ALGORITHMIC COMPUTATION OF DE RHAM COHOMOLOGY OF COMPLEMENTS OF COMPLEX AFFINE VARIETIES

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ABSTRACT. Let $X = \mathbb{C}^n$. In this paper we present an algorithm that computes the de Rham cohomology groups $H^i_{dR}(U, \mathbb{C})$ where U is the complement of an arbitrary Zariski-closed set Y in X.

Our algorithm is a merger of the algorithm given by T. Oaku and N. Takayama in [12], who considered the case where Y is a hypersurface, and our methods from [14] for the computation of local cohomology. We further extend the algorithm to compute de Rham cohomology groups with supports $H^i_{dR,Z}(U, \mathbb{C})$ where again U is an arbitrary Zariski-open subset of X and Z is an arbitrary Zariski-closed subset of U.

Our main tool is a generalization of the restriction process from [11] to complexes of modules over the Weyl algebra. The restriction rests on an existence theorem on V_d-strict resolutions of complexes that we prove by means of an explicit construction via Cartan-Eilenberg resolutions.

All presented algorithms are based on Gröbner basis computations in the Weyl algebra and the examples are carried out using the computer system Kan by N. Takayama [13].

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1. Introduction

De Rham cohomology on smooth varieties in a purely algebraic context seems to have been introduced by A. Grothendieck [5]. In the famous paper [6] R. Hartshorne developed the concept of algebraic de Rham cohomology of arbitrary algebraic varieties as an analog to classical (singular) cohomology. Results of A. Grothendieck and P. Deligne prove that it agrees with classical cohomology if the base field is $\mathbb{C}$. Moreover, Hartshorne also developed the notion of algebraic de Rham cohomology with supports and proved that it fits into certain natural long exact sequences related to inclusion maps (7.1).

In [12], the authors give an algorithm that computes (by Gröbner basis computations in the Weyl algebra) the algebraic de Rham cohomology of the complement U of any given hypersurface Y of $X = \mathbb{C}^n$. Their method is based on the initial definition of Hartshorne, as the hypercohomology of the de Rham complex on U. They show that this complex is in the derived category the same as the tensor product over $\mathcal{O}_X$ of the sheaf of differential n-forms on X with a resolution of $\mathcal{O}_U$, $\mathcal{O}_U$ considered as a module over the sheaf of differential operators on X. The computation of the hypercohomology of the latter complex reduces to computation of usual cohomology of the global sections since U is affine and the sheaves involved are quasi-coherent. An algorithm to compute the cohomology of complexes of the type one gets after taking global sections was given in [11]. The strategy is to use the method of restriction of a D-module to a linear subvariety ([10] and Section 5 of [11]).

In this note we shall prove

Theorem 6.1. *The de Rham cohomology groups of the complement of an affine complex variety are effectively computable by means of Gröbner basis computations in Weyl algebras.*

In fact, we shall first prove a general existence theorem about finite V_d-strict D_n-free complexes that are quasi-isomorphic to a given D_n-free complex $C^\bullet$. We give a constructive proof. Then we generalize the restriction process to the restriction of a complex to a linear subvariety. As applications we obtain an algorithm that computes de Rham cohomology of arbitrary Zariski-open U, and an algorithm that computes de Rham cohomology of Zariski open sets with supports in a Zariski closed subset Z of U.

Now we shall give a detailed overview of the structure of this paper. Let $D_n = \mathbb{C}[x_1, \dots, x_n]\langle\partial_1, \dots, \partial_n\rangle$ be the n-th Weyl algebra over $\mathbb{C}$.

First of all, in Section 2, we show that if U is the complement of any Zariski closed set Y defined by $F = \{f_0, \ldots, f_r\}$ in X then computation of the de Rham cohomology of U can be performed by computing the cohomology of the tensor product over D_n of a D_n-free resolution of $D_n/(\partial_1, \ldots, \partial_n) \cdot D_n$ with the Mayer-Vietoris complex $MV^\bullet(F)$ associated to $f_0, \ldots, f_r$ (cf. Subsection 2.4). This will require a review of some of the algorithms given in [11] and [12].

In the following two sections we compute a certain D_n-free complex that is quasi-isomorphic to $MV^\bullet(F)$. In fact, let $C^\bullet$ be an arbitrary complex of finitely generated D_n-modules such that the cohomology of $C^\bullet$ is specializable to the subspace $x_1 = \ldots = x_d = 0$ for fixed $0 \leq d \leq n$ (cf. Definition 3.2). We present a method that computes a D_n-free complex $A^\bullet$ that is quasi-isomorphic to $C^\bullet$ and has certain properties related to the V_d-filtration (for facts about the V_d-filtration, see also [11]). Section 3 concentrates on the algorithmic aspects of this construction while Section 4 contains the required lemmas and proofs.

Section 5 is devoted to the explicit computation of the derived tensor product $(D_n/(x_1, \ldots, x_d) \cdot D_n) \otimes^L_{D_n} C^\bullet$, $0 \leq d \leq n$, where $C^\bullet$ is required to have cohomology that is specializable to the subspace $x_1 = \ldots = x_d = 0$ but otherwise is arbitrary. As a corollary of this computation we give in Section 6 an algorithm that computes $H^\bullet_{dR}(U, \mathbb{C})$, Algorithm (6.1).

In Section 7 we review the definition of de Rham cohomology with supports and derive an algorithm that computes $H^i_{dR,Z}(X \setminus Y, \mathbb{C})$ for arbitrary closed subvarieties Y, Z of X. The idea here is similar to the original argument in [12], twisted with the Čech complex associated to Z.

2. Algebraic de Rham Cohomology

2.1. **Notation.** Throughout this article, we shall use the following notation. $\mathbb{C}$ will stand for the field of complex numbers, X denotes the affine n-dimensional space $\mathbb{C}^n$ and $Y = \mathrm{Var}(f_0, \ldots, f_r)$ will be a closed subvariety of X cut out by polynomials $\{f_0, \ldots, f_r\} \subseteq R_n$ where $R_n = \mathbb{C}[x_1, \ldots, x_n]$. We set $U = X \setminus Y$.

D_n will be the ring of differential operators on X (also called the n-th Weyl algebra) generated by the multiplications by the x_i (which we will call also x_i) and the partial derivatives $\partial_i = \partial/\partial x_i$. Set $\mathcal{O}_X$ to be the structure sheaf on X. $\mathcal{D}_X$ will be the sheaf version of D_n, $\mathcal{D}_X = \mathcal{O}_X \otimes_{R_n} D_n$. Let

$$\Omega_d = D_n/D_n \cdot (\partial_1, \ldots, \partial_d) \quad \text{and} \quad \tilde{\Omega}_d = D_n/D_n \cdot (x_1, \ldots, x_d)$$

for $0 \le d \le n$. In the special case $n = d$ we write Ω for Ω_n and define $\Omega(\mathcal{D}_X) = \Omega \otimes_{D_n} \mathcal{D}_X$.

If $\mathcal{M}$ is a $\mathcal{D}_X$- or D_n-module, $\Omega^\bullet(\mathcal{M})$ will throughout stand for the de Rham complex of $\mathcal{M}$. In other words, $\Omega^k(\mathcal{M}) = \mathcal{M} \otimes_{\mathbb{Z}} \bigwedge^k(\mathbb{Z}^n)$ where $\bigwedge^k(\mathbb{Z}^n)$ is spanned by the symbols $dx_{i_1} \wedge \ldots \wedge dx_{i_k}$ and the differential δ is defined in the usual way: $\delta(u \otimes dx_{i_1} \wedge \ldots \wedge dx_{i_k}) = \sum_{j=1}^n (\partial_j \cdot u) \otimes dx_j \wedge dx_{i_1} \wedge \ldots \wedge dx_{i_k}$. If V is a variety, $\Omega^\bullet(V)$ will denote the de Rham complex on V. Furthermore, set $\Omega^\bullet = \Gamma(X, \Omega^\bullet(X))$.

