ has the right A-module structure given by $S(-\bar{\mathbf{e}}) \cong \frac{A(-\bar{\mathbf{e}})}{(\partial_1, \ldots, \partial_d) \cdot A}$.

5. The Characteristic Variety

In this section, we use the relationship between $\mathcal{D}$-modules on X and graded A-modules to describe the characteristic varieties. In particular, we relate the dimensions of F and $\widetilde{F}$. For simplicity, we restrict our attention to left modules.

We start by recalling the quotient construction of X; see [C] or [M1]. Let T be the torus $\operatorname{Hom}\big(\operatorname{Cl}(X), k^*\big) \cong (k^*)^{d-n}$. The group T can be embedded into $(k^*)^d$ by the projection $\mathbb{Z}^d \longrightarrow \operatorname{Cl}(X)$. The diagonal action of $(k^*)^d$ on the affine space $\mathbb{A}^d$ induces an action of T on $\mathbb{A}^d$ such that the open subset $U = \mathbb{A}^d \setminus \operatorname{Var}(\mathfrak{b})$ is T-invariant — $\operatorname{Var}(\mathfrak{b})$ denotes the subscheme associated to the ideal $\mathfrak{b}$. Since X is smooth (and hence simplicial), there is a canonical morphism $U \longrightarrow X$ such that X is a geometric quotient of U with respect to the action of T. Furthermore, we have

Lemma 5.1. *For every $z \in U$, $\operatorname{Stab}_T(z) = \{1\}$. In particular, all the T-orbits in U have dimension $d - n$.*

Proof. Consider a point z in U and $t \in T$ satisfying $t \cdot u = u$. Writing $z = (z_1, \dots, z_d)$, we have $t \cdot z = \big(t(\bar{\mathbf{e}}_1)z_1, \dots, t(\bar{\mathbf{e}}_d)z_d\big)$ and we deduce that $t(\bar{\mathbf{e}}_i) = 1$, for all i such that $z_i \neq 0$. Because there is $\sigma \in \Delta$ such that $x^{\widehat{\sigma}}(z) \neq 0$, we conclude that $t(\bar{\mathbf{e}}_i) = 1$ for every i with $\mathbf{v}_i \notin \sigma$.

On the other hand, t belongs to $\operatorname{Hom}\big(\operatorname{Cl}(X), k^*\big)$ so we have

$$t(\langle \mathbf{h}, \mathbf{v}_1\rangle \bar{\mathbf{e}}_i + \cdots + \langle \mathbf{h}, \mathbf{v}_d\rangle \bar{\mathbf{e}}_d) = t(\bar{\mathbf{e}}_1)^{\langle \mathbf{h}, \mathbf{v}_1\rangle} \cdots t(\bar{\mathbf{e}}_d)^{\langle \mathbf{h}, \mathbf{v}_d\rangle} = 1\,,$$

for every $\mathbf{h} \in N^\vee$. It follows that $\prod_{\mathbf{v}_i \in \sigma} t(\bar{\mathbf{e}}_i)^{\langle \mathbf{h}, \mathbf{v}_i\rangle} = 1$. Because this holds for each $\mathbf{h} \in N^\vee$ and the $\mathbf{v}_i$ form part of a basis of N, we also conclude that $t(\bar{\mathbf{e}}_i) = 1$ when $v_i \in \sigma$ and therefore $t = 1$. $\square$

We next identify $\mathbb{A}^d \times \mathbb{A}^d$ with the cotangent bundle of $\mathbb{A}^d$ and consider the natural T-action on it. Let $S' = k[x_1, \dots, x_d, \xi_1, \dots, \xi_d]$, with the $\operatorname{Cl}(X)$-grading given by $\deg(x_i) = -\deg(\xi_i) = \bar{\mathbf{e}}_i$, be the coordinate ring of $\mathbb{A}^d \times \mathbb{A}^d$. Since the action of T on $\mathbb{A}^d$ is linear, it follows that the action of T on $\mathbb{A}^d \times \mathbb{A}^d$ is given by $t \cdot (z_1, z_2) = (t \cdot z_1, t^{-1} \cdot z_2)$. It is clear that $V = U \times \mathbb{A}^d \subset \mathbb{A}^d \times \mathbb{A}^d$ is invariant under the action of T. We construct its quotient as follows.

