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# On the $b$-functions of hypergeometric systems

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For any integer $d \times (n+1)$ matrix $A$ and parameter $\beta \in \mathbb{C}^d$ let $M_A(\beta)$ be the associated $A$-hypergeometric (or GKZ) system in the variables $x_0, \ldots, x_n$. We describe bounds for the (roots of the) $b$-functions of both $M_A(\beta)$ and its Fourier transform along the hyperplanes $(x_j = 0)$. We also give an estimate for the $b$-function for restricting $M_A(\beta)$ to a generic point.

## Contents



Let $D$ be the ring of algebraic $\mathbb{C}$-linear differential operators on $\mathbb{C}^{n+1}$ with coordinates $x_0, \ldots, x_n$.

**Definition 0.1** (Compare Kashiwara [1976/77], Maisonobe and Mebkhout [2004])**.** Let $M$ be a left $D$-module and pick an element $m \in M$ with annihilator $I \subseteq D$. If $(V^i D)$ is the vector space spanned by the monomials $x^\alpha \partial^\beta$ with $\alpha_0 - \beta_0 \geq i$ then the *$b$-function* of $m \in M$ along the coordinate hyperplane $x_0 = 0$ is the minimal monic polynomial $b(s)$ that satisfies: $b(x_0\partial_0) \cdot m \in (V^1 D) \cdot m$ in $M$, which is to say $b(x_0\partial_0) \in I + (V^1 D)$ in $D$.

If $M$ is cyclic, i.e., $M = D/I$, then we call $b$-function of $M$ the $b$-function in the above sense of the element $1 + I \in M$. □

The $b$-function exists in greater generality along any hypersurface $(f = 0)$, as long as the module $M$ is holonomic, cf. Kashiwara [1976/77]. The $V$-filtration of Kashiwara and Malgrange then takes the form $(V^i D) = \{P \in D \mid f^{i+k} \text{ divides } P \bullet f^k \text{ for } k \gg 0\}$. Both the $V$-filtration and the $b$-function are intimately connected to the restriction of the given $D$-module to the hypersurface. The purpose of this note is to give, for any $A$-hypergeometric system as well as its Fourier transform, an explicit arithmetic description of a bound for the root set of the $b$-function along any coordinate hyperplane that involves the parameter $\beta$ in a very elementary way.

We have several applications in mind: first, it is a longstanding question to understand the monodromy of $A$-hypergeometric systems, and for this purpose the roots of the $b$-function as considered above can be of some use. On the other hand, the Fourier transform of an $A$-hypergeometric system often (see Schulze and Walther [2009b]) appears as a direct image module under a natural torus embedding given by the columns of the matrix $A$. This point of view turns out to be extremely useful for Hodge theoretic considerations of $A$-hypergeometric systems (see Reichelt [2014]). It is one of the fundamental insights of Morihiko Saito (see Saito [1988, Section 3.2]) that the boundary behavior of variations of Hodge structures (or, more generally, of mixed Hodge modules) is controlled by the Kashiwara–Malgrange filtration along such a boundary divisor. In the case of a cyclic $D$-module, such as $A$-hypergeometric systems or their Fourier transforms, one can often deduce a large part of this filtration from the values of the $b$-function. We refer to Reichelt and Sevenheck [2015] for an

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immediate application of our results. In a third direction, one can also see our calculation of the $b$-function of the Fourier transform as a refinement of Schulze and Walther [2009b], Fernández-Fernández and Walther [2011] geared towards restriction of $A$-hypergeometric systems.

In the last part we compute an upper bound for the $b$-function of restriction of the $A$-hypergeometric system to a generic point, again in elementary terms of $A$ and $\beta$. Since the restriction of a $D$-module to a point is a dual object to the 0-th level solution functor, our estimate can be viewed as a step towards a sheafification in $\beta$ of the solution space, a problem that remains unsolved.

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## 1 Basic notions and results

**Notation.** Throughout, the base field is $\mathbb{C}$ and we consider a $\mathbb{C}$-vector space $V$ of dimension $n+1$. □

In this introductory section we review basic facts on $A$-hypergeometric systems as well as the Euler–Koszul functor. Readers are advised to refer to Matusevich et al. [2005] for more detailed explanations.

**Notation 1.1.** For any integer matrix $A$, let $R_A$ (resp. $O_A$) be the polynomial ring over $\mathbb{C}$ generated by the variables $\partial_j$ (resp. $x_j$) corresponding to the columns $\mathbf{a}_j$ of $A$. We identify $O_A$ with the symmetric algebra on $\mathrm{Hom}_{\mathbb{C}}(V,\mathbb{C}) \cong \bigoplus \mathbb{C}\cdot x_j$. Further, let $D_A$ be the ring of $\mathbb{C}$-linear differential operators on $O_A$, where we identify $\frac{\partial}{\partial x_j}$ with $\partial_j$ and multiplication by $x_j$ with $x_j$ so that both $R_A$ and $O_A$ become subrings of $D_A$. □

### 1.1 $A$-hypergeometric systems

Let $A = (\mathbf{a}_0, \ldots, \mathbf{a}_n)$ be an integer $d \times (n+1)$ matrix, $d \leq n+1$. For convenience we assume that $\mathbb{Z}A = \mathbb{Z}^d$. For $(v_1, \ldots, v_r) = \mathbf{v} \in \mathbb{Z}^r$ we denote by $\mathbf{v}_+, \mathbf{v}_-$ the vectors given by

$$(\mathbf{v}_+)_j = \max(0, v_j) \qquad \text{and} \qquad (\mathbf{v}_-)_j = \max(0, -v_j).$$

For the complex parameter vector $\beta \in \mathbb{C}^d$ consider the system of $d$ *homogeneity equations*

$$E_i \bullet \phi \quad = \quad \beta_i \cdot \phi, \tag{1.1}$$

where $E_i = \sum_{j=0}^{n} a_{i,j} x_j \partial_j$ is the $i$-th *Euler operator*, together with the *toric* (partial differential) *equations*

$$\{(\underbrace{\partial^{\mathbf{v}_+} - \partial^{\mathbf{v}_-}}_{:=\Delta_{\mathbf{v}}}) \bullet \phi = 0 \quad | \quad A \cdot \mathbf{v} = 0\}. \tag{1.2}$$

In $R_A$, the toric operators $\{\Delta_{\mathbf{v}} | A \cdot \mathbf{v} = 0\}$ generate the *toric ideal* $I_A$. The quotient

$$S_A := R_A / I_A$$

is naturally isomorphic to the semigroup ring $\mathbb{C}[\mathbb{N}A]$. In $D_A$, the left ideal generated by all equations (1.1) and (1.2) is the *hypergeometric ideal* $H_A(\beta)$. We put

$$M_A(\beta) := D_A / H_A(\beta);$$

this is the $A$-hypergeometric system introduced and first investigated by Gelfand, Graev, Kapranov, and Zelevinsky, in Gel′fand [1986] and a string of other papers. ⋄

### 1.2 $A$-degrees

If the rowspan of $A$ contains $\mathbf{1}_A$ we call $A$ *homogeneous*. Homogeneity is equivalent to $I_A$ defining a projective variety, and also to the system $H_A(\beta)$ having only regular singularities Hotta [1998], Schulze and Walther [2008]. A more general *$A$-degree function* on $R_A$ and $D_A$ is induced by:

$$-\deg_A(x_j) := \mathbf{a}_j =: \deg_A(\partial_j).$$

We denote $\deg_{A,i}(-)$ the $A$-degree function associated to the weight given by the $i$-th row of $A$, so $\deg_A = (\deg_{A,1}, \ldots, \deg_{A,d})$.

An $R_A$- (resp. $D_A$-)module $M$ is *$A$-graded* if it has a decomposition $M = \bigoplus_{\alpha\in\mathbb{Z}^d} M_\alpha$ such that the module structure respects the grading $\deg_A(-)$ on $R_A$ (resp. $D_A$) and $M$. If $N$ is an $A$-graded $R_A$-module, then we denote $\deg_A(N) \subseteq \mathbb{Z}^d$ the set of all degrees of all non-zero homogeneous elements of $N$. The *quasi-degrees* $\operatorname{qdeg}_A(N)$ of $N$ are the points in the Zariski closure in $\mathbb{C}^d$ of $\deg_A(N)$.

As is common, if $M$ is $A$-graded then $M(\mathbf{b})$ denotes for each $\mathbf{b} \in \mathbb{Z}A$ its shift with graded structure $(M(\mathbf{b}))_{\mathbf{b}'} = M_{\mathbf{b}+\mathbf{b}'}$.