If I is a finite set, $|I|$ will denote its cardinality.

Remark 2.1. Eventually we will be interested in implementations of our algorithms. With this in mind we will assume that the coefficients of all f_i belong to a subfield K of $\mathbb{C}$ that is *computable*. That is to say, elements of K can be represented with a finite set of data, their sum, product and quotient can be calculated in a finite number of steps, and there is a finite procedure that determines whether a given expression of elements of K is zero or not.

2.2. **Definition of de Rham Cohomology.** Let A be a smooth scheme over the field K of characteristic zero, and B a closed subscheme defined by the sheaf of ideals $\mathcal{I}$. Recall the notion of completion $\hat{\mathcal{G}}$ of a quasi-coherent sheaf $\mathcal{G}$ on A with respect B: if V is open in A, $\hat{\mathcal{G}}(V \cap B) = \varprojlim_k (\mathcal{G}(V)/\mathcal{I}^k(V) \cdot \mathcal{G}(V))$.

Algebraic de Rham cohomology of an arbitrary closed subset B of an arbitrary smooth scheme A over any field K of characteristic zero is defined as the hypercohomology of the complex $\hat{\Omega}^\bullet(A)$ where the hat denotes completion of $\Omega^\bullet(A)$ with respect to the system of ideals $\mathcal{I}$ which defines B in A. (For a precise definition of the maps in $\hat{\Omega}^\bullet(A)$ see [6], page 22.) It is shown in [6] that this definition does not depend on the embedding of B in A nor in fact on A itself. The Comparison Theorems of A. Grothendieck and P. Deligne state among other things that if $K = \mathbb{C}$ then algebraic de Rham cohomology of the $\mathbb{C}$-scheme B agrees with its singular cohomology with coefficients in $\mathbb{C}$.

In the special case where B is smooth, one may take $B = A$ and then the sheaf of ideals $\mathcal{I}$ is the zero sheaf. In particular, for open subsets of X, $H^i_{dR}(U, \mathbb{C})$ is the hypercohomology of the complex $\Omega^\bullet(U)$.

2.3. **The Idea of Oaku and Takayama.** For this subsection, assume that $r = 0$ and $F = \{f\}$ so that $Y = \operatorname{Var}(f)$ is a hypersurface. Let $j : U \hookrightarrow X$ be the embedding. We will review some of the algorithms in [12].

The basic observation is the following

Lemma 2.2. *The complex $\Omega^\bullet(D_n) = \Omega^\bullet$ is (a complex in the category of right D_n-modules and in that category) quasi-isomorphic to the complex that is zero except in position n and whose n-th entry is the right D_n-module $D_n/(\partial_1, \dots, \partial_n) \cdot D_n = \Omega$. A corresponding statement holds for $\mathcal{D}_X$.* □

The hypercohomology $\mathbb{H}^\bullet(X, \Omega^\bullet(\mathcal{O}_U))$ of $\Omega^\bullet(\mathcal{O}_U)$ equals the cohomology of the global sections because U is affine and $\Omega^\bullet(\mathcal{O}_U)$ consists of quasi-coherent sheaves ([7], Theorem III.3.5 together with the spectral sequence $H^q(X, \Omega^p(\mathcal{O}_U)) \Rightarrow \mathbb{H}^{p+q}(X, \Omega^\bullet(\mathcal{O}_U)))$. Also, $\Omega^\bullet(\mathcal{O}_U)$ and $j_*(\Omega^\bullet(\mathcal{O}_U))$ have the same cohomology and hence $H^\bullet_{dR}(U, \mathbb{C})$ is the cohomology of $j_*(\Omega^\bullet(\mathcal{O}_U)) = \Omega^\bullet(\mathcal{D}_X) \otimes_{\mathcal{D}_X} j_*(\mathcal{O}_U)$.

It follows from the lemma that since $\Omega^\bullet(\mathcal{D}_X)$ is a complex of free $\mathcal{D}_X$-modules, $j_*(\Omega^\bullet(\mathcal{O}_U))$ is the complex that computes the torsion sheaves $\mathcal{T}or^{\mathcal{D}_X}_{n-\bullet}(\Omega(\mathcal{D}_X), j_*(\mathcal{O}_U))$. The cohomology of the global sections of this complex will simply be $\mathrm{Tor}^{D_n}_{n-\bullet}(\Omega, R_n[f^{-1}])$.

A crucial point is now to use the fact that the Tor-functor is balanced. Let $A^\bullet$ be a truncated finite free resolution of length greater than n for $R_n[f^{-1}]$ in the category of left D_n-modules. That this is possible follows for example from the fact that D_n is left-Noetherian and that $R_n[f^{-1}]$ is D_n-cyclic ([2]).

Then the cohomology of $\Omega \otimes_{D_n} A^\bullet$ is the de Rham cohomology of U with coefficients in $\mathbb{C}$ shifted by n, since $H^{-i}(\Omega \otimes_{D_n} A^\bullet) = \mathrm{Tor}^{D_n}_i(\Omega, R_n[f^{-1}])$ and $\mathrm{Tor}^{D_n}_i(\Omega, R_n[f^{-1}]) = 0$ for $i < 0$ and $i > n$.

T. Oaku and N. Takayama gave an algorithm in [11] for the computation of the cohomology groups of this kind of complex. It is in fact explained how one can find the cohomology groups of the complex $D_n/(x_1, \dots, x_n) \cdot D_n \otimes^L_{D_n} M$ where M is an arbitrary holonomic D_n-module and the tensor product is considered as an element in the derived category. The algorithm is based on the concepts of V-filtration and V-strictness which are considered in detail in Sections 3-5. The present problem can be reduced to that case by applying the Fourier automorphism:

Definition 2.3. The d-th *Fourier transform* $\mathcal{F}_d$ of a free D_n-module is defined as follows. $\mathcal{F}_d(x_i) = \partial_i, \mathcal{F}_d(\partial_i) = -x_i$ for $1 \leq i \leq d$ while $\mathcal{F}_d(x_i) = x_i$ and $\mathcal{F}_d(\partial_i) = \partial_i$ for all $i > d$.

We write simply $\mathcal{F}$ for $\mathcal{F}_n$. If $M = {D_n}^t/I$ then $\mathcal{F}_d(M) := {D_n}^t/\mathcal{F}_d(I)$.

Computation of $H^i_{dR}(U, \mathbb{C})$ for a hypersurface can be summarized as follows (compare [11], Algorithm 2.1 for restriction):

Algorithm 2.4.

Input: $f \in R_n$; $i \in \mathbb{N}$.

Output: $H^i_{dR}(U, \mathbb{C})$ where $U = \mathbb{C}^n \setminus \mathrm{Var}(f)$.

Begin

1. Find a V_n-strict finite free resolution $A^\bullet$ of the D_n-module $\mathcal{F}(R_n[f^{-1}])$, $\mathcal{F}(R_n[f^{-1}])$ positioned in degree n.
2. Replace each D_n by the right D_n-module $\mathcal{F}(\Omega) \cong \mathbb{C}[\partial_1, \ldots, \partial_n]$ in that resolution.
3. Truncate the resolution using the method of [11] to a complex of finite dimensional $\mathbb{C}$-vector spaces.
4. Take the i-th cohomology.

End.