Proposition 5.2. *There is a morphism $\pi \colon V \longrightarrow X'$ such that X' is the geometric quotient of V by the action of T. In addition, for every $z \in V$, we have $\operatorname{Stab}_T(z) = \{1\}$ implying that all the T-orbits in V have dimension $d - n$.*

Proof. The first step is to construct the morphism $\pi \colon V \longrightarrow X'$ as a categorical quotient — this is a local problem. For every $\sigma \in \Delta$, let $V_\sigma \subseteq V$ be the open subset defined by the non-vanishing of $x^{\widehat{\sigma}}$. In other words, we have $V_\sigma = \big(U \setminus \mathrm{Var}(x^{\widehat{\sigma}})\big) \times \mathbb{A}^d \subseteq V$, which is clearly T-invariant. Thus, the categorical quotient is locally $V_\sigma = \mathrm{Spec}\,\big(S'[(x^{\widehat{\sigma}})^{-1}]^T\big)$. Since $t \cdot x^{\mathbf{a}}\xi^{\mathbf{b}} = t(\bar{\mathbf{a}} - \bar{\mathbf{b}})x^{\mathbf{a}}\xi^{\mathbf{b}}$, for every $t \in T$ and $\mathbf{a}$, $\mathbf{b} \in \mathbb{Z}^d$, we have $S'[(x^{\widehat{\sigma}})^{-1}]^T = S'[(x^{\widehat{\sigma}})^{-1}]_{\bar{\mathbf{0}}}$. Now, if σ_0 is a face of σ such that $\sigma_0 = \sigma \cap \mathbf{h}^\perp$ for $\mathbf{h} \in N^\vee \cap \sigma^\vee$, then we set $\mathbf{c} = \langle \mathbf{h}, \mathbf{v}_1\rangle \mathbf{e}_1 + \cdots + \langle \mathbf{h}, \mathbf{v}_d\rangle \mathbf{e}_d \in \mathbb{Z}^d$. It follows that $S'[(x^{\widehat{\sigma_0}})^{-1}]_{\bar{\mathbf{0}}} = \big(S'[(x^{\widehat{\sigma}})^{-1}]_{\bar{\mathbf{0}}}\big)_{x^{\mathbf{c}}}$ which provides an open immersion $V_{\sigma_0} \hookrightarrow V_\sigma$. Thus, we obtain morphisms $\pi_\sigma \colon V_{\sigma_0} \longrightarrow V_\sigma$ which glue together to give the categorical quotient $\pi \colon V \longrightarrow X'$.

In the second step, we establish that $\pi \colon V \longrightarrow X'$ is in fact a geometric quotient. By Amplification 1.3 in [MFK], it suffices to show that every T-orbit in V is closed. Consider $z = (z_1, z_2) \in V$. Since $U \longrightarrow X$ is a geometric quotient, the projection $V \longrightarrow U$ induces a morphism $\chi \colon \overline{Tz} \longrightarrow \overline{Tz_1} = Tz_1$. By Lemma 5.1, the morphism $\gamma \colon T \longrightarrow Tz_1$ given by $\gamma(t) = tz_1$ is bijective. Because the characteristic of the ground field k is zero and both T and Tz_1 are smooth, the map γ is an isomorphism. Let $\gamma' \colon T \longrightarrow Tz \subseteq V$ be defined by $\gamma'(t) = tz$. Hence, the map $\gamma' \circ \gamma^{-1} \circ \chi \colon \overline{Tz} \longrightarrow Tz \subseteq \overline{Tz}$ is the identity on Tz. It follows that $\gamma' \circ \gamma^{-1} \circ \chi$ is the identity map on $\overline{Tz}$ and we conclude that $Tz = \overline{Tz}$.