### 1.3 Euler–Koszul complex

Since

$$\begin{aligned} x^{\mathbf{u}}E_i - E_i x^{\mathbf{u}} &= -(A\cdot\mathbf{u})_i x^{\mathbf{u}},\\ \partial^{\mathbf{u}}E_i - E_i\partial^{\mathbf{u}} &= (A\cdot\mathbf{u})_i \partial^{\mathbf{u}}, \end{aligned}$$

we have

$$E_i P = P(E_i - \deg_{A,i}(P)) \tag{1.3}$$

for any $A$-homogeneous $P \in D_A$ and all $i$.

On the $A$-graded $D_A$-module $M$ one can thus define commuting $D_A$-linear endomorphisms $E_i$ via

$$E_i \circ m := (E_i + \deg_{A,i}(m)) \cdot m$$

for $A$-homogeneous elements $m \in M$. In particular, if $N$ is an $A$-graded $R_A$-module one obtains commuting sets of $D_A$-endomorphisms on the left $D_A$-module $D_A \otimes_{R_A} N$ by

$$E_i \circ (P \otimes Q) := (E_i + \deg_{A,i}(P) + \deg_{A,i}(Q))P \otimes Q.$$

The *Euler–Koszul complex* $\mathscr{K}_\bullet(N;\beta)$ of the $A$-graded $R_A$-module $N$ is the homological Koszul complex induced by $E - \beta := \{(E_i - \beta_i)\circ\}_1^d$ on $D_A \otimes_{R_A} N$. In particular, the terminal module $D_A \otimes_{R_A} N$ sits in homological degree zero. We denote the homology groups of $\mathscr{K}_\bullet(N;\beta)$ by $\mathscr{H}_\bullet(N;\beta)$. Implicit in the notation is "$A$": different presentations of semigroup rings that act on $N$ yield different Euler–Koszul complexes.

If $N(\mathbf{b})$ denotes the usual shift-of-degree functor on the category of graded $R_A$-modules, then $\mathscr{K}_\bullet(N;\beta)(\mathbf{b})$ and $\mathscr{K}_\bullet(N(\mathbf{b});\beta - \mathbf{b})$ are identical.

### 1.4 The toric category

There is a bijection between faces $\tau$ of the cone $\mathbb{R}_{\geq 0}A$ and $A$-graded prime ideals $I_A^\tau = I_A + R_A\{\partial_j \mid j \notin \tau\}$ of $R_A$ containing $I_A$. If the origin is a face of $\mathbb{R}_{\geq 0}A$, it corresponds to the ideal $I_A^\emptyset = (\partial_0, \ldots, \partial_n)$. In general, $R_A/I_A^\tau \cong \mathbb{C}[\mathbb{N}\tau]$.

An $R_A$-module $N$ is *toric* if it is $A$-graded and has a (finite) $A$-graded composition chain

$$0 = N_0 \subsetneq N_1 \subsetneq N_2 \cdots \subsetneq N_k = N$$

such that each composition factor $N_i/N_{i-1}$ is isomorphic as $A$-graded $R_A$-module to an $A$-graded shift $(R_A/I_A^\tau)(\mathbf{b})$ for some $\mathbf{b} \in \mathbb{Z}A$ and some face $\tau$. The category of toric modules is closed under the formation of subquotients and extensions.

For toric input $N$, the modules $\mathscr{H}_\bullet(N;\beta)$ are holonomic. As $D_A$ is $R_A$-free, any short exact sequence $0 \longrightarrow N' \longrightarrow N \longrightarrow N'' \longrightarrow 0$ of $A$-graded $R_A$-modules produces a long exact sequence of Euler–Koszul homology. If $\beta$ is not a quasi-degree of $N$ then the complex $\mathscr{K}_\bullet(N;\beta)$ is exact, and if $N$ is a maximal Cohen–Macaulay module then $\mathscr{K}_\bullet(N;\beta)$ is a a resolution of $\mathscr{H}_0(N;\beta)$.

### 1.5 The Euler space

**Notation 1.2.** The $\mathbb{C}$-linear span of the Euler operators $\{E_i\}_1^d$ is called the *Euler space*. Let $E$ be in the Euler space. Then $E$ is in a unique fashion (as $\mathrm{rk}(A) = d$) a linear combination $E = \sum c_i E_i$. With $\beta_E := \sum c_i \beta_i$ we have $E - \beta_E \in H_A(\beta)$. We further write $\deg_E(-)$ for the degree function $\sum c_i \deg_{A,i}(-)$. □

Denote $\theta_j = x_j \partial_j$ and $\theta = (\theta_0, \ldots, \theta_n)$. A linear combination $\sum_j v_j \theta_j$ is in the Euler space if and only if the coefficient vector $\mathbf{v} = (v_0, \ldots, v_n)$, interpreted as a linear functional on $\mathbb{C}^{n+1}$ via $\mathbf{v}((q_0, \ldots, q_n)) := \sum v_i q_i$, is the pull-back via $A$ of a linear functional on $\mathbb{C}^d$. In other words,

$$[\mathbf{v} \cdot \theta^T = \sum_j v_j \theta_j \text{ is in the Euler space}] \Leftrightarrow [\mathbf{v} = \mathbf{c} \cdot A \text{ for some } \mathbf{c} \in \mathbb{C}^d].$$

If $L\colon \mathbb{C}^d \longrightarrow \mathbb{C}$ is a linear functional then the Euler operator in $H_A(\beta)$ corresponding to its image under $\mathrm{Hom}_{\mathbb{C}}(\mathbb{C}^d, \mathbb{C}) \xrightarrow{\cdot A} \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}^{n+1}, \mathbb{C})$ is denoted $E_L - \beta_L$.

**Lemma 1.3.** *For any set $F$ of columns of $A$ contained in a hyperplane that passes through the origin of $\mathbb{C}^d$ but does not contain $\mathbf{a}_k$, there is an Euler operator $E_F - \beta_F$ in $H_A(\beta)$ such that the coefficient of $\theta_j$ in $E_F$ is zero for all $j \in F$, and equal to 1 for $j = k$. If $\mathbb{R}_{\geq 0}F$ is a facet of $\mathbb{R}_{\geq 0}A$ then $E_F - \beta_F$ is unique.* □

**Proof**. Choose for any such set $F$ a linear functional $L\colon \mathbb{Q}^d \longrightarrow \mathbb{Q}$ that vanishes on $F$ while $L(\mathbf{a}_k) = 1$. The corresponding Euler operator $E_L - \beta_L$ has the desired properties, and if we define numbers $a_{L,j}$ by

$$E_L =: \sum_j a_{L,j} x_j \partial_j$$

then $a_{L,j} = L(\mathbf{a}_j)$. The uniqueness in the facet case is obvious. ■

## 2 Restricting the Fourier transform

The Fourier transform $\mathscr{F}(-)$ is a functor from the category of $D$-modules on $V$ to the category of $D$-modules on the dual space $V^* = \mathrm{Hom}_{\mathbb{C}}(V, \mathbb{C})$. In this section we bound the $b$-function along a coordinate hyperplane of the Fourier transform $\mathscr{F}(M_A(\beta))$ of the hypergeometric system. Note that this module is called $\check{M}_A^\beta$ in Reichelt and Sevenheck [2015].

The square of the Fourier transform is the involution induced by $x \mapsto -x$, which has no effect on the analytic properties of the modules we study. In particular, $b$-functions along coordinate hyperplanes are unaffected by this involution and we therefore consider $\mathscr{F}^{-1}(M_A(\beta))$ without harm.

We start with introducing some notation.

**Notation 2.1.** Let $\{y_j\}$ be the coordinates on $V^*$ such that $\mathscr{F}^{-1}(\partial_j) = y_j$ on the level of differential operators. We let $\tilde{D}_A$ be the ring of $\mathbb{C}$-linear differential operators on $\tilde{O}_A := \mathbb{C}[y_0, \ldots, y_n]$, generated by $\{y_j, \delta_j\}_0^n$ where $\delta_j$ denotes $\frac{\partial}{\partial y_j}$. Then $\mathscr{F}^{-1}(x_j) = -\delta_j$. The subring $\mathbb{C}[\delta_1, \ldots, \delta_n]$ of $\tilde{D}_A$ is denoted $\tilde{R}_A$. The isomorphism $(\tilde{-})\colon D_A \longrightarrow \tilde{D}_A$ induced by $\tilde{\partial}_j := y_j$ and $\tilde{x}_j = \delta_j$ sends $O_A$ to $\tilde{R}_A$ and $R_A$ to $\tilde{O}_A$.