2.4. **Computing de Rham Cohomology for arbitrary** Y**.** Let Y now be cut out by the $r+1$ polynomials $f_0, \ldots, f_r$. The central problem arises from the fact that computation of the hypercohomology of $\Omega^\bullet(\mathcal{D}_U) \otimes_{\mathcal{D}_U} \mathcal{O}_U$ is not just $\mathrm{Tor}^{D_n}_{n-\bullet}(\Omega, \Gamma(U, \mathcal{O}_U))$ anymore, due to the existence of higher cohomology of quasi-coherent sheaves on U. The strategy is to find an open covering of U such that each of the open sets in the covering is acyclic for cohomology of quasi-coherent sheaves.

Definition 2.5. Let $\Re :=$ the ordered nonempty subsets of $\{0, \ldots, r\}$. Singleton subsets $\{j\} \in \Re$ we shall denote by just j. Set $U_i = X \setminus \mathrm{Var}(f_i)$. More generally for $I \in \Re$, we define $U_I = \bigcap_{i \in I} U_i$.

Similarly, set $f_I = \prod_{i \in I} f_i$ with the special cases $f_I = f_i$ if $I = \{i\}$ and abbreviate $\mathcal{O}_{U_I}$ as $\mathcal{O}_I$.

To get started, notice that $U_I = X \setminus \mathrm{Var}(f_I)$. This means in particular, that by Oaku–Takayama the de Rham cohomology groups of U_I with coefficients in $\mathbb{C}$ are computable as the cohomology of $\Omega^\bullet(D_n) \otimes_{D_n} R_n[{f_I}^{-1}]$. Notice also that $U = X \setminus Y$ is just the union of all the U_I.

In [6], page 28, Hartshorne explains how de Rham cohomology of schemes may be recovered from the de Rham complexes on the open sets in a finite covering. For our U that works as follows.

For each $I \in \Re$ let $X_I = \prod_{i \in I} U_i$. Then U_I embeds in X_I as the diagonal. As X_I is smooth, $\hat{\Omega}^\bullet(X_I)$ computes de Rham cohomology of U_I, the hat denoting completion at the closed subscheme $U_I \subseteq X_I$.

Consider the direct image $\mathcal{M}_I^\bullet$ of $\hat{\Omega}^\bullet(X_I)$ in U, induced by the inclusion $j_I : U_I \hookrightarrow U$.

Since U_I is smooth, $\hat{\Omega}^\bullet(X_I)$ is naturally quasi-isomorphic to $\Omega^\bullet(\mathcal{O}_I)$, cf. [6], Proposition II.1.1. Hence $\mathcal{M}_I^\bullet$ is naturally quasi-isomorphic to $j_{I*}(\Omega^\bullet(\mathcal{O}_I))$, the direct image of $\Omega^\bullet(\mathcal{O}_I) = \Omega^\bullet(D_n) \otimes_{D_n} \mathcal{O}_I$.

For $j \notin I$, the natural maps $X_{I \cup \{j\}} \twoheadrightarrow X_I$ and $U_{I \cup \{j\}} \hookrightarrow U_I$ give a natural map $\hat{\Omega}^\bullet(X_I) \to \hat{\Omega}^\bullet(X_{I \cup \{j\}})$. Similarly, we get chain maps $\phi^\bullet_{I,j} : \Omega^\bullet(\mathcal{O}_I) \to \Omega^\bullet(\mathcal{O}_{I \cup \{j\}})$ induced from the inclusion $U_{I \cup \{j\}} \hookrightarrow U_I$.

It is not hard to see that the natural quasi-isomorphism from $\hat{\Omega}^\bullet(X_I)$ to $\Omega^\bullet(\mathcal{O}_I)$ transforms the map $\hat{\Omega}^\bullet(X_I) \to \hat{\Omega}^\bullet(X_{I\cup\{j\}})$ into $\phi^\bullet_{I,j}$. So the same is true for the direct images in U, the induced maps between which we also denote by $\phi^\bullet_{I,j}$.

Multiply $\phi^\bullet_{I,j}$ by $(-1)^{\mathrm{sgn}(I,j)}$, $\mathrm{sgn}(I,j)$ being the number of elementary permutations that are needed to make the string (I,j) an actual element of $\Re$ (that is, an ordered set).

Let us write $\mathcal{J}^\bullet_I := j_{I*}(\Omega^\bullet(D_n) \otimes_{D_n} \mathcal{O}_I)$, a complex of sheaves on U that is naturally quasi-isomorphic to $\mathcal{M}^\bullet_I$. We will now construct a double complex $\mathcal{MV}(\mathcal{J})$ out of all the $\mathcal{J}^\bullet_I$. Let $\mathcal{MV}(\mathcal{J})^{k,l} = \bigoplus_{|I|=l} \mathcal{J}^k_I$. The maps in the horizontal (k-) direction are simply the direct sums of the differentials of the $\mathcal{J}^\bullet_I$ involved, while the vertical (l-) maps are defined to be the sums of all maps which are composed as follows:

$$\bigoplus_{|I|=l} \mathcal{J}^k_I \xrightarrow{nat} \mathcal{J}^k_I \xrightarrow{\phi^k_{I,j}} \mathcal{J}^k_{I\cup\{j\}} \hookrightarrow \bigoplus_{|I'|=l+1} \mathcal{J}^k_{I'}.$$

Notice that this is in fact a double complex (and in particular anti-commutative) due to the sign rule that applies to the $\phi^k_{I,j}$.

Then, according to Hartshorne, the de Rham cohomology of U is the hypercohomology of the associated total complex $Tot^\bullet(\mathcal{MV}(\mathcal{J}))$. Of course, $\mathcal{MV}(\mathcal{J})$ is just the origin of the usual Mayer-Vietoris spectral sequence of de Rham cohomology and sometimes called the Čech-de Rham complex .

If Y is a hypersurface then U is affine, so hypercohomology on U is necessarily cohomology of the global sections. In our more general situation we claim

Lemma 2.6. *The complex $j_{I*}(\Omega^\bullet(\mathcal{O}_I))$ consists entirely of sheaves that have no higher cohomology on U.*

Proof. In order to see this observe that it is sufficient to show that $j_{I*}(\mathcal{O}_I)$ has this property, because $\Omega^i(D_n)$ is D_n-free. If $\mathcal{E}^\bullet_I$ is an $\mathcal{O}_I$-injective resolution of $\mathcal{O}_I$ on U_I, then $j_{I*}(\mathcal{E}^\bullet_I)$ is a complex of flasque sheaves on U as direct images of flasque sheaves are flasque. Moreover, as U_I is affine, j_{I*} is an exact functor on quasi-coherent sheaves (and $\mathcal{O}_{U_I}$-morphisms), because $R^i j_{I*}(-)$ is the sheaf associated to the presheaf $V \to H^i(V \cap U_I, -)$ for open subsets V of U (Proposition III.8.1 in [7]). Hence we actually get a flasque resolution of $j_{I*}(\mathcal{O}_I)$. Taking global sections we see that $j_{I*}(\mathcal{O}_I)$ has no higher cohomology on U, as $\Gamma(U, j_{I*}(\mathcal{E}^i_I)) = \Gamma(U_I, \mathcal{E}^i_I)$. □

Remark 2.7. We note in passing that the proof actually shows that $H^i(j_{I*}(\mathcal{G}), U) = 0$ for positive i and all quasi-coherent $\mathcal{G}$ on U_I.