Since the projection from V onto U is T-equivariant, the second assertion follows from Lemma 5.1. □

Before discussing characteristic varieties, we review some properties of the order filtration. Recall that the sheaf $\mathcal{D}$ is naturally filtered by the order of the differential operators. In particular, this makes $H^0(U_\sigma, \mathcal{D})$ into a filtered ring. We can also filter the ring A by the order of the differential operators (in fact, $S' = \mathrm{gr}(A)$) and this induces a filtration on the quotient $\frac{(A_{x^{\widehat{\sigma}}})_{\bar{\mathbf{0}}}}{(A_{x^{\widehat{\sigma}}})_{\bar{\mathbf{0}}} \cdot (\theta_{\bar{\mathbf{u}}} : \bar{\mathbf{u}} \in \mathrm{Cl}(X)^\vee)}$. As Musson observed, we have

Lemma 5.3 (Musson). *For every $\sigma \in \Delta$, the isomorphism $\bar{\varphi}_\sigma$ [see equation* (3.C)] *preserves the filtrations induced by the order of differential operators.*

Proof. See Section 4 in [M1]. □

A filtration of an A-module F is called good if the associated graded module $\mathrm{gr}(F)$ is finitely generated over S'. Every finitely generated A-module has a good filtration and, conversely, any module with a good filtration is necessarily finitely generated over A. For a good filtration of F, we define the characteristic ideal $\mathfrak{i}(F)$ to be the radical of $\operatorname{Ann}_{S'}\big(\operatorname{gr}(F)\big)$. Since any two good filtrations are equivalent, the characteristic ideal $\mathfrak{i}(F)$ is independent of the choice of good filtration. The characteristic variety of F is $\operatorname{Ch}(F) = \operatorname{Var}\big(\mathfrak{i}(F)\big) \subseteq \mathbb{A}^d \times \mathbb{A}^d$. Analogously, for a $\mathcal{D}$-module $\mathcal{F}$ with a good filtration, we define the characteristic variety $\operatorname{Ch}(\mathcal{F})$ to be the support of the associated graded sheaf $\operatorname{gr}(\mathcal{F})$.

We first describe the characteristic variety associated to the graded left A-modules $D_L(\bar{\mathbf{b}})$. Let $p_{\bar{\mathbf{u}}} = \langle \bar{\mathbf{u}}, \bar{\mathbf{e}}_1\rangle x_1\xi_1 + \cdots + \langle \bar{\mathbf{u}}, \bar{\mathbf{e}}_d\rangle x_d\xi_d \in S'$, for all $\bar{\mathbf{u}} \in \operatorname{Cl}(X)^\vee$. We consider the ideal $\mathfrak{p} = (p_{\bar{\mathbf{u}}} : \bar{\mathbf{u}} \in \operatorname{Cl}(X)^\vee)$ and the corresponding variety $Z = \operatorname{Var}(\mathfrak{p}) \subseteq \mathbb{A}^d \times \mathbb{A}^d$. It is clear that Z is invariant under the T-action.

Proposition 5.4. *The variety Z is a normal complete intersection of dimension $d + n$. Moreover, Z is equal to the characteristic variety $\operatorname{Ch}\big(D_L(\bar{\mathbf{b}})\big)$, for every $\bar{\mathbf{b}} \in \operatorname{Cl}(X)$.*

Proof. By choosing a basis $\bar{\mathbf{u}}_1, \dots, \bar{\mathbf{u}}_{d-n}$ for $\operatorname{Cl}(X)^\vee$, we can write $\mathfrak{p} = (p_{\bar{\mathbf{u}}_i} : 1 \le i \le d - n)$. For each $\bar{\mathbf{u}}_i$, we pick a representative $\mathbf{u}_i \in (\mathbb{Z}^d)^\vee$ for $\bar{\mathbf{u}}_i$ such that $\mathbf{u}_1, \dots, \mathbf{u}_{d-n}$ are linearly independent. We then enlarge this collection to obtain a basis $\mathbf{u}_1, \dots, \mathbf{u}_d$ for $(\mathbb{Z}^d)^\vee$. Setting $q_{\mathbf{u}} = \langle \mathbf{u}, \mathbf{e}_1\rangle x_1\xi_1 + \cdots + \langle \mathbf{u}, \mathbf{e}_d\rangle x_d\xi_d \in S'$, for all $\mathbf{u} \in (\mathbb{Z}^d)^\vee$, it follows that the ideal $(q_{\mathbf{u}_i} : 1 \le i \le d)$ equals $(x_i\xi_i : 1 \le i \le d)$, which has dimension d. Hence, the $q_{\mathbf{u}_i}$ and $p_{\bar{\mathbf{u}}_i}$ form a regular sequence and we deduce $\dim(Z) = d + n$.