Thus, $\tilde{I}_A := \mathscr{F}^{-1}(I_A)$ is an ideal of $\tilde{O}_A$; the advantage of considering $\mathscr{F}^{-1}$ rather than $\mathscr{F}$ is that $\tilde{I}_A$ retains the shape of the generators of $I_A$ as differences of monomials. For each $j$ set $\tilde{\theta}_j := \mathscr{F}^{-1}(\theta_j) = -\delta_j y_j$. The $i$-th level $V$-filtration on $\tilde{D}_A$ along $y_t$ is spanned by $\delta^\alpha y^\beta$ with $\beta_t - \alpha_t \geq i$. □

Before we get into the technical part, let us show by example an outline of what is to happen.

*Example* 2.2. Let $A = \begin{pmatrix} -1 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}$, a matrix whose associated semigroup ring is a normal complete intersection. We will estimate the $b$-function for restriction to the hyperplane $y_1 = 0$ (corresponding to the middle column) of $\mathscr{F}^{-1}(M_A(\beta))$.

The ideal $\tilde{H}_A(\beta) := \mathscr{F}^{-1}(H_A(\beta))$ is generated by

$$-\tilde{\theta}_0 + \tilde{\theta}_2 - \beta_1, \qquad \tilde{\theta}_0 + \tilde{\theta}_1 + \tilde{\theta}_2 - \beta_2, \qquad y_0 y_2 - y_1^2. \tag{2.1}$$

Since $y_1 \in (V^1\tilde{D}_A)$, $y_0 y_2$ and hence also $\tilde{\theta}_0\tilde{\theta}_2$ are in $(V^1\tilde{D}_A) + \tilde{H}_A(\beta)$. The strategy of the example, and of the theorem in this section, is to multiply the element $1 \in \tilde{D}_A$ by suitable Euler operators so that the result is a sum of a polynomial $p(\tilde{\theta}_1)$ with an element of $\mathbb{C}[\tilde{\theta}_0, \tilde{\theta}_1, \tilde{\theta}_2] \cdot \tilde{\theta}_0\tilde{\theta}_2$; this certifies $p(\tilde{\theta}_1)$ to be in $\tilde{H}_A(\beta) + (V^1\tilde{D}_A)$.

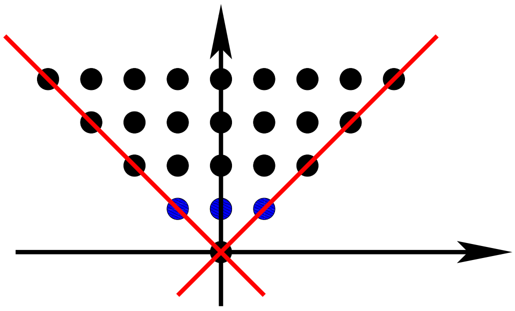

Fig. 1: Restriction of the Fourier transform to $y_1 = 0$.

In the case at hand, the relevant Euler operators are $2\tilde{\theta}_0 + \tilde{\theta}_1 + \beta_1 - \beta_2$ and $\tilde{\theta}_1 + 2\tilde{\theta}_2 - \beta_1 - \beta_2$. Modulo $\tilde{H}_A(\beta)$ we can rewrite $(V^1\tilde{D}_A) \ni 4\delta_0\delta_2 y_1^2 \equiv 4\tilde{\theta}_0\tilde{\theta}_2 \equiv (-\tilde{\theta}_1 - \beta_1 + \beta_2)(-\tilde{\theta}_1 + \beta_1 + \beta_2)$. It follows that $(\tilde{s} + \beta_1 - \beta_2)(\tilde{s} - \beta_1 - \beta_2)$ is a multiple of the $b$-function, where $\tilde{s} = \tilde{\theta}_1 = -y_1\delta_1 - 1$. This Fourier twist in the argument of the $b$-function occurs naturally throughout and we will make our computations in this section in terms of $b(\tilde{s})$.

The expressions $\tilde{\theta}_1 + 2\tilde{\theta}_2$ and $2\tilde{\theta}_0 + \tilde{\theta}_1$ that appear in the Euler operators we used can be found systematically as follows. Let $d_1, d_2$ denote the coordinates on the degree group $\mathbb{Z}^2$ corresponding to $E_1$ and $E_2$; compare the discussion following Notation 1.2. An element of $S_A$ has degree on the facet $\mathbb{R}_{\geq 0}\mathbf{a}_0$ if and only if the functional $L_1(d_1, d_2) = d_1 + d_2$ vanishes, and the Euler field that corresponds to this functional in the spirit of Lemma 1.3 is exactly $\theta_1 + 2\theta_2 - \beta_1 - \beta_2$. The elements of $S_A$ with degree on the facet $\mathbb{R}_{\geq 0}\mathbf{a}_2$ are determined by the vanishing of $L_2(d_1, d_2) = d_2 - d_1$ and the Euler field corresponding to this functional is exactly $2\theta_0 + \theta_1 + \beta_1 - \beta_2$. It is no coincidence that the union of the kernels of these two functionals is exactly the set of quasi-degrees of $S_A/\partial_1 \cdot S_A$. The point is that modulo $\tilde{H}_A(\beta)$ all monomials in $\tilde{S}_A$ with degree in $\mathbb{R}_+ A$ are already in $(V^1\tilde{D}_A)$. The task is then to deal with those with degree on the boundary through multiplication with suitable expressions.

The picture shows in blue the elements of $A$, in black the other elements of $\mathbb{N}A$, and in red the quasi-degrees of $S_A/\partial_1 \cdot S_A$. Note finally that $(\beta_2 - \beta_1)\mathbf{a}_1$ and $(\beta_1 + \beta_2)\mathbf{a}_1$ are the intersections of $\mathbb{R} \cdot \mathbf{a}_1$ with $\operatorname{qdeg}_A(S_A) + \beta$. $\square$

We now generalize the computation of the example to the general case.

**Convention 2.3.** *For the remainder of this section we consider restriction to the hyperplane $y_0$ in order to save overhead (in terms of a further index variable).* $\square$

Consider the toric module $N = S_A/\partial_0 S_A$, and take a toric filtration

$$(N) \qquad 0 = N_0 \subsetneq N_1 \subsetneq \ldots \subsetneq N_k = N$$

with composition factors

$$\overline{N}_\alpha := N_\alpha / N_{\alpha-1},$$

each isomorphic to some shifted face ring $S_{F'_\alpha}(\mathbf{b}_\alpha)$, $F'_\alpha = \tau_\alpha \cap A$, attached to a face $\tau_\alpha$ of $\mathbb{R}_{\geq 0}A$. (We will call such $F'_\alpha$ also a face.) Lifting the $N_\alpha$ to $S_A$ yields an increasing sequence of $A$-graded ideals $J_\alpha \ni \partial_0$ of $S_A$ with $N_\alpha = J_\alpha/\partial_0 \cdot S_A$.

Choose for each composition factor a facet $F_\alpha$ containing $F'_\alpha$. Note that none of the faces $F'_\alpha$ will contain $\mathbf{a}_0$ (as $\partial_0$ is zero on $N$ but not nilpotent on any face ring of a face containing $\mathbf{a}_0$) and hence we can arrange that the corresponding facets do not contain $\mathbf{a}_0$ either.

Lemma 1.3 produces for each $\overline{N}_\alpha$ a facet $F_\alpha$ and corresponding functional $L_{F_\alpha}$ (which we abbreviate to $L_\alpha$) that vanishes on the facet and evaluates to 1 on $\mathbf{a}_0$. The associated Euler operator in $H_A(\beta)$ is $E_{F_\alpha} - \beta_{F_\alpha}$. Since $L_\alpha$ is zero on all $A$-columns in $F_\alpha$ and since $\overline{N}_\alpha$ is a shifted quotient of $S_{F_\alpha}$, there is a unique value for $L_\alpha$ on the $A$-degrees of all nonzero $A$-homogeneous elements of $\overline{N}_\alpha$. We denote this value by $L_\alpha(\overline{N}_\alpha)$. Note, however, that $L_\alpha(\overline{N}_\alpha)$ does very much depend on the choice of the facet $F_\alpha$ even though the notation does not remember this.

Now let $T_\alpha$ be the image in $\mathscr{F}^{-1}(M_A(\beta))$ of $\mathscr{F}^{-1}(J_\alpha)$ under the map induced by $\tilde{O}_A \longrightarrow \tilde{D}_A \longrightarrow \mathscr{F}^{-1}(M_A(\beta))$. Note that the image of $T_0 = y_0\tilde{O}_A$ in $\mathscr{F}^{-1}(M_A(\beta))$ is in $(V^1\tilde{D}_A) \cdot \overline{1}$, the bar denoting cosets in $\mathscr{F}^{-1}(M_A(\beta))$.