So the complex $Tot^\bullet(\mathcal{MV}(\mathcal{J}))$ consists of $\Gamma(U,-)$-acyclic sheaves. Thus, in order to compute its hypercohomology it suffices to compute the cohomology of the global sections of that complex. We arrive at

Proposition 2.8. *The de Rham cohomology of U with coefficients in $\mathbb{C}$, which may be computed as the hypercohomology of the complex $Tot^\bullet(\mathcal{MV}(\mathcal{J}))$, agrees with the cohomology of the global sections of $Tot^\bullet(\mathcal{MV}(\mathcal{J}))$ and can be computed as $H^\bullet(\Omega^\bullet \otimes_{D_n} MV^\bullet)$, where*

$$MV^\bullet : 0 \to \underbrace{\bigoplus_{|I|=1} R_n[{f_I}^{-1}]}_{\text{degree } 0} \to \ldots \to \underbrace{\bigoplus_{|I|=r+1} R_n[{f_I}^{-1}]}_{\text{degree } r} \to 0. \tag{2.1}$$

Proof. This follows from the discussion before the proposition, since the global sections on U of $j_{I*}(\Omega^\bullet(\mathcal{O}_I))$ are exactly the elements of $\Omega^\bullet(D_n) \otimes_{D_n} R_n[{f_I}^{-1}]$ and hence $\Gamma(U, Tot^\bullet(\mathcal{MV}(\mathcal{J}))) = \Omega^\bullet(D_n) \otimes_{D_n} MV^\bullet$. $\square$

Remark 2.9. Recall that for any set of polynomials $\{p_i\}_0^m$, the Čech complex $\check{C}^\bullet(R_n; p_0, \ldots, p_m) := \bigotimes_0^m \check{C}^\bullet(p_i)$ is defined by

$$\check{C}^\bullet(p_i) = (0 \to R_n \xrightarrow{1 \to \frac{1}{1}} R_n[{p_i}^{-1}] \to 0).$$

Thus, MV^i is the $(i+1)$-st entry of the Čech complex $\check{C}^\bullet(R_n; f_0, \ldots, f_r)$ if $i \geq 0$ and zero otherwise.

In the special case where $r = 0$ one sees that the complex $MV^\bullet$ degenerates to $(0 \to R_n[{f_0}^{-1}] \to 0)$ reducing to the case from [12].

In [14], Algorithm 5.1 we gave an algorithm that explicitly computes the Čech complex to a finite set of polynomials as a complex of finitely generated left D_n-modules by means of Gröbner basis computations. Using this algorithm we may explicitly compute $MV^\bullet$ for given $f_0, \ldots, f_r$. For the convenience of the reader we reproduce that algorithm here:

Algorithm 2.10.
INPUT: $f_0, \ldots, f_r \in R_n$.
OUTPUT: The Čech complex $\check{C}^\bullet(R_n; f_0, \ldots, f_r)$ in terms of generators and relations as finitely generated D_n-modules.

Begin

1. Compute the annihilator ideal $J^\Delta((F_I)^s)$ and the Bernstein polynomial $b^\Delta_{F_I}(s)$ for all k-fold products F_I of $f_0, \ldots, f_r$, k running through $0, \ldots, r$ (so I runs through $\Re$).
2. Compute the smallest integer root a_I for each $b^\Delta_{F_I}(s)$, let a be the minimum of all a_I and replace s by a in all the annihilator ideals.

3. Set $\check{C}^k = \oplus_{|I|=k} D_n / J^{\Delta}((F_I)^s)|_{s=a}$.
4. Compute the matrices M_k representing the D_n-linear maps $\check{C}^k \to \check{C}^{k+1}$.
5. Return these matrices and the presentations for the modules $\check{C}^k$.

End.

For more details the reader is encouraged to look at [14].

It will now be our task to develop an algorithm that computes the cohomology of $\Omega^\bullet \otimes_{D_n} MV^\bullet = \Omega \otimes^L_{D_n} MV^\bullet$.

To this end we devote the next section to the description of an algorithm that turns a complex of finitely generated left D_n-modules into a quasi-isomorphic V_d-strict complex of finitely generated free D_n-modules. In Section 4 we provide the proofs for the correctness of the algorithm. Then, in Section 5, we prove the restriction theorem, which allows to give an algorithm that computes the restriction (in the derived category) of a complex of D_n-modules to a hyperplane.

3. An Algorithm to construct a V-strict free Complex

In the next three sections we will develop algorithms that find the cohomology of $\Omega \otimes^L_{D_n} C^\bullet$ where $C^\bullet$ is an arbitrary bounded complex of finitely generated D_n-modules with specializable cohomology (cf. Definitions 3.1, 3.2). In particular, in this section we find a free D_n-complex with special properties related to the so-called V-filtration, quasi-isomorphic to a given complex $C^\bullet$ while in the next section we give the proofs for the correctness of the algorithms of this section.

3.1. We need to introduce some terminology from [11] related to the V-filtration.

Definitions 3.1. Fix an integer d with $0 \leq d \leq n$ and set $H = \mathrm{Var}(x_1, \ldots, x_d)$. For $\alpha \in \mathbb{Z}^n$, we set $\alpha_H = (\alpha_1, \ldots, \alpha_d, 0, \ldots, 0)$.

On the ring D_n we define the V_d-*filtration* $F^k_H(D_n)$ as the K-linear span of all operators $x^\alpha \partial^\beta$ for which $|\alpha_H| + k \geq |\beta_H|$. More generally, on a free D_n-module $A = \oplus_{j=1}^t D_n \cdot e_j$ we define

$$F^k_H(A[\mathfrak{m}]) = \sum_{j=1}^t F_H^{k-\mathfrak{m}(j)}(D_n) \cdot e_j,$$

where $\mathfrak{m}$ is an element of $\mathbb{Z}^m$. We shall call $\mathfrak{m}$ the *shift vector*. A shift vector is tied to a fixed set of generators.

We define the V_d-*degree* of an operator $P \in A[\mathfrak{m}]$, $V_d\deg(P[\mathfrak{m}])$, to be the smallest k such that $P \in F^k_H(A[\mathfrak{m}])$.

If M is a quotient of the free D_n-module $A = \oplus_1^t D_n \cdot e_j$, $M = A/I$, we define the V_d-filtration on M by $F_H^k(M[\mathfrak{m}]) = F_H^k(A[\mathfrak{m}]) + I$. For submodules N of A we define the V_d-filtration by intersection: $F_H^k(N[\mathfrak{m}]) = F_H^k(A[\mathfrak{m}]) \cap N$.

If $A^\bullet$ is a free D_n-resolution of the module M, M being positioned in degree b, we say that $A^\bullet[\mathfrak{m}_\bullet]$ is *V_d-strict* if there exist shift vectors $\mathfrak{m}_i$ such that $F_H^k(A^i[\mathfrak{m}_i]) \to F_H^k(A^{i+1}[\mathfrak{m}_{i+1}]) \to F_H^k(A^{i+2}[\mathfrak{m}_{i+2}])$ is exact for all $i < b-1$ and all k, and $F_H^k(A^{b-1}[\mathfrak{m}_{b-1}]) \to F_H^k(A^b[\mathfrak{m}_b]) \to F_H^k(M[\mathfrak{m}_b]) \to 0$ is exact for all k.

If M is a submodule of the free module $A[\mathfrak{m}]$, then a *V_d-strict Gröbner basis* or a *V_d-Gröbner basis* for M is a set of generators $\{m_1, \ldots, m_\tau\}$ for M which satisfies: for all $m \in M$ we can find $\{\alpha_i\}_1^\tau \in D_n$ such that $m = \sum \alpha_i m_i$ and $V_d\deg(\alpha_i m_i[\mathfrak{m}]) \leq V_d\deg(m[\mathfrak{m}])$ for all i.

It has been shown by T. Oaku and N. Takayama in [11] (Proposition 3.8 and following remarks) how to compute V_d-strict Gröbner bases, and for any D_n-module M positioned in degree b a free V_d-strict resolution $(A^\bullet[\mathfrak{m}_\bullet], \phi^\bullet)$ of $M[\mathfrak{m}_b]$, $A^i = \oplus_1^{r_i} D_n, r_i = 0$ if $i > b$. The construction given in [11] allows for arbitrary $\mathfrak{m}_b$.