To prove that Z is normal, we apply Serre's criterion. Because being a complete intersection implies the $(S2)$ condition, it suffices to show that Z satisfies condition $(R1)$, which we check by using the Jacobian criterion. The Jacobian matrix $\operatorname{Jac}(x, \xi)$ of $(p_{\bar{\mathbf{u}}_1}, \dots, p_{\bar{\mathbf{u}}_{d-n}})$ is given by

$$\begin{pmatrix} \langle \bar{\mathbf{u}}_1, \bar{\mathbf{e}}_1\rangle\xi_1 & \cdots & \langle \bar{\mathbf{u}}_1, \bar{\mathbf{e}}_d\rangle\xi_d & \langle \bar{\mathbf{u}}_1, \bar{\mathbf{e}}_1\rangle x_1 & \cdots & \langle \bar{\mathbf{u}}_1, \bar{\mathbf{e}}_d\rangle x_d \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \langle \bar{\mathbf{u}}_{d-n}, \bar{\mathbf{e}}_1\rangle\xi_1 & \cdots & \langle \bar{\mathbf{u}}_{d-n}, \bar{\mathbf{e}}_d\rangle\xi_d & \langle \bar{\mathbf{u}}_{d-n}, \bar{\mathbf{e}}_1\rangle x_1 & \cdots & \langle \bar{\mathbf{u}}_{d-n}, \bar{\mathbf{e}}_d\rangle x_d \end{pmatrix}.$$

Observe that, for $1 \le i \le d$, the restriction $\bigoplus_{j \ne i} \mathbb{Z}\mathbf{e}_j \longrightarrow \operatorname{Cl}(X)$ is surjective. Indeed, if σ_i is the cone generated by $\mathbf{v}_i$, then every element in $\operatorname{Cl}(X)$ can be represented by a divisor whose support does not intersect U_{σ_i}. We deduce that if the rank of $\operatorname{Jac}(x, \xi)$ is strictly less than $d - n$, then at least two of the pairs of coordinates $(x_1, \xi_1), \dots, (x_d, \xi_d)$ are zero. By cutting with n extra quadrics $q_{\mathbf{u}_{d-n+1}}, \dots, q_{\mathbf{u}_d}$, we see that the codimension of the singular locus of Z is at least two. Therefore, the variety Z is normal. Moreover Z is a cone which implies it is connected and hence integral.

Recall that, for $\bar{\mathbf{b}} \in \mathrm{Cl}(X)^\vee$, we have $D_L(\bar{\mathbf{b}}) = \frac{A(\bar{\mathbf{b}})}{(\theta_{\bar{\mathbf{u}}_i} + \langle \bar{\mathbf{u}}_i, \bar{\mathbf{b}} \rangle : 1 \le i \le d-n)}$, and for the order filtration the initial term (or principal symbol) of the above elements is $\mathrm{in}(\theta_{\bar{\mathbf{u}}_i} + \langle \bar{\mathbf{u}}_i, \bar{\mathbf{b}} \rangle) = \theta_{\bar{\mathbf{u}}_i}$. Since the $\theta_{\bar{\mathbf{u}}_i}$ for $1 \le i \le d-n$ form a regular sequence in $S' = \mathrm{gr}(A)$, it follows that $\mathrm{gr}\big(D_L(\bar{\mathbf{b}})\big) = S'/\mathfrak{p}$. On the other hand, we have already seen that $\mathfrak{p}$ is reduced, so we have $\mathrm{Ch}\big(D_L(\bar{\mathbf{b}})\big) = \mathrm{Var}(\mathfrak{p}) = Z$. □

Corollary 5.5. *If $F \in A\text{-}\mathsf{GrMod}_\theta^f$, then we have* $\mathrm{Ch}(F) \subseteq Z$.