**Lemma 2.4.** *In the context above, let $\kappa_\alpha$ be the constant $L_\alpha(\overline{N}_\alpha)$. Then in $\mathscr{F}^{-1}(M_A(\beta))$, modulo the image of $(V^1\tilde{D}_A)$,*

$$(\tilde{\theta}_0+\kappa_\alpha-\beta_\alpha)\cdot(V^0\tilde{D}_A)\cdot T_\alpha=(V^0\tilde{D}_A)\cdot(\tilde{\theta}_0+\kappa_\alpha-\beta_\alpha)\cdot T_\alpha\subseteq(V^0\tilde{D}_A)\cdot T_{\alpha-1}.$$

□

**Proof**. Since the commutators $[\tilde{\theta}_0,(V^0\tilde{D}_A)]$ are in $(V^1\tilde{D}_A)$, it suffices to show that $(\tilde{\theta}_0+\kappa_\alpha-\beta_\alpha)\cdot T_\alpha\subseteq(V^0\tilde{D}_A)\cdot T_{\alpha-1}$ modulo $\mathscr{F}^{-1}(H_A(\beta))$.

By definition, $\tilde{E}_\alpha-\beta_\alpha:=\mathscr{F}^{-1}(E_\alpha-\beta_\alpha)$ is zero in $\mathscr{F}^{-1}(M_A(\beta))$. Take a monomial $\tilde{m}\in\tilde{O}_A$ whose coset lies in $T_\alpha\setminus T_{\alpha-1}$. By Equation (1.3), $\tilde{E}_\alpha\cdot\tilde{m}=\tilde{m}(\tilde{E}_\alpha-\kappa_\alpha)$ since $\mathscr{F}^{-1}(-)$ is a homomorphism. Now write $E_\alpha=\sum a_{\alpha,j}\theta_j$; as before we have $a_{\alpha,j}=L_\alpha(\mathbf{a}_j)$.

Since the coefficient of $\theta_0$ in $E_\alpha$ is 1, it follows that in $\mathscr{F}^{-1}(M_A(\beta))$:

$$\begin{aligned}\tilde{\theta}_0\tilde{m}&=(-\tilde{E}_\alpha+\tilde{\theta}_0)\tilde{m}+\tilde{E}_\alpha\tilde{m}\\&=\sum_{\substack{j\neq0\\L_\alpha(\mathbf{a}_j)\neq0}}a_{\alpha,j}\delta_jy_j\tilde{m}+\tilde{m}(\tilde{E}_\alpha-\kappa_\alpha)\\&=\sum_{\substack{j\neq0\\\mathbf{a}_j\notin F_\alpha}}a_{\alpha,j}\delta_jy_j\tilde{m}+\tilde{m}(\beta_\alpha-\kappa_\alpha).\end{aligned}$$

Recall that $F_\alpha$ contains $F'_\alpha$ and that $\overline{N}_\alpha$ is a $\mathbb{Z}A$-shift of $S_{F'_\alpha}=R_A/I^\tau_A$, whence each $y_j$ with $\mathbf{a}_j\notin F'$ annihilates $\mathscr{F}^{-1}(\overline{N}_\alpha)$. Therefore, each term $a_{\alpha,j}\delta_j(y_jm)$ in the last sum of the display is in $(V^0D_A)T_{\alpha-1}$. It follows that in $\mathscr{F}^{-1}(M_A(\beta))$ we have $(\tilde{\theta}_0+\kappa_\alpha-\beta_\alpha)T_\alpha\subseteq(V^0\tilde{D}_A)T_{\alpha-1}$ as claimed. ■

**Theorem 2.5.** *For $t=0,\ldots,n$, the number $\varepsilon\in\mathbb{C}$ is a root of the b-function $b(\tilde{s})$ (with $\tilde{s}=\tilde{\theta}_t=-\delta_ty_t$) of $\mathscr{F}^{-1}(M_A(\beta))$ along $y_t=0$, only if $\varepsilon\cdot\mathbf{a}_t$ is a point of intersection of the line $\mathbb{C}\cdot\mathbf{a}_t$ with the set $\beta-\operatorname{qdeg}_A(N)$, the quasi-degrees of the toric module $N=S_A/\partial_tS_A$ multiplied by $-1$ and shifted by $\beta$.*

□

**Proof**. Without loss of generality we shall suppose that $t=0$ by way of re-indexing.

We will show that a divisor of $\prod_\alpha(\tilde{\theta}_0+\kappa_\alpha-\beta_\alpha)$ is inside $H_A(\beta)+(V^1\tilde{D}_A)$, in notation from the previous lemma.

Indeed, it follows from Lemma 2.4 that $\prod_\alpha(\tilde{\theta}_0+\kappa_\alpha-\beta_\alpha)$ multiplies $\overline{1}\in\mathscr{F}^{-1}(M_A(\beta))$ into $(V^0\tilde{D}_A)\cdot y_0\cdot\overline{1}\subseteq(V^1\tilde{D}_A)\cdot\overline{1}$. Hence the root set of the $b$-function $b(\tilde{\theta}_0)$ in question is a subset of $\{\beta_\alpha-\kappa_\alpha\}$, $\alpha$ running through the indices of the chosen composition series of $N$. This set is determined by the composition series $(N)$ and the choices of the facets $F_\alpha$ for each $N_\alpha$. Varying over all choices of facets $\{F_\alpha\}$ for a given chain $(N)$, the root set of $b(\tilde{\theta}_0)$ is in the intersection $\rho_N$ of all possible sets $\{\beta_\alpha-\kappa_\alpha\}_{\alpha\in(N)}$.

Since $L_\alpha(\mathbf{a}_0)=1$, the point $(\beta_\alpha-\kappa_\alpha)\cdot\mathbf{a}_0$ is the intersection of the hyperplane $L_\alpha=\beta_\alpha-\kappa_\alpha$ with the line $\mathbb{C}\cdot\mathbf{a}_0$. Thus, $\rho_N$ is inside the intersection of $\mathbb{C}\cdot\mathbf{a}_0$ with all arrangements $\operatorname{Var}\prod_\alpha(L_\alpha-\beta_\alpha+\kappa_\alpha)$. The intersection of the arrangements $\operatorname{Var}\prod_\alpha(L_\alpha-\beta_\alpha+\kappa_\alpha)$ is the union of the quasi-degrees of all $\overline{N}_\alpha$ of the composition chain $(N)$, multiplied by $-1$ and shifted by $-\beta_\alpha$. As $N$ is finitely generated, $\operatorname{qdeg}_A(N)=\bigcup_\alpha\operatorname{qdeg}_A(\overline{N}_\alpha)$. Hence the root set of $b(\tilde{\theta}_0)$ is contained in the intersection $-\operatorname{qdeg}_A(S_A/\partial_0S_A)+\beta$ with $\mathbb{C}\cdot\mathbf{a}_0$.

■

*Remark* 2.6. The quantity $\tilde{\theta}_t$ is the more natural argument for the $b$-function here. Note that the roots of $b(y_t\delta_t)$ are those of $b(\tilde{\theta}_t)$ shifted up by 1 and then multiplied by $-1$. □

*Example* 2.7. Let $A=(\mathbf{a}_0,\mathbf{a}_1,\mathbf{a}_2)=\begin{pmatrix}-1&0&3\\1&1&1\end{pmatrix}$ and $\beta=\binom{\beta_1}{\beta_2}$. The ring $S_A$ is a complete intersection but not normal.

Consider restriction to $y_1=0$ (the middle column). Then $N=S_A/\partial_1\cdot S_A$ has a toric filtration involving 4 steps, given by the ideals $0\subsetneq\partial_0^3\cdot N\subsetneq\partial_0^2\cdot N\subsetneq\partial_0\cdot N\subsetneq N$. The corresponding $A$-graded composition factors are $S_A(-3\cdot\mathbf{a}_0)/(\partial_1,\partial_2)S_A$ and $\{S_A(-\alpha\cdot\mathbf{a}_0)/(\partial_0,\partial_1)S_A\}_{\alpha=0}^2$. The $b$-function $b(\tilde{\theta}_1)$ for the inverse Fourier transform is $(\tilde{\theta}_1-\beta_1-\beta_2)\prod_{\alpha=0}^2(\tilde{\theta}_1-\frac{3\beta_2-\beta_1-4\alpha}{3})$.

Explicitly, $y_1^4-y_0^3y_2\in\tilde{H}_A(\beta)$ gives $(V^1\tilde{D}_A)\ni\delta_0^3\delta_2y_0^3y_2=\tilde{\theta}_2\tilde{\theta}_0(\tilde{\theta}_0-1)(\tilde{\theta}_0-2)$ which modulo $\tilde{H}_A(\beta)$ equals $(-1)^4(\tilde{\theta}_1-\beta_1-\beta_2)\prod_{\alpha=0}^2(\tilde{\theta}_1-\frac{3\beta_2-\beta_1-4\alpha}{3})$. The relevant Euler operators are $\theta_1+4\theta_2-\beta_1-\beta_2$ and $3\theta_1+4\theta_0-3\beta_2+\beta_1$.