The method employed is to construct a free resolution with the usual technique of finding a Gröbner basis for $\ker(A^i \to A^{i+1})$ and calculating the syzygies on this basis. The trick is to impose an order that refines the partial ordering given by V_d-degree, together with a homogenization technique.

The vectors $\mathfrak{m}_i$ are obtained for each A^i with decreasing i: if A^i maps its generators on a Gröbner basis of $\ker(A^{i+1} \to A^{i+2})$ then the shift component $\mathfrak{m}_i(j)$ corresponding to the j-th generator e_j of A^i is defined as $V_d\deg(\phi^i(e_j)[\mathfrak{m}_{i+1}])$.

We need to generalize the definitions of [11] to the case where the complex $A^\bullet$ is not a resolution and M is a complex rather than a module.

Definitions 3.2. A complex of free D_n-modules $\cdots \to A^{i-1} \stackrel{\phi^{i-1}}{\to} A^i \stackrel{\phi^i}{\to} A^{i+1} \to \cdots$ is said to be *V_d-adapted at A^i* with respect to certain shift vectors $\mathfrak{m}_{i-1}, \mathfrak{m}_i, \mathfrak{m}_{i+1}$ if

$$\phi^i\left(F_H^k(A^i[\mathfrak{m}_i])\right) \subseteq F_H^k(A^{i+1}[\mathfrak{m}_{i+1}])$$

and also

$$\phi^{i-1}\left(F_H^k(A^{i-1}[\mathfrak{m}_{i-1}])\right) \subseteq F_H^k(A^i[\mathfrak{m}_i])$$

for all k.

We shall say that the complex is V_d*-strict at* A^i if it is V_d-adapted at A^i and moreover

$$\operatorname{im}(\phi^{i-1}) \cap F_H^k(A^i[\mathfrak{m}_i]) = \operatorname{im}(\phi^{i-1}|_{F_H^k(A^{i-1}[\mathfrak{m}_{i-1}])})$$

for all k.

For $1 \leq d \leq n$ we set $\theta_d = x_1\partial_1 + \ldots + x_d\partial_d$ and $\theta_0 = 0$. Recall that a D_n-module $M[\mathfrak{m}] = A[\mathfrak{m}]/I$ is called *specializable to* H if there is a polynomial $b(s)$ in a single variable such that

$$b(\theta_d + k) \cdot F_H^k(M[\mathfrak{m}]) \quad \subseteq \quad F_H^{k-1}(M[\mathfrak{m}]) \tag{3.1}$$

for all k (cf. [12]). Introducing

$$\operatorname{gr}_H^k(M[\mathfrak{m}]) = (F_H^k(M[\mathfrak{m}]))/(F_H^{k-1}(M[\mathfrak{m}])),$$

this can be written as

$$b(\theta_d + k) \cdot \operatorname{gr}_H^k(M[\mathfrak{m}]) = 0.$$

The monic polynomial $b(\theta)$ of least degree satisfying an equation of the type (3.1) is called the *b-function for restriction of* $M[\mathfrak{m}]$ *to* H.

Remark 3.3. Specializability descends to quotients and submodules. Namely, assume that $M[\mathfrak{m}] = (A/I)[\mathfrak{m}]$ is specializable and $N[\mathfrak{m}] = (A'/I)[\mathfrak{m}]$ is a submodule of M (where $I \subseteq A' \subseteq A$). Let $b(s)$ be a polynomial that satisfies $b(\theta_d+k)\cdot F_H^k(A[\mathfrak{m}]) \subseteq F_H^{k-1}(A[\mathfrak{m}])+I$. Then clearly $b(\theta_d+k)\cdot F_H^k(A[\mathfrak{m}]) \subseteq F_H^{k-1}(A[\mathfrak{m}]) + A'$ as well and hence $(M/N)[\mathfrak{m}]$ is specializable to H. On the other hand, if $P' \in F_H^k(A'[\mathfrak{m}]) = F_H^k(A[\mathfrak{m}]) \cap A'$ then $b(\theta+k)\cdot P = Q + Q'$ where $Q \in F_H^{k-1}(A[\mathfrak{m}])$ and $Q' \in I$ and hence $Q \in F_H^{k-1}(A[\mathfrak{m}]) \cap A' = F_H^k(A'[\mathfrak{m}])$. This implies that N is also specializable and we see that the b-functions for restriction of $N[\mathfrak{m}]$ and for $(M/N)[\mathfrak{m}]$ divide the b-function for restriction of $M[\mathfrak{m}]$ to H.

Notice that independently of d, $\operatorname{gr}_H^\bullet(D_n[0]) \cong D_n$, as a ring.

The main purpose of this section is to construct for a given D_n-finite complex $0 \to C^0 \to \ldots \to C^r \to 0$ a quasi-isomorphic free V_d-strict complex $A^\bullet[\mathfrak{m}_\bullet]$.

Remark 3.4. If $A^\bullet[\mathfrak{m}_\bullet]$ is a free resolution of M and V_d-strict in our sense it is also V_d-strict in the sense of [12]. In fact our definition is a natural generalization to complexes that are not resolutions.

Let $\cdots \to A^{i-1}[\mathfrak{m}_{i-1}] \xrightarrow{\phi^{i-1}} A^i[\mathfrak{m}_i] \xrightarrow{\phi^i} A^{i+1}[\mathfrak{m}_{i+1}] \to \cdots$ be a free V_d-strict complex. Then the V_d-filtration on A^i induces a filtration on the i-cycles $Z^i = \ker \phi^i$ and since the complex is V_d-strict this gives a natural filtration on the cohomology module H^i,

$$F_H^k(H^i[\mathfrak{m}_i]) = F_H^k(Z^i[\mathfrak{m}_i])/\operatorname{im} F_H^k(A^{i-1}[\mathfrak{m}_{i-1}]).$$

That this is a filtration follows from

$$F^k_H(Z^i[\mathfrak{m}_i]) \cap \operatorname{im}(F^{k+1}_H(A^{i-1}[\mathfrak{m}_{i-1}])) = \operatorname{im}(F^k_H(Z^{i-1}[\mathfrak{m}_{i-1}])).$$

Let B^i be $\operatorname{im}(\phi^{i-1})$, the i-boundaries. The short exact sequences $0 \to Z^i \to A^i \to B^{i+1} \to 0$ give rise to short exact sequences of groups $0 \to F^k_H(Z^i[\mathfrak{m}_i]) \to F^k_H(A^i[\mathfrak{m}_i]) \to F^k_H(B^{i+1}[\mathfrak{m}_{i+1}]) \to 0$ since the complex is V_d-strict. Similarly, the short exact sequences $0 \to B^i \to Z^i \to H^i \to 0$ induce short exact sequences $0 \to F^k_H(B^i[\mathfrak{m}_i]) \to F^k_H(Z^i[\mathfrak{m}_i]) \to F^k_H(H^i[\mathfrak{m}_i]) \to 0$.

This in turn induces short exact sequences of the graded objects,

$$0 \to \operatorname{gr}^k_H(B^i[\mathfrak{m}_i]) \to \operatorname{gr}^k_H(Z^i[\mathfrak{m}_i]) \to \operatorname{gr}^k_H(H^i[\mathfrak{m}_i]) \to 0 \tag{3.2}$$

and

$$0 \to \operatorname{gr}^k_H(Z^i[\mathfrak{m}_i]) \to \operatorname{gr}^k_H(A^i[\mathfrak{m}_i]) \to \operatorname{gr}^k_H(B^{i+1}[\mathfrak{m}_{i+1}]) \to 0. \tag{3.3}$$

These sequences are the main feature of V_d-strict complexes. They say that forming graded objects and taking cohomology commutes for V_d-strict complexes.