Proof. By Corollary 4.8, F is a quotient of $\bigoplus_{i=1}^r D_L(\bar{\mathbf{b}}_i)$, for some r and some $\bar{\mathbf{b}}_i$. It follows that $\mathrm{Ch}(F) \subseteq \bigcup_{i=1}^r \mathrm{Ch}\big(D_L(\bar{\mathbf{b}}_i)\big) = Z$. □

We next relate the cotangent bundle of X to the variety Z. Consider the following diagram:

$$\begin{array}{ccccc} Z \cap V & \longrightarrow & V & \longrightarrow & U \\ \big\downarrow & & {\scriptstyle\pi}\big\downarrow & & \big\downarrow \\ \pi(Z \cap V) & \longrightarrow & X' & \longrightarrow & X \end{array}$$

where $X' \longrightarrow X$ arises from the universal property of the categorical quotient.

Proposition 5.6. *There is a canonical isomorphism of varieties over X between $\zeta \colon \pi(Z \cap V) \longrightarrow X$ and the cotangent bundle T^*X over X.*

Proof. Since T^*X is naturally isomorphic to $\mathrm{Ch}(\mathcal{D})$, we see that T^*X is isomorphic to $\mathrm{Spec}\big(\mathrm{gr}\, H^0(U_\sigma, \mathcal{D})\big)$ over U_σ. On the other hand, from the local description of X', we know that the inverse image of U_σ is $\mathrm{Spec}(S'_{x^{\hat{\sigma}}})_{\bar{\mathbf{0}}}$ and therefore

$$\zeta^{-1}(U_\sigma) = \mathrm{Spec}\left(\frac{(S'_{x^{\hat{\sigma}}})_{\bar{\mathbf{0}}}}{(p_{\bar{\mathbf{u}}} : \bar{\mathbf{u}} \in \mathrm{Cl}(X)^\vee)}\right).$$

By Lemma 5.3, we have an isomorphism of filtered rings:

$$\bar{\varphi}_\sigma \colon \frac{(A_{x^{\hat{\sigma}}})_{\bar{\mathbf{0}}}}{(A_{x^{\hat{\sigma}}})_{\bar{\mathbf{0}}} \cdot (\theta_{\bar{\mathbf{u}}} : \bar{\mathbf{u}} \in \mathrm{Cl}(X)^\vee)} \longrightarrow H^0(U_\sigma, \mathcal{D}).$$

Notice that the graded ring associated to left hand side is $\frac{(S'_{x^{\hat{\sigma}}})_{\bar{\mathbf{0}}}}{(p_{\bar{\mathbf{u}}} : \bar{\mathbf{u}} \in \mathrm{Cl}(X)^\vee)}$. Indeed, following the proof of Proposition 5.4, the initial terms of $\theta_{\bar{\mathbf{u}}_1}, \ldots, \theta_{\bar{\mathbf{u}}_{d-n}}$ are equal to $p_{\bar{\mathbf{u}}_1}, \ldots, p_{\bar{\mathbf{u}}_{d-n}}$ and form a regular sequence in $(S'_{x^{\hat{\sigma}}})_{\bar{\mathbf{0}}}$. Therefore, by passing to the associated graded rings, $\bar{\varphi}_\sigma$ induces the required isomorphism. Because the $\bar{\varphi}_\sigma$ are compatible with restriction, these local isomorphisms glue together to give the required isomorphism □

We now present the main result in this section.