The picture shows in blue the columns of $A$, in black the other elements of $\mathbb{N}A$, in red the quasi-degrees of $N=S_A/\partial_1\cdot S_A$. The roots of $b(\delta_1y_1)$ (which are opposite to the roots of $b(\tilde{\theta}_1)$) are the intersections of the line $\mathbb{C}\cdot\binom{0}{1}$ with the shift of the red lines by $-\beta$.

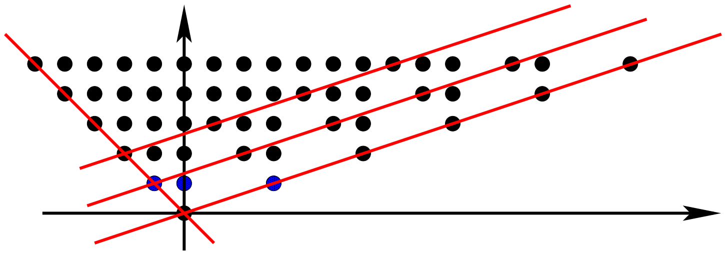

Fig. 2: Restriction of the Fourier transform to $y_1 = 0$.

In this example, each composition factor corresponds to facet and to a component of the quasi-degrees of $N$. One checks that each composition chain must have these four lines as quasi-degrees. Note, however, that composition chains are far from unique and in general such correspondence will not exist. $\square$

*Remark* 2.8. The $b$-function for $\mathscr{F}^{-1}(M_A(\beta))$ along a coordinate hyperplane is generally not reduced, and its degree may be lower than the length of the shortest toric filtration for $N = S_A/\partial_t \cdot S_A$ would suggest. (Not every component of $\beta - \mathrm{qdeg}_A(N)$ needs to meet the line $\mathbb{C} \cdot \mathbf{a}_t$). $\square$

**Corollary 2.9.** *The roots of the b-function $b(\delta_t y_t)$ of $\mathscr{F}^{-1}(M_A(\beta))$ along $y_t = 0$ are in the field $\mathbb{Q}(\beta)$.*
*Consider $\mathscr{F}^{-1}(M_A(0))$; then:*
1. *the roots of the b-function $b(\tilde{\theta}_t)$ are non-negative rationals;*
2. *if $S_A$ is normal, all roots are in the interval $[0, 1)$;*
3. *if the interior ideal of $S_A$ is contained in $\partial_t \cdot S_A$ then zero is the only root.*

$\square$

**Proof**. The first claim is a consequence of the intersection property in Theorem 2.5: the defining equations for the quasi-degrees are rational.

Let $N = S_A/\partial_t S_A$. For items 1.-3., we need to study the intersection of $\mathrm{qdeg}_A(N)$ with $\mathbb{C} \cdot \mathbf{a}_t$, since $\beta = 0$ and $\delta_t y_t = -\tilde{\theta}_t$. The quasi-degrees of $N$ are covered by hyperplanes of the sort $L_\alpha = \varepsilon$ where $L_\alpha$ is a rational supporting functional of the facet $F_\alpha$. In particular, we can arrange $L_\alpha$ to be zero on $F_\alpha$, positive on the rest of $A$, and $L_\alpha(\mathbf{a}_t) = 1$. As $\deg_A(N) \subseteq \deg_A(S_A)$, $\varepsilon \geq 0$. Hence $\mathrm{Var}(L_\alpha - \varepsilon)$ meets $\mathbb{C} \cdot \mathbf{a}_t$ in the non-negative rational multiple $\varepsilon \mathbf{a}_t$ of $\mathbf{a}_t$. If $S_A$ is normal, $\deg_A(S_A/\partial_A S_A)$ is covered by hyperplanes $\mathrm{Var}(L_\alpha - \varepsilon)$ that do not meet the cone $\mathbf{a}_t + \mathbb{R}_{\geq 0} A$. These are precisely the ones for which $\varepsilon < 1$.

If $\partial_t \cdot S_A$ contains the interior ideal then $\deg_A(N)$, and hence $\mathrm{qdeg}_A(N)$, is inside the supporting hyperplanes of the cone, which meet $\mathbb{C} \cdot \mathbf{a}_t$ at the origin. $\blacksquare$

*Remark* 2.10. One special case in which case 3 of Corollary 2.9 applies is when $S_A$ is Gorenstein and where further $\partial_t$ generates the canonical module. The matrix $A = (\mathbf{a}_0, \ldots, \mathbf{a}_3) = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$, with the interior ideal being generated by $\partial_1 \partial_3$, provides an example that case (3) can occur in a Gorenstein situation without the boundary of $\mathbb{N}A$ being saturated. See Schulze and Walther [2009a] for a discussion on Cohen–Maculayness of face rings of Cohen–Macaulay semigroup rings. $\square$

## 3 $b$-functions for the hypergeometric system

### 3.1 Restriction along a hyperplane

We are here interested in the $b$-function for the hypergeometric module $M_A(\beta)$ along the hyperplane $x_t = 0$. As in the previous section, apart from examples, we actually carry out all computations for $t = 0$, in order to have as few variables around as possible. On the other hand, the natural argument for expressing the $b$-function will be $s = x_0 \partial_0$.

**Notation 3.1.** With $A = (\mathbf{a}_0, \ldots, \mathbf{a}_n)$ and distinguished index 0, we denote $A' := (\mathbf{a}_1, \ldots, \mathbf{a}_n)$. Via $\mathbb{N}A' \subseteq \mathbb{N}A$ we consider $S_{A'}$ as a subring of $S_A$.

For $k \in \mathbb{N}$ let $\overline{J}_{A,0;k} \subseteq S_{A'}$ be the vector space spanned by the monomials $\partial^{\mathbf{u}}$ with $u_0 = 0$ (so that $\partial^{\mathbf{u}} \in S_{A'}$) that satisfy $\partial_0^k \cdot \partial^{\mathbf{u}} \in S_{A'}$. We denote $J_{A,0;k} \subseteq R_{A'}$ the preimage of $\overline{J}_{A,0;k}$ under the natural surjection $R_{A'} \twoheadrightarrow S_{A'}$. Put $J_{A,0} = \sum_{k \geq 1} J_{A,0;k}$ and $\overline{J}_{A,0} = J_{A,0}/I_{A'} \subseteq S_{A'}$ . $\square$

Each $\overline{J}_{A,0;k}$ is a monomial ideal of $S_{A'}$ since $\partial_0^k(\partial^{\mathbf{v}}\partial^{\mathbf{u}}) = \partial^{\mathbf{v}}(\partial_0^k\partial^{\mathbf{u}})$. Note, however, that $\overline{J}_{A,0;k}$ need not be contained in $\overline{J}_{A,0;k+1}$. If $\mathbf{a}_0 \in \mathbb{R}_{\geq 0}A'$ then some power of $\partial_0$ is in $S_{A'}$ and so $\overline{J}_{A,0} = S_{A'}$.

**Definition 3.2.** For $\mathbf{a}_0 \in \mathbb{R}^d$ outside $\mathbb{R}_{\geq 0}A'$, a point $\mathbf{a} \in \mathbb{R}_{\geq 0}A'$ is $\mathbf{a}_0$-*visible* if $\mathbf{a} + \lambda \cdot \mathbf{a}_0$, $0 < \lambda \ll 1$ is outside $\mathbb{R}_{\geq 0}A'$. (The idea behind the choice of language is that the observer stands at the point of projective space given by the line $\mathbb{R}\mathbf{a}_0$.)

By abuse of notation, we say that $\partial^{\mathbf{a}}$ is $\mathbf{a}_0$-visible if $\mathbf{a}$ is. □

**Lemma 3.3.** *Assume that $\mathbf{a}_0$ is not in the cone $\mathbb{R}_{\geq 0}A'$. Then the radical of $J_{A,0}$ is generated by the $\mathbf{a}_0$-invisible elements of $S_{A'}$, and in consequence the quasi-degrees of $S_{A'}/J_{A,0}$ are a union of shifted face spans where each face is in its entirety visible from $\mathbf{a}_0$.* □

**Proof**. If $\mathbb{Z}A/\mathbb{Z}A'$ has positive rank then all points of $\mathbb{N}A$ are $\mathbf{a}_0$-visible while $J_{A,0}$ is clearly zero, so that in this case there is nothing to prove. We therefore assume that $\mathbb{Z}A/\mathbb{Z}A'$ is finite.

It is immediate that $\mathbf{a}$ is $\mathbf{a}_0$-visible if and only if any positive integer multiple of it is. This implies that no power of an $\mathbf{a}_0$-visible element $\partial^{\mathbf{a}}$ of $S_{A'}$ can be in the radical of $J_{A,0}$ since $\partial^{m\cdot\mathbf{a}+k\mathbf{a}_0}$ can't have its degree in the cone of $A'$.