Remark 3.5. The previous remark supplies a more systematic proof for the fact that specializability is inherited by subquotients. Namely, in the notation of Remark 3.4, consider the exact sequence

$$0 \to (A'/I)[\mathfrak{m}] \to (A/I)[\mathfrak{m}] \to (A/A')[\mathfrak{m}] \to 0,$$

which is easily seen to be V_d-strict for all d. Thus we get an associated sequence of graded modules

$$0 \to \operatorname{gr}^k_H((A'/I)[\mathfrak{m}]) \to \operatorname{gr}^k_H((A/I)[\mathfrak{m}]) \to \operatorname{gr}^k_H((A/A')[\mathfrak{m}]) \to 0.$$

If the middle module is annihilated by $b(\theta + k)$ then necessarily so are the two outer ones.

3.2. We shall use *matrices* to represent submodules, quotient modules, and left D_n-morphisms in our algorithms. In particular, we shall think of free modules as row vectors. We note that since we are dealing with left modules, the matrices that represent D_n-morphisms will act by right multiplication.

Definitions 3.6. If I is a matrix, then $\operatorname{rows}(I)$ stands for the number of rows of I and $\operatorname{cols}(I)$ for the number of columns of I. Assuming that $I : {D_n}^p \to {D_n}^q$ is a map of left D_n-modules given by multiplication by the matrix I,

$${D_n}^p \ni (P_1, \dots , P_p) \mapsto (P_1, \dots , P_p) \cdot I \in {D_n}^q,$$

then ${D_n}^p$ is the *domain* of I and ${D_n}^q$ its *range*.

Let I be a $p \times q$ matrix with entries in D_n. The row vectors in I are denoted by $I_{[i]}$, $(i = 1, \ldots, p)$, i.e., I is of the form

$$I = \left.\begin{pmatrix} I_{[1]} \\ I_{[2]} \\ \vdots \\ I_{[p]} \end{pmatrix}\right\} p$$

The image of the morphism induced by I in ${D_n}^q$ is the module generated by the rows of I and denoted by $\operatorname{im}(I)$. The quotient module ${D_n}^q / \operatorname{im}(I)$ is $\operatorname{coker}(I)$.

Let M and M' be $p \times q$ and $p' \times q$ matrices respectively. We introduce two basic operations for two matrices. These will be building blocks of our algorithms. Define $\operatorname{div}(M, M') :=$ a set of generators for the left D_n-module

$$\left\{ (c_1, \ldots, c_p) \in {D_n}^p \mid \exists (d_1, \ldots, d_{p'}) \in {D_n}^{p'}, \sum_{i=1}^{p} c_i M_{[i]} + \sum_{j=1}^{p'} d_j M'_{[j]} = 0 \right\}.$$

We regard $\operatorname{div}(M, M')$ as an $r \times p$ matrix where r is the number of generators. $\operatorname{coker}(\operatorname{div}(M, M'))$ is isomorphic to $\operatorname{im}(M)/(\operatorname{im}(M) \cap \operatorname{im}(M'))$. If M' is the zero matrix we write $\operatorname{syz}(M)$ for $\operatorname{div}(M, M')$, the syzygy module on the rows of M.

We next define $\operatorname{sol}(M, M')_{[i]}$ to be a solution of

$$(c_1, \ldots, c_p) \in {D_n}^p \text{ and } \sum_{j=1}^{p} c_j M_{[j]} = e_i \quad \text{modulo} \quad \operatorname{im}(M'),$$

where e_i is the i-th basis element on the common range of M and M'. $\operatorname{sol}(M, M')$ is regarded as a $q \times p$ matrix. Note that if M is surjective onto $\operatorname{coker}(M')$ (i.e., $\operatorname{im}(M) + \operatorname{im}(M') = {D_n}^q$), then there exists a solution for the linear indefinite equation above. div and sol can be computed by standard techniques of computing syzygies (see, e.g., [1]).

We also define $\operatorname{pro}(k, l)$ to be the matrix $\left(\dfrac{0_{k \times l}}{\mathrm{id}_{l \times l}}\right)$, which is the projection matrix on modules of rank $k + l$ to the last l components.

We assume that the given bounded complex $C^\bullet$ is expressed in terms of matrices. This means that for all i, $C^i \cong \operatorname{coker}(I^i)$ where I^i is an $m_i \times n_i$ matrix, and $C^i \to C^{i+1}$ is induced by the $n_{i-1} \times n_i$ matrix f^{i-1} which satisfies ${D_n}^{m_{i-1}} \xrightarrow{I^{i-1}} {D_n}^{n_{i-1}} \xrightarrow{f^{i-1}} {D_n}^{n_i} \to \operatorname{coker} I^i$ is the zero map. (This is equivalent to requiring that $\operatorname{im}(I^{i-1}) \subseteq \operatorname{im}(\operatorname{div}(f^{i-1}, I^i))$, or that $\operatorname{coker}(I^{i-1}) \xrightarrow{f^{i-1}} \operatorname{coker}(I^i)$ is well defined.)

We observe that since the matrices act by right multiplication then the matrix $M_{\phi'\circ\phi}$ corresponding to a composition of the maps ϕ and ϕ' is the product of the corresponding matrices, $M_{\phi'\circ\phi} = M_\phi \cdot M_{\phi'}$.

Our algorithm for the construction of a V_d-strict free complex which is quasi-isomorphic to the given complex consists of four subprocedures which are explained below. These procedures will be presented with examples for better understanding.

Example 3.7. Our running example is to construct a V_3-strict free complex that is quasi-isomorphic to $\left(\operatorname{coker}(I^0) \xrightarrow{f^0} \operatorname{coker}(I^1)\right)$ with

$$(3.4)\quad I^0 = \begin{pmatrix} x\partial_x & 0 \\ y\partial_y & 0 \\ z & 0 \\ 0 & x\partial_x \\ 0 & y \\ 0 & z\partial_z \end{pmatrix}, \quad I^1 = \begin{pmatrix} x\partial_x \\ y\partial_y \\ z\partial_z \end{pmatrix}, \quad f^0 = \begin{pmatrix} \partial_z \\ -\partial_y \end{pmatrix}.$$

(One can see that the requirement $\operatorname{im}(I^0 \cdot f^0) \subseteq \operatorname{im}(I^1)$ is satisfied.) This example is geometrically motivated: it will serve to compute the de Rham cohomology groups of the complement of the union of a (complex) line and a plane in $\mathbb{C}^3$ (Example 6.3).

Now, let us describe the main algorithm. First we give an algorithm that breaks the complex into a family of short exact sequences. The main task will be to determine the matrices representing the short exact sequences from those representing the complex. The second step will be to establish an algorithm that turns short exact sequences into V_d-strict short exact sequences. In Step 3, for each short exact V_d-strict sequence, we construct a double complex of V_d-strict free resolutions, whose total complex is quasi-isormorphic to the given short exact sequence. The final step assembles the resolutions to a combined double complex whose associated total complex gives a V_d-strict free complex quasi-isomorphic to the initial complex. More concisely, we have the following algorithm whose steps are explained in detail below. (We denote as before by B^i the i-boundaries, by Z^i the i-cycles and by H^i the i-cohomology of the complex $C^\bullet$.)

Algorithm 3.8. (From a complex to a V_d-strict free complex.)
INPUT: A complex $C^\bullet = \{\operatorname{coker}(I^i), f^i\}$, $i = 0, \dots, r$. (I^i and f^i are matrices.) A starting shift vector $\mathfrak{m}_r$ on the range of I^r.
OUTPUT: A bounded V_d-strict free complex $({D_n}^{t_i}, g^i)$ and shift vectors $\mathfrak{m}_i$. Here, g^i are matrices representing the maps ${D_n}^{t_i} \to {D_n}^{t_{i+1}}$.