Theorem 5.7. *If $F \in A\text{-}\mathsf{GrMod}_\theta^f$, then the characteristic variety of F is T-invariant and $\pi\big(\operatorname{Ch}(F) \setminus \operatorname{Var}(\mathfrak{b}) \times \mathbb{A}^d\big) = \operatorname{Ch}(\widetilde{F})$.*

Proof. Since $F \in A\text{-}\mathsf{GrMod}_\theta^f$, we may choose a finite set $f_1, \dots, f_r$ of homogeneous generators for F. By using these homogeneous elements to define a good filtration of F, it follows that $\operatorname{gr}(F)$ is a graded finitely generated S'-module. Therefore, both $\mathfrak{j}(F) = \operatorname{Ann}_{S'}\big(\operatorname{gr}(F)\big)$ and its radical $\mathfrak{i} = \sqrt{\mathfrak{j}(F)}$ are graded ideals of S'. Recall that for every $t \in T$, we have $t \cdot x^{\mathbf{a}}\xi^{\mathbf{b}} = t(\bar{\mathbf{a}} - \bar{\mathbf{b}})x^{\mathbf{a}}\xi^{\mathbf{b}}$. We deduce that every subscheme defined by a graded ideal is T-invariant; in particular, $\operatorname{Ch}(F) = \operatorname{Var}(\mathfrak{i})$ is T-invariant.

To prove the second assertion, we argue locally and use the identification in Proposition 5.6. Over the open subset U_σ, the ideal defining $\operatorname{Ch}(F) \setminus \operatorname{Var}(\mathfrak{b}) \times \mathbb{A}^d \subseteq \operatorname{Spec}(S'_{x^{\hat{\sigma}}})$ is $\mathfrak{j} \cdot S'_{x^{\hat{\sigma}}}$. On the other hand, the characteristic variety of $\widetilde{F}$ over U_σ can be computed as follows: If $H^0(U_\sigma, \widetilde{F}) = (F_{x^{\hat{\sigma}}})_{\bar{\mathbf{0}}}$ has the good filtration induced by the images of $f_1, \dots, f_r$, then we obtain $\operatorname{gr}\big(H^0(U_\sigma, \widetilde{F})\big) \cong (\operatorname{gr}(F)_{x^{\hat{\sigma}}})_{\bar{\mathbf{0}}}$. To see that the annihilator of $\operatorname{gr}\big(H^0(U_\sigma, \widetilde{F})\big)$ in $(S'_{x^{\hat{\sigma}}})_{\bar{\mathbf{0}}}$ is $(\mathfrak{j}_{x^{\hat{\sigma}}})_{\bar{\mathbf{0}}}$, it is enough to observe that $\operatorname{gr}(F)_{x^{\hat{\sigma}}}$ can be generated by elements of degree zero. Since $\sqrt{(\mathfrak{i}(F)_{x^{\hat{\sigma}}})_{\bar{\mathbf{0}}}} = (\mathfrak{j}_{x^{\hat{\sigma}}})_{\bar{\mathbf{0}}}$, we may identify $\pi(\operatorname{Ch}(F) \cap V_\sigma)$ with $\operatorname{Ch}(\widetilde{F}|_{U_\sigma})$. Because these identifications are compatible with restriction, they glue together to give the required isomorphism. $\square$

As a corollary, we obtain

Proof of Theorem 1.2. Follows immediately from Theorem 5.7. $\square$

We end by relating the dimension of the A-module F and its associated $\mathcal{D}$-module $\widetilde{F}$. By definition, the local dimension of a $\mathcal{D}$-module $\mathcal{F}$ at a point $p \in X$ is equal to the Krull dimension of the associated graded module of $\mathcal{F}_p$ with respect to a good filtration respecting the order filtration of $\mathcal{D}_p$. It is also equal to the dimension of the characteristic variety of $\mathcal{F}_p$. The dimension of F is by definition the maximum of the local dimensions and is equivalently the dimension of $\operatorname{Ch}(F)$.

By Theorem 5.7, the dimension of $\widetilde{F}$ is equal to the maximum of the local dimensions of F over the open set $\mathbb{A}^d \setminus \operatorname{Var}(\mathfrak{b})$, minus the dimension $d - n$ of the orbits under the group action. Before showing that when F has no $\mathfrak{b}$-torsion, we can express this dimension in terms of $\dim(F)$, we collect two lemmas.