For the converse, suppose $\mathbf{a}$ is not $\mathbf{a}_0$-visible, so that there are positive integers $p < q$ with $\mathbf{a} + (p/q)\cdot\mathbf{a}_0 \in \mathbb{R}_{\geq 0}A'$. Then a high power of $\partial^{q\cdot\mathbf{a}+p\cdot\mathbf{a}_0}$ is in $\mathbb{C}[\mathbb{Z}A \cap \mathbb{R}_{\geq 0}A']$ and a suitable power $\partial^{\mathbf{b}}$ of that will be in $\mathbb{C}[\mathbb{Z}A' \cap \mathbb{R}_{\geq 0}A']$ because of the finiteness of $\mathbb{Z}A/\mathbb{Z}A'$. Now let $\tau$ be the smallest face of $\mathbb{R}_{\geq 0}A'$ that contains $\mathbf{b}$; this makes $\mathbf{b}$ an interior point of $\tau$. Since $\mathbb{C}[\tau \cap \mathbb{Z}A']$ is a finitely generated $\mathbb{C}[\tau \cap \mathbb{N}A']$-module, some power of $\partial^{\mathbf{b}}$ is in $\mathbb{C}[\tau \cap \mathbb{N}A'] \subseteq S_{A'}$. This shows that some power of $\partial^{q\cdot\mathbf{a}}$ times some power of $\partial^{p\cdot\mathbf{a}_0}$ is in $S_{A'}$, establishing the first claim of the lemma.

In every composition chain for $S_{A'}/J_{A,0}$, each composition factor is an $S_{A'}/\sqrt{J_{A,0}}$-module. Thus the quasi-degrees of $S_{A'}/J_{A,0}$ are inside a union of shifted quasi-degrees of $S_{A'}/\sqrt{J_{A,0}}$ and hence all $\mathbf{a}_0$-visible, which implies the second claim. ■

Our main theorem in this section is:

**Theorem 3.4.** *The root locus of the b-function $b(x_0\partial_0)$ for restriction of $M_A(\beta)$ along $x_0 = 0$ is, up to inclusion of non-negative integers, contained in the locus of intersection $(-\operatorname{qdeg}_{A'}(S_{A'}/\overline{J}_{A,0}) + \beta) \cap \mathbb{C}\cdot\mathbf{a}_0$. The set of integers needed can be taken to be the integers $0, \ldots, k-1$ such that $J_{A,0} = \sum_{1\leq i\leq k} J_{A,0;i}$.*

*In two extreme cases one can be explicit:*

1. *if $\dim S_A - 1 = \dim S_{A'}$ then the b-function is linear with root given by the intersection of $(-\operatorname{qdeg}_A(S_{A'}) + \beta) \cap \mathbb{C}\cdot\mathbf{a}_0$;*
2. *if $\mathbf{a}_0 \in \mathbb{R}_{\geq 0}A'$ then the b-function has integer roots in $\{0, 1, \ldots, k-1\}$ where $k = \min\{t \in \mathbb{N} \mid 0 \neq t\cdot\mathbf{a}_0 \in \mathbb{N}A'\}$.*

□

**Proof**. We first dispose of the extreme cases. If $\dim S_A - 1 = \dim S_{A'}$, then $S_A$ is the polynomial ring $S_{A'}[\partial_0]$ and $A'$ is a facet of $A$. By Lemma 1.3 there is $\mathbf{v} = (v_1, \ldots, v_d)$ such that the Euler operator

$$E - \beta_E = \sum v_i(E_i - \beta_i)$$

is in $H_A(\beta)$ and equals $\theta_0 - \beta_E$. In particular, the $b$-function is $s - \beta_E$. On the other hand: $\overline{J}_{A,0}$ is zero in this case, $\mathbf{v} = (v_1, \ldots, v_d)$ is in the kernel of ${A'}^T$, and $\mathbf{a}_0^T\mathbf{v} = 1$. Therefore, the quasi-degrees of $S_{A'}/\overline{J}_{A,0}$ form the hyperplane given as the kernel of $\mathbf{v}$ and $(\mathbf{v}^T\beta)\mathbf{a}_0 = \beta_E\mathbf{a}_0$ is the intersection of $-\operatorname{qdeg}_A(S_{A'}) + \beta$ with $\mathbb{C}\mathbf{a}_0$.

If $\mathbf{a}_0 \in \mathbb{R}_{\geq 0}A'$ then $\mathbb{N}\mathbf{a}_0$ meets $\mathbb{N}A'$ and so $\partial_0^k = \partial^{\mathbf{u}}$ with $\mathbf{u} = (0, u_1, \ldots, u_n) \in \mathbb{N}A'$. In particular, $J_{A,0} = S_{A'}$ in this case. Moreover, $(x_0\partial_0)(x_0\partial_0 - 1)\cdots(x_0\partial_0 - k + 1) = x_0^k\partial_0^k = x_0^k(\partial_0^k - \partial^{\mathbf{u}}) + x_0^k\partial^{\mathbf{u}} \in H_A(\beta) + V^1(D_A)$ shows the claim made in this case.

Now suppose that $A$ and $A'$ have equal rank but $\mathbf{a}_0 \notin \mathbb{R}_{\geq 0}A'$. In that case, $\overline{J}_{A,0}$ is a non-trivial ideal of $S_{A'}$. We shall use a toric filtration

$$(N) \quad : \quad 0 = N_0 \subsetneq N_1 \subsetneq \ldots \subsetneq N_t = S_{A'}/\overline{J}_{A,0}$$

and let $J_\alpha \supseteq J_{A,0}$ be the $R_{A'}$-ideal such that $N_\alpha = J_\alpha/J_{A,0}$. We will view $J_\alpha$ as subset of $D_{A'}$ or even $D_A$. In analogy to the previous case, for any $\partial^{\mathbf{u}}$ in $J_{A,0;k}$ the $b$-function along $x_0$ of the coset of $\partial^{\mathbf{u}}$ in $M_A(\beta)$ divides $s(s-1)\cdots(s-k+1)$. Indeed, $\partial^{\mathbf{u}} \in J_{A,0;k}$ implies that $\partial_0^k\partial^{\mathbf{u}} - \partial^{\mathbf{v}} \in I_A$ for some $\mathbf{v}$ with $v_0 = 0$, and so

$x_0^k\partial_0^k\partial^{\mathbf{u}} \in H_A(\beta)+V^1(D_A)$. In particular, the root set of the $b$-function of the coset of $\partial^{\mathbf{u}}$ in $M_{A'}(\beta)$ is inside the set of integers described in the statement of the theorem.

For each composition factor $\overline{N}_\alpha = N_\alpha/N_{\alpha-1}$ choose now a facet $\tau_\alpha$ of $A'$ and an element $\partial^{\mathbf{u}_\alpha}$ of $S_{A'}$ $\mathbf{u}_\alpha \in \{0\}\times\mathbb{N}^n$ such that $N_\alpha$ is a quotient of $S_{A'}\cdot\partial^{\mathbf{u}_\alpha}$ and such that the annihilator of $\partial^{\mathbf{u}_\alpha}$ in $\overline{N}_\alpha$ contains the toric ideal $I_{A'}^{\tau_\alpha}$. Then $\operatorname{qdeg}_{A'}(\overline{N}_\alpha)$ is contained in $A'\cdot\mathbf{u}_\alpha+\operatorname{qdeg}_{A'}(S_{\tau_\alpha})$.

Since $\mathbf{a}_0$ is not in $\mathbb{R}_{\geq 0}A'$, Lemma 3.3 shows that the facet $\tau_\alpha$ can be chosen such that $\mathbf{a}_0 \notin \mathbb{Q}\cdot\tau_\alpha$. Indeed, if an entire face of $\mathbb{R}_{\geq 0}A'$ is visible from $\mathbf{a}_0$ then it sits in at least one facet whose span does not contain $\mathbf{a}_0$. By Lemma 1.3 there is an element $E_\alpha$ of the Euler space of $A$ that does not involve any element of $\tau_\alpha$, but which has coefficient 1 for $\theta_0$. Notation 1.2 then associates a degree function $\deg_{E_\alpha}(-)$ to $\alpha$.