1. Break the complex into short exact pieces $0 \to B^i \to Z^i \to H^i \to 0$ and $0 \to Z^i \to C^i \to B^{i+1} \to 0$ with $0 \leq i \leq r$. (Subsection 3.3.)
2. For each of these pairs of short exact sequences, starting with $0 \to B^r \to Z^r \to H^r \to 0$ and $0 \to Z^r \to C^r \to B^{r+1} = 0 \to 0$, change the pair to a pair of V_d-strict short exact sequences such that the shift vectors for Z^i and B^i agree in both short exact sequences where these modules occur. (Algorithm 3.11.)
3. Over each short exact V_d-strict sequence obtained in the previous step construct a double complex with V_d-strict rows and columns such that

   a) the resolutions for B^i and Z^i are identical in the two instances where B^i and Z^i are resolved, and

   b) the total complex of the double complex is quasi-isomorphic to the initial three term sequence. (Algorithm 3.15.)
4. From the double complexes constructed in Step 3, assemble a V_d-strict Cartan-Eilenberg resolution over $C^\bullet$. (Algorithm 3.18.)
5. Take the total complex of the double complex obtained in the previous step.

End.

3.3. Now, let us explain the subprocedures called from Algorithm 3.8. Recall that f^i is a $n_{i-1} \times n_i$ matrix, and I^i is a $m_i \times n_i$ matrix. Since B^i is the submodule of C^i generated by the cosets (modulo I^i) of the rows of f^{i-1}, $B^i = \operatorname{coker}(\operatorname{div}(f^{i-1}, I^i))$, a $r_{i-1} \times n_{i-1}$ matrix.

Z^i is the kernel of the map $C^i \to C^{i+1}$ induced by f^i. Generators of Z^i are given by the cosets of the rows of $\operatorname{div}(f^i, I^{i+1})$ inside C^i. Hence Z^i is the cokernel of $\operatorname{div}(\operatorname{div}(f^i, I^{i+1}), I^i)$ which we assume to be a $s_i \times r_i$ matrix. The map from ${D_n}^{r_i} \to {D_n}^{n_i}$ that induces the embedding $Z^i \hookrightarrow C^i$ is given by $\operatorname{div}(f^i, I^{i+1})$, which is an $r_i \times n_i$ matrix.

There is a natural map from B^i to C^i given by inclusion. The $n_{i-1} \times n_i$ matrix f^{i-1} lifts this map, by assumption. Since the image of f^{i-1} is contained in the image of $\operatorname{div}(f^i, I^{i+1})$ (which stems from the fact that $f^{i-1} \cdot f^i = 0$ modulo the image of I^{i+1}) we can find an $n_{i-1} \times r_i$ matrix lift^i with $\operatorname{lift}^i \cdot \operatorname{div}(f^i, I^{i+1}) = f^{i-1}$.

Lemma 3.9. $\operatorname{im}(I^{i-1} \cdot \operatorname{lift}^i) \subseteq \operatorname{im}(\operatorname{div}(\operatorname{div}(f^i, I^{i+1}), I^i)$ *for any* lift^i *with* $\operatorname{lift}^i \cdot \operatorname{div}(f^i, I^{i+1}) = f^{i-1}$.

Proof. We have $\operatorname{im}(I^{i-1} \cdot f^{i-1}) \subseteq \operatorname{im}(I^i)$ by assumption on the input. Hence $\operatorname{im}(I^{i-1} \cdot \operatorname{lift}^i \cdot \operatorname{div}(f^i, I^{i+1})) \subseteq \operatorname{im}(I^i)$. The claim follows. □

Thus, lift^i represents the natural inclusion $B^i \hookrightarrow Z^i$.

$H^i = Z^i/B^i$ has the same generators as Z^i, but extra relations. These are given by by the image of lift^i. We arrive at the following two presentations of families of short exact sequences derived from I^i, f^i:

$$\begin{tikzcd}
B^i & & Z^i & & H^i \\
{D_n}^{n_{i-1}} \arrow[u] \arrow[rr, "\mathrm{lift}^i"] & & {D_n}^{r_i} \arrow[u] \arrow[rr, "\mathrm{id}_{r_i \times r_i}"] & & {D_n}^{r_i} \arrow[u] \\
{D_n}^{r_{i-1}} \arrow[u, "\mathrm{div}(f^{i-1}, I^i)"'] & & {D_n}^{s_i} \arrow[u, "\mathrm{div}(\mathrm{div}(f^i, I^{i+1}), I^i)"'] & & {D_n}^{n_{i-1}+s_i} \arrow[u]
\end{tikzcd}$$

where $\left(\dfrac{\mathrm{lift}^i}{\mathrm{div}(\mathrm{div}(f^i, I^{i+1}), I^i))}\right)$ is the missing matrix in the rightmost column, and

$$\begin{tikzcd}
Z^i & & C^i & B^{i+1} \\
{D_n}^{r_i} \arrow[u] \arrow[rr, "\mathrm{div}(f^i, I^{i+1})"] & & {D_n}^{n_i} \arrow[u] \arrow[r, "\mathrm{id}_{n_i \times n_i}"] & {D_n}^{n_i} \arrow[u] \\
{D_n}^{s_i} \arrow[u, "\mathrm{div}(\mathrm{div}(f^i, I^{i+1}), I^i)"'] & & {D_n}^{m_i} \arrow[u, "I^i"'] & {D_n}^{r_i} \arrow[u, "\mathrm{div}(f^i, I^{i+1})"']
\end{tikzcd} .$$

Example 3.10. Since the sequence (3.4) has only two terms, Step 1 of Algorithm 3.8 will result in exactly two short exact sequences, namely $0 \to Z^0 \to C^0 \to B^1 \to 0$ and $0 \to B^1 \to Z^1 \to H^1 \to 0$ together with equalities $B^0 = 0$, $Z^0 = H^0$, $Z^1 = C^1$, $B^2 = 0$.

Let us first consider the sequence $Z^0 \to C^0 \to B^1$. By the remarks in 3.3, $Z^0 = \mathrm{coker}(\mathrm{div}(\mathrm{div}(f^0, I^1), I^0))$ while $B^1 = \mathrm{coker}(\mathrm{div}(f^0, I^1))$ and of course $C^0 = \mathrm{coker}(I^1)$.

For the second sequence we note that $Z^1 = \mathrm{coker}(I^1)$. Therefore $H^1 = \mathrm{coker}\left(\dfrac{f^0}{I^1}\right)$.

Using Kan [13], one computes

$$\mathrm{div}(f^0, I^1) = \begin{pmatrix} -\partial_y & -\partial_z \\ 0 & -y \\ -z & 0 \\ -x\partial_x & 0 \\ 0 & x\partial_x \\ y\partial_y & 0 \end{pmatrix}$$

and

$$\operatorname{div}(\operatorname{div}(f^0, I^1), I^0) = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ y & 0 & 0 & 0 & 0 & 0 \\ -z & 0 & 0 & 0 & 0 & 0 \\ x\partial_x & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

We have $\text{lift}^1 = f^0$ because $Z^1 = C^1$.

3.4. In this subsection we give an algorithm that accomplishes Step 2 of Algorithm 3.8. The correctness of the algorithm is shown in Lemma 4.1 and 4.2.