Lemma 5.8. *If F is a finitely generated left A-module, f is an element of S, and F' is a finitely generated A-submodule of $F[f^{-1}]$, then $\dim(F') \leq \dim(F)$.*

Proof. This claim follows immediately from standard results about Gelfand-Kirillov dimension; see Propositions 8.3.2 (i) and 8.3.14 (iii) in [MR]. ☐

Proposition 5.9. *Let $F \in A\text{-}\mathsf{GrMod}_\theta^f$ and recall that $\mathfrak{b}$ is the irrelevant ideal in S. If F has no $\mathfrak{b}$-torsion, then we have*

$$\dim(F) = \max\{\dim(F_p) : p \in \mathbb{A}^d \setminus \mathrm{Var}(\mathfrak{b})\} .$$

Proof. Suppose otherwise; then we have $\dim(F) > \dim(F_z)$ for all $z \in \mathbb{A}^d \setminus \mathrm{Var}(\mathfrak{b})$. Let F' be the maximum submodule of F of dimension strictly less than $\dim(F)$. In other words, F' is the submodule consisting of all $f \in F$ such that $A \cdot f$ has dimension strictly less than $\dim(F)$. Since F' is a submodule, there is a short exact sequence

$$0 \longrightarrow F' \longrightarrow F \longrightarrow \tfrac{F}{F'} \longrightarrow 0 .$$

By construction, F/F' has no nonzero submodules of dimension strictly less than $\dim(F)$ and, hence, the irreducible components of $\mathrm{Ch}(F/F')$ have dimension at least $\dim(F)$; see [S]. By hypothesis, the irreducible components of $\mathrm{Ch}(F)$ of dimension $\dim(F)$ are contained inside $\zeta^{-1}(\mathrm{Var}(\mathfrak{b}))$ where $\zeta : T^*X \longrightarrow X$. Since $\mathrm{Ch}(F) = \mathrm{Ch}(F') \cup \mathrm{Ch}(F/F')$, it follows that the characteristic variety of F/F' is contained inside $\zeta^{-1}(\mathrm{Var}(\mathfrak{b}))$. Moreover, the support of an A-module equals the projection of its characteristic variety (see [GM]) which implies that F/F' is supported on $\mathrm{Var}(\mathfrak{b})$. Taking the long exact sequence in local cohomology, we have

$$0 \longrightarrow H^0_{\mathfrak{b}}(F') \longrightarrow H^0_{\mathfrak{b}}(F) \longrightarrow H^0_{\mathfrak{b}}\left(\tfrac{F}{F'}\right) \longrightarrow H^1_{\mathfrak{b}}(F') \longrightarrow \cdots .$$

By assumption F has no $\mathfrak{b}$-torsion, so we have $H^0_{\mathfrak{b}}(F') = H^0_{\mathfrak{b}}(F) = 0$. Because F/F' is supported on $\mathrm{Var}(\mathfrak{b})$, we have $H^0_{\mathfrak{b}}(F/F') = F/F'$. Choosing a set of generators $\mathfrak{b} = (s_1, \ldots, s_r)$, the long exact sequence induces

$$0 \longrightarrow \tfrac{F}{F'} \longrightarrow \tfrac{\bigoplus_{i=1}^r F'[s_i^{-1}]}{F'} \longrightarrow \cdots .$$

Hence, F/F' is a finitely generated A-subquotient of $\bigoplus_{i=1}^r F'[s_i^{-1}]$. Lemma 5.8 then implies that $\dim(F/F') \leq \dim(F') < \dim(F/F')$ which is a contradiction. ☐

Finally, we have

Theorem 5.10. *If $F \in A\text{-}\mathsf{GrMod}_\theta^f$ has no $\mathfrak{b}$-torsion, then we have $\dim(\widetilde{F}) = \dim(F) - d + n$.*

Proof. Applying Proposition 5.9, we see that $\dim(F)$ is the maximum of the local dimensions of F over $\mathbb{A}^d \setminus \mathrm{Var}(\mathfrak{b})$. Hence, the claim follows from Proposition 5.2. ☐

A coherent $\mathcal{D}$-module $\mathcal{F}$ is holonomic if $\dim(\mathcal{F}) = \dim X$.