As $\partial_j\cdot\partial^{\mathbf{u}_\alpha}\in N_{\alpha-1}$ for $j\notin\tau_\alpha$ it follows that the difference of $(E_\alpha-\beta_\alpha)\cdot\partial^{\mathbf{u}_\alpha}$ and $(\theta_0-\beta_\alpha)\cdot\partial^{\mathbf{u}_\alpha}$ is inside $(V^0D_A)N_{\alpha-1}$. Since $E_\alpha-\beta_\alpha$ is in $H_A(\beta)$, so is $\partial^{\mathbf{u}_\alpha}(E_\alpha-\beta_\alpha)=(E_\alpha-\beta_\alpha+\deg_{E_\alpha}(\partial^{\mathbf{u}_\alpha}))\partial^{\mathbf{u}_\alpha}$. Therefore, $(\theta_0-\beta_\alpha+\deg_{E_\alpha}(\partial^{\mathbf{u}_\alpha}))\partial^{\mathbf{u}_\alpha}$ is in $H_A(\beta)+(V^0D_A)N_{\alpha-1}$. Then, in parallel to how Lemma 2.4 was used in the proof of Theorem 2.5, the product

$$\prod_\alpha(\theta_0-\beta_\alpha+\deg_{E_\alpha}(\partial^{\mathbf{u}_\alpha}))$$

multiplies $1\in D_A$ into $H_A(\beta)+(V^0D_A)J_{A,0}+(V^1D_A)$. Multiplying by $x_0^k\partial_0^k$ for suitable $k$ one obtains the desired bound for the $b$-function as in the second paragraph of the proof.

It follows as in Theorem 2.5 (with the modification that we have here $\theta_0$ rather than $\mathscr{F}^{-1}(\theta_0)$, which affects signs) that the intersection of the roots of all such bounds is the intersection of $(-\operatorname{qdeg}_{A'}(S_{A'}/\overline{J}_{A,0})+\beta)$ with the line $\mathbb{C}\cdot\mathbf{a}_0$. ∎

*Example* 3.5. With $A=(\mathbf{a}_0,\mathbf{a}_1,\mathbf{a}_2)=\begin{pmatrix}-1&0&3\\1&1&1\end{pmatrix}$, consider the $b$-function along $x_1$ of the $A$-hypergeometric system. The ideal $J_{A,1}$ is generated by $1\in S_{A'}=\mathbb{C}[\mathbb{N}(\mathbf{a}_0,\mathbf{a}_2)]$ since $\partial_1^4$ is in $S_{A'}$. The set of necessary integer roots is then $\{0,1,2,3\}$. No other roots are needed since $S_A/J_{A,1}$ is zero, irrespective of $\beta$.

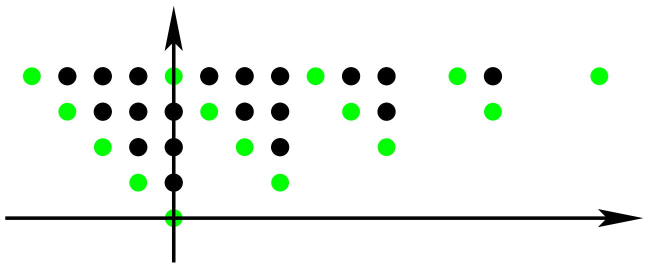

Fig. 3: The elements of $S_A\smallsetminus S_{A'}$ (black) and $S_{A'}$ (green) for restriction to $x_1$

Restriction to $(x_2=0)$ behaves differently. As $S_{A'}=\mathbb{C}[\mathbb{N}(\mathbf{a}_0,\mathbf{a}_1)]$ now, $J_{A,2}=J_{A,2;1}$ is generated by $\partial_0^3$, and the quasi-degrees of $S_{A'}/J_{A,2}$ are the lines $\mathbb{C}\cdot(0,1)+(i,0)$ with $i=0,-1,-2$. The intersection of the negative

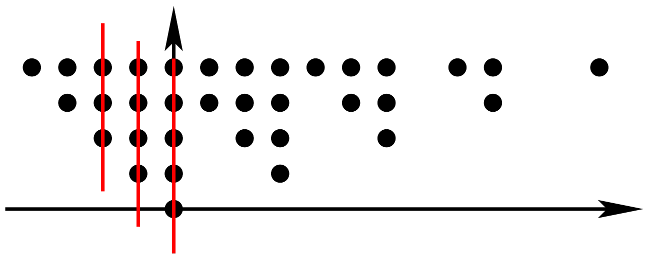

Fig. 4: The quasi-degrees of $S_A/J_{A,2}$ form three parallel lines.

of these three lines, shifted by $\beta$, with the line $\mathbb{C}\cdot\mathbf{a}_2$ is $\mathbf{a}_2\cdot\{(i+\beta_1)/3\}_{i=0,1,2}$. So the $b$-function has (at worst) roots $\{0,\beta_1,\beta_1+1,\beta_1+2\}/3$. □

*Remark* 3.6. We believe that both bounds in Theorems 2.5 (as is) and 3.4 (up to integers) are sharp. □

### 3.2 Restriction to a generic point

We suppose here that $A$ is homogeneous; in other words, the Euler space contains a homothety. Let $p = (p_0, \ldots, p_n)$ be a point of $\mathbb{C}^{n+1}$. We wish to estimate here the $b$-function for restriction of $M_A(\beta)$ to the point $-p$ if $p$ is generic. As a holonomic module is a connection near any generic point, this restriction yields a vector space isomorphic to the space of solutions to $H_A(\beta)$ near $-p$, see Saito et al. [2000, Sec. 5.2].

**Definition 3.7.** Let $\theta_p = (x_0 + p_0)\partial_0 + \ldots + (x_n + p_n)\partial_n$ and write $\theta$ for $\theta_p$ if $p = 0$. The $b$-function for restriction of a principal $D$-module $M = D/I$ to the point $x + p = 0$ is the minimal polynomial $b_p(s)$ such that $b_p(\theta_p) \in I + (V_p^1 D)$ where $V_p^k D$ is the Kashiwara–Malgrange $V$-filtration along $\mathrm{Var}(x + p)$:

$$V_p^k D = \mathbb{C} \cdot \{(x+p)^{\mathbf{u}} \partial^{\mathbf{v}} \quad | \quad |\mathbf{u}| - |\mathbf{v}| \geq k\}.$$

□

*Remark* 3.8. 1. For any pair of manifolds $Y \subseteq X$ and given a $D$-module $M$ on $X$ one can define a $b$-function of restriction for the section $m \in M$ along $Y$ by a formula generalizing both Definition 0.1 and Definition 3.7. Kashiwara proved their existence for holonomic $M$.

2. The roots of this $b$-function here relate to restriction of solution sheaves as follows. Near a generic point $x + p = 0$, a $D$-module $M$ is a connection whose solution space has a basis consisting of a certain number of holomorphic functions. The germs of these functions form a vector space that can be identified with the dual of the 0-th homology group of $(D/(x+p)D) \otimes_D^L M$. Filtering this complex by $V_p^\bullet D$, $b_p(k)$ annihilates the $k$-th graded part of its homology, compare Oaku [1997], Oaku and Takayama [2001], Walther [2000]. In particular, $b_p(s)$ carries information on the starting terms of the solution sheaf of $M$ near $x + p = 0$.

□

The purpose of this section is to bound $b_p(s)$ for $I = H_A(\beta)$ and generic $p$ with the following strategy. We first show that a polynomial $b(s)$ is a multiple of $b_p(s)$ if $b(\theta)$ is in $D_A(I_A, A \cdot \mathscr{E} \cdot \partial)$ where

$$\mathscr{E} = \begin{pmatrix} p_0 & 0 & \cdots & 0 \\ 0 & p_1 & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & p_n \end{pmatrix},$$

provided that $p$ is component-wise nonzero. The generators of $D_A(I_A, A \cdot \mathscr{E} \cdot \partial)$ are independent of $x$ and we next observe that the radical of $R_A(I_A, A \cdot \mathscr{E} \cdot \partial)$ is $R_A \cdot \partial$, provided that $p$ is generic. Thus, $b_p(s)$ will be a factor of any polynomial that annihilates the finite length module $R_A/(I_A, A \cdot \mathscr{E} \cdot \partial)$ as long as $p$ is generic. We exhibit a particular such polynomial with all roots integral. In the case of a normal semigroup ring, we show that the (necessarily integral) roots of $b_p(s)$ are in the interval $[0, d-1]$.

We begin with pointing out that $b(\theta_p) \in I + (V_p^1 D)$ is equivalent to $b(\theta) \in I_p + (V_0^1 D)$ where $I_p$ is the image of $I$ under the morphism induced by $x \mapsto x - p$, $\partial \mapsto \partial$ and $(V_0^k D)$ is the Kashiwara–Malgrange filtration along the origin. Among the generators of $I = H_A(\beta)$, only the Euler operators depend on $x$ while $(I_A)_p = I_A$ for any $p$; one has $(E_i - \beta_i)_p = \sum a_{i,j}(x_j - p_j)\partial_j - \beta_i = E_i - \beta_i - \sum a_{i,j} p_j \partial_j$. We hence seek a relation $b(\theta) \in D_A \cdot (I_A, E - \beta - A \cdot \mathscr{E} \cdot \partial) + (V_0^1 D_A)$ with $\mathscr{E}$ as above.