Corollary 5.11. *If $F \in A\text{-}\mathsf{GrMod}_\theta^f$ is holonomic, then $\widetilde{F}$ is holonomic. Furthermore, every holonomic $\mathcal{D}$-module is of the form $\widetilde{F}$ for some holonomic $F \in A\text{-}\mathsf{GrMod}_\theta^f$.*

Proof. Follows immediately from Theorem 5.10 and Remark 4.4. □

References

[B$^+$] Armand Borel, Pierre-Paul Grivel, Burchard Kaup, André Haefliger, Bernard Malgrange, and Fritz Ehlers, *Algebraic $\mathcal{D}$-modules*, Perspectives in Mathematics 2, Academic Press Inc., Boston, MA, 1987.

[C] David A. Cox, *The homogeneous coordinate ring of a toric variety*, Journal of Algebraic Geometry **4** (1995), no. 1, 17–50.

[EMS] David Eisenbud, Mircea Mustaţă, and Michael Stillman, *Cohomology on toric varieties and local cohomology with monomial supports*, Journal of Symbolic Computation **29** (2000), no. 4-5, 583–600.

[F] William Fulton, *Introduction to toric varieties*, The William H. Roever Lectures in Geometry, Princeton University Press, Princeton, NJ, 1993.

[GM] Michel Granger and Philippe Maisonobe, *A basic course on differential modules*, Éléments de la théorie des systèmes différentiels. $\mathcal{D}$-modules cohérents et holonomes (Nice, 1990), Travaux en Cours 45, Hermann, Paris, 1993, pp. 103–168.

[J1] Andrew G. Jones, *Differential operators on semigroup algebras*, Ph.D. thesis, Department of Pure Mathematics, The University of Sheffield, May 1993.

[J2] Andrew G. Jones, *Rings of differential operators on toric varieties*, Proceedings of the Edinburgh Mathematical Society, Series II **37** (1994), no. 1, 143–160.

[MFK] David Mumford, John Fogarty, and Frances C. Kirwan, *Geometric invariant theory*, third ed., Ergebnisse der Mathematik und ihrer Grenzgebiete (2) 34, Springer-Verlag, Berlin, 1994.

[MR] John C. McConnell and J. Chris Robson, *Noncommutative Noetherian rings*, With the cooperation of Lance W. Small, A Wiley-Interscience Publication, John Wiley & Sons Ltd., Chichester, 1987.

[M1] Ian M. Musson, *Rings of differential operators on invariant rings of tori*, Transactions of the American Mathematical Society **303** (1987), no. 2, 805–827.

[M2] Ian M. Musson, *Differential operators on toric varieties*, Journal of Pure and Applied Algebra **95** (1994), no. 3, 303–315.

[O] Toshinori Oaku, *An algorithm of computing b-functions*, Duke Mathematical Journal **87** (1997), no. 1, 115–132.

[OT] Toshinori Oaku and Nobuki Takayama, *Algorithms for D-modules — restriction, tensor product, localization, and local cohomology groups*, `arXiv:math.AG/9805006`.

[S] Gregory G. Smith, *Irreducible components of characteristic varieties*, `arXiv:math.AG/9912066`.

[SST] Mutsumi Saito, Bernd Sturmfels, and Nobuki Takayama, *Gröbner deformations of hypergeometric differential equations*, Algorithms and Computation in Mathematics 6, Springer-Verlag, Berlin, 2000.

[W] Uli Walther, *Algorithmic computation of local cohomology modules and the local cohomological dimension of algebraic varieties*, Effective methods in algebraic geometry (Saint-Malo, 1998), Journal of Pure and Applied Algebra **139** (1999), no. 1-3, 303–321.

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