Generally, a statement $b(\theta) \in I + (V_0^1 D_A)$ is equivalent to $b(\theta)$ being in the degree zero part $\mathrm{gr}^0_{V_0}(I)$ of the associated graded object. Note that $\mathrm{gr}_{V_0}(D_A)$ is a Weyl algebra again (although of course the symbol map $D_A \longrightarrow \mathrm{gr}_{V_0}(D_A)$ is not an isomorphism). Abusing notation, we denote $x$ and $\partial$ also the symbols in $\mathrm{gr}_{V_0}(D_A)$ of the respective elements of $D_A$. By the previous paragraph then, the graded ideal $\mathrm{gr}_{V_0}(H_A(\beta)_p)$ contains the elements that generate $I_A$ (since $I_A$ is homogeneous!), as well as the elements $A \cdot \mathscr{E} \cdot \partial$ which arise as the $V_0$-symbols of $E_p - \beta$.

We need the following folklore result ) for which we know no explicit reference.

*Claim.* The $R_A$-ideal generated by $I_A$ and $A \cdot \mathscr{E} \cdot \partial$ has, for generic $\mathscr{E}$, radical $R_A \cdot \partial$. □

A sequence of $d$ generic linear forms is of course a system of parameters on $S_A$; the issue is to show that linear forms of the type $A \cdot \mathscr{E} \cdot \partial$ are sufficiently generic.

**Proof.** As $I_A$ and $A \cdot \mathscr{E} \cdot \partial$ are standard graded, $\mathrm{Var}(I_A, A \cdot \mathscr{E} \cdot \partial)$ is a conical variety. It thus suffices to show that the ideal $\mathrm{Var}(I_A, A \cdot \mathscr{E} \cdot \partial)$ is of height $n+1$.

The ideal $R_A[x](I_A, A \cdot \theta)$ in the polynomial ring $R_A[x]$ defines in the cotangent bundle $\mathrm{Spec}(R_A[x])$ of $\mathbb{C}^{n+1}$ the union of the conormals to each torus orbit since the Euler fields are tangent to the torus and span a space of the correct dimension in each orbit point. Suppose the claim is false, so that there is a nonzero point $y \in \mathrm{Var}(I_A)$ such that (the generically chosen vector) $p$ is a conormal vector to the orbit of $y$. If $y$ is in a torus orbit $O_\tau$ associated to a proper face $\tau$ of $A$ then its coordinates corresponding to $A \smallsetminus \tau$ are zero and we can reduce the question to the case where $A = \tau$. It is hence enough to show that there is $p \in \mathbb{C}^{n+1}$ such that $p$ is not a conormal vector to any smooth point of $\mathrm{Var}(I_A)$.

Let $X \subseteq \mathbb{C}^{n+1}$ be any reduced affine variety and denote $X_0$ its smooth locus. We define a set $C(X)$ inside $\mathbb{C}^{n+1}$ by setting

$$[\eta \in C(X)] \iff [\exists y \in X_0, \quad \eta \in (T^*_{X_0}(\mathbb{C}^{n+1}))_y]$$

where $(T^*_{X_0}(\mathbb{C}^{n+1}))_y$ is the fiber of the conormal bundle at $y$ of the pair $X_0 \subseteq \mathbb{C}^{n+1}$. This is a constructible, analytically parameterized union of a $\dim(X)$-dimensional family of vector spaces of dimension $n+1-\dim(X)$, which hence might fill $\mathbb{C}^{n+1}$.

Now suppose that $X$ is a conical variety; then the conormals of $y$ and $\lambda y$ agree for all $\lambda \in \mathbb{C}^*$. In particular,

$$C(X) = \bigcup_{\overline{y} \in \mathrm{Proj}(X)} (T^*_{X_0}(\mathbb{C}^{n+1}))_y$$

where $\mathrm{Proj}(X)$ is the associated projective variety. But this is now an analytically parameterized union of a $(\dim(X)-1)$-dimensional family of vector spaces of dimension $n+1-\dim(X)$. It follows that most elements of $\mathbb{C}^{n+1}$ are outside $C(X)$ in this case, and the claim follows. ■

It follows from the Claim that $\mathrm{gr}_{V_0}(H_A(\beta)_p)$ contains all monomials in $\partial$ of a certain degree $k$ that depends on $A$. Let $E = \theta_0 + \ldots + \theta_n$; by hypothesis $E - \beta_E \in H_A(\beta)$.

**Lemma 3.9.** *Denote $\partial_A^k$ be the set of all monomials of degree $k$ in $\partial_0, \ldots, \partial_n$, and $D_A \cdot \partial_A^k$ the left $D_A$-ideal generated by $\partial_A^k$. Then in $D_A/D_A \cdot \partial_A^k$, the identity $E(E-1)\cdots(E-k+1) \cong 0$ holds.* □

**Proof**. This is clear if $k=1$. In general, by induction,

$$E(E-1)\cdots(E-k+1) \in D_A \cdot \partial_A^{k-1} \cdot (E-k+1) = D_A \cdot E \cdot \partial_A^{k-1} \subseteq D_A \cdot \partial_A^k.$$

■

*Remark* 3.10. The homogeneity of $X$ is necessary in the Claim, since otherwise $C(X)$ does not need to be contained in a hypersurface. Consider, for example, $A = (2,1)$ in which case the union of all tangent lines (nearly) fills the plane, and where the zero locus of $I_A$ and $A \cdot \mathscr{E} \cdot \partial$ contains always at least two points. □

The lemma implies that $\mathrm{gr}^0_{V_0}(H_A(\beta)_p)$ contains $E(E-1)\cdots(E-k+1)$ if $p$ is generic. In other words, the $b$-function for restriction of $M_A(\beta)$ to a generic point divides $s(s-1)\cdots(s-k+1)$.

In some cases one can be more explicit about $k-1$, the top degree in which $R_A/R_A(I_A, A\cdot\mathscr{E}\cdot\partial)$ is nonzero. Suppose $S_A$ is a Cohen–Macaulay ring, then systems of parameters are regular sequences. In particular, the Hilbert series of $Q_A := R_A/R_A(I_A, A\cdot\mathscr{E}\cdot\partial)$ is that of $S_A$ multiplied by $(1-t)^d$. Suppose in addition, that $S_A$ is normal. Since we already assume that $S_A$ is standard graded, let $P$ be the polytope that forms the convex hull of the columns of $A$. The Hilbert series of $S_A$ is then of the form $\sum_{m=0}^\infty p_m \cdot t^m$ where $p_m$ is the number of lattice points in the dilated polytope $m \cdot P$. This number of lattice points is counted by the Erhart polynomial $E_P(m)$ of $P$, a polynomial of degree $d-1 = \dim(P)$. If one writes the Hilbert series of $S_A$ in standard form $Q(t)/(1-t)^d$ then the Hilbert series of $Q_A$ is just the polynomial $Q(t)$. In particular, the highest degree of a non-vanishing element of $Q_A$ is the degree of $Q(t)$.

In order to determine $\deg(Q(t))$ let $E_P(m) = e_{d-1}m^{d-1} + \ldots + e_0$. Now in

$$\sum_{m=0}^{\infty} E_P(m)t^m = \sum_{i=0}^{d-1}\left(e_i \cdot \sum_{m=0}^{\infty} m^i \cdot t^m\right),$$

each term $\sum_{m=0}^\infty m^i \cdot t^m$, for $m > 0$, is a polylogarithm $\mathrm{Li}_{-i}(t)$ given by $(t\frac{\mathrm{d}}{\mathrm{d}t})^n(\frac{t}{1-t})$. A simple calculation shows that $\mathrm{Li}_{-i}(t)$ is the quotient of a polynomial of degree $i-1$ by $(1-t)^i$. Hence the sum in the display is the quotient of a polynomial of degree at most $d-1$ by $(1-t)^d$. The degree is truly $d-1$ as one can check from the differential expression for $\mathrm{Li}_{-i}(t)$ above.

Therefore, the Hilbert series $Q(t)$ of $Q_A$ is a polynomial of degree $d-1$. We have proved

**Theorem 3.11.** *Let $S_A$ be standard graded. The $b$-function for restriction of $M_A(\beta)$ to a generic point $x+p=0$ divides $s(s-1)\cdots(s-k+1)$ where $k$ denotes the highest degree in which the quotient $S_A/S_A\cdot(A\cdot\mathscr{E}\cdot\partial)$ is nonzero. If, in addition, $S_A$ is normal then one may take $k=d$.* □ ■

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