We assume that an arbitrary short exact sequence of D_n-modules

$$0 \longrightarrow P_A/M_A \longrightarrow P_B/M_B \longrightarrow P_C/M_C \longrightarrow 0$$

(where P_A, P_B, P_C are free D_n-modules of rank n_A, n_B, n_C and M_A, M_B, M_C are submodules) is given to us via the set of the following 5 matrices which we call the *data structure of the short exact sequence*:

$$0 \longrightarrow \operatorname{coker}(I_A) \xrightarrow{f_{AB}} \operatorname{coker}(I_B) \xrightarrow{f_{BC}} \operatorname{coker}(I_C) \longrightarrow 0$$

where

- I_A, I_B and I_C are $m_A \times n_A, m_B \times n_B$ and $m_C \times n_C$ matrices respectively whose rows generate M_A, M_B, M_C;
- f_{AB} and f_{BC} are $n_A \times n_B$ and $n_B \times n_C$ matrices to represent the left D_n-morphisms $P_A/M_A \to P_B/M_B$ and $P_B/M_B \to P_C/M_C$ respectively. (In particular, $\operatorname{im}(I_A \cdot f_{AB}) \subseteq \operatorname{im}(I_B)$ and $\operatorname{im}(I_B \cdot f_{BC}) \subseteq \operatorname{im}(I_C)$ are part of the definition.)

We shall often abbreviate the data structure to $\left[I_A \xrightarrow{f_{AB}} I_B \xrightarrow{f_{BC}} I_C\right]$. The proof of Lemma 4.1 implies the following algorithm to transform a given short exact sequence into a V_d-strict short exact sequence.

Algorithm 3.11. (Making a short exact sequence V_d-strict.)
INPUT: $\left[I_A \xrightarrow{f_{AB}} I_B \xrightarrow{f_{BC}} I_C\right]$ and a shift vector $\mathfrak{m}_C$ on the range of I_C.
OUTPUT: $\left[J_A \xrightarrow{g_{AB}} J_{AC} \xrightarrow{g_{BC}} J_C\right]$ and shift vectors $\mathfrak{m}_A$, $\mathfrak{m}_C$ on the range of J_A and J_C.

Begin

1. Let the rows of J_C be a V_d-strict Gröbner basis for $\operatorname{im}(I_C) \subseteq P_C[\mathfrak{m}_C]$ and set $J_A = I_A$.

2. Put $f_{CB} = \mathrm{sol}(f_{BC}, I_C)_{[\bullet]}$, i.e.,

$$f_{CB} := \begin{pmatrix} \mathrm{sol}(f_{BC}, I_C)_{[1]} \\ \vdots \\ \mathrm{sol}(f_{BC}, I_C)_{[n_C]} \end{pmatrix}.$$

The matrix f_{CB} represents a lift from P_C to P_B for the map f_{BC}, $P_C \xrightarrow{f_{CB}} P_B \xrightarrow{f_{BC}} P_C \to \mathrm{coker}(I_C)$ is the projection $P_C \to \mathrm{coker}(I_C)$.

3. Set

$$\begin{aligned} f_{ACB} &:= \begin{array}{c} n_A\{ \\ n_C\{ \end{array} \left(\frac{\overbrace{\quad f_{AB} \quad}^{n_B}}{f_{CB}} \right), \\ J_{AC} &:= \mathrm{div}(f_{ACB}, I_B). \end{aligned}$$

(Then $\mathrm{coker}(J_{AC}) \cong \mathrm{coker}(I_B)$.)

4. Determine the shift $\mathfrak{m}_A$ as follows. Compute elements $\{\rho_i\}_1^{m_C}$ in $\mathrm{im}(J_{AC})$ such that $\rho_i \cdot \mathrm{pro}(n_A, n_C) = (J_C)_{[i]}$. Then set

$$\mathfrak{m}_A(k) = \min_{i|\rho_i(k)\neq 0} \{V_d \deg(\rho_i \cdot \mathrm{pro}(n_A, n_C)[\mathfrak{m}_C]) - V_d \deg(\rho_i(k)\}.$$

Here, $\rho_i(k)$ is the k-th element of the vector ρ_i, considered without shift.

5. Return

$$0 \longrightarrow \mathrm{coker}(J_A)[\mathfrak{m}_A] \xrightarrow{g_{AB}} \mathrm{coker}(J_{AC})[\mathfrak{m}_A, \mathfrak{m}_C] \xrightarrow{g_{BC}} \mathrm{coker}(J_C)[\mathfrak{m}_C] \longrightarrow 0,$$

and $\mathfrak{m}_A, \mathfrak{m}_C$ where

$$g_{AB} := \left(\mathrm{id}_{n_A \times n_A} \;\middle|\; 0_{n_A \times n_C} \right), \quad g_{BC} := \left(\frac{0_{n_A \times n_C}}{\mathrm{id}_{n_C \times n_C}} \right).$$

End.

Remark 3.12.

3.12.1. The output of Algorithm 3.11 fits into a diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{coker}(J_A[\mathfrak{m}_A]) \arrow[r] & \mathrm{coker}(J_{AC}[\mathfrak{m}_A, \mathfrak{m}_C]) \arrow[r] & \mathrm{coker}(J_C[\mathfrak{m}_C]) \arrow[r] & 0 \\ 0 \arrow[r] & {D_n}^{\mathrm{cols}(J_A)}[\mathfrak{m}_A] \arrow[r] \arrow[u] & {D_n}^{\mathrm{cols}(J_{AC})}[\mathfrak{m}_A, \mathfrak{m}_C] \arrow[r] \arrow[u] & {D_n}^{\mathrm{cols}(J_C)}[\mathfrak{m}_C] \arrow[r] \arrow[u] & 0 \\ 0 \arrow[r] & \mathrm{im}(J_A)[\mathfrak{m}_A] \arrow[r] \arrow[u] & \mathrm{im}(J_{AC})[\mathfrak{m}_A, \mathfrak{m}_C] \arrow[r] \arrow[u] & \mathrm{im}(J_C)[\mathfrak{m}_C] \arrow[r] \arrow[u] & 0 \end{tikzcd}$$

where $\mathrm{cols}(J_{AC}) = \mathrm{cols}(J_A) + \mathrm{cols}(J_C)$ and the rows and columns are V_d-strict and exact.

Another interpretation of the output is the following diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{coker}(I_A) \arrow[r] & \mathrm{coker}(I_B) \arrow[r] & \mathrm{coker}(I_C) \arrow[r] & 0 \\
0 \arrow[r] & {D_n}^{\mathrm{cols}(J_A)}[\mathfrak{m}_A] \arrow[r] \arrow[u] & {D_n}^{\mathrm{cols}(J_{AC})}[\mathfrak{m}_A, \mathfrak{m}_C] \arrow[r] \arrow[u] & {D_n}^{\mathrm{cols}(J_C)}[\mathfrak{m}_C] \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & {D_n}^{\mathrm{rows}(J_A)} \arrow[r] \arrow[u, "\cdot J_A"'] & {D_n}^{\mathrm{rows}(J_{AC})} \arrow[r] \arrow[u, "\cdot J_{AC}"'] & {D_n}^{\mathrm{rows}(J_C)} \arrow[r] \arrow[u, "\cdot J_C"'] & 0.
\end{tikzcd}$$

where $\mathrm{rows}(J_{AC}) = \mathrm{rows}(J_A) + \mathrm{rows}(J_C)$. (This will be proved in Lemma 4.4.) We can define a shift vector $\mathfrak{n}_A$ on ${D_n}^{\mathrm{rows}\, J_A}$ by $\mathfrak{n}_A(i) = V_d \deg((J_A)_{[i]}[\mathfrak{m}_A])$, and proceed in a similar fashion with J_{AC}, J_C.

3.12.2. In Lemma 4.2 we will show how this algorithm can be modified to accomplish the following: assume given are two sequences $\left[I_A \xrightarrow{f_{AB}} I_B \xrightarrow{f_{BC}} I_C\right]$ and $\left[I_D \xrightarrow{f_{DA}} I_A \xrightarrow{f_{AF}} I_F\right]$ together with a shift vector $\mathfrak{m}_C$ on the range of I_C. Then one can rewrite $\mathrm{coker}(I_A)$, $\mathrm{coker}(I_B)$, $\mathrm{coker}(I_D)$ and $\mathrm{coker}(I_F)$ as well as find shift vectors $\mathfrak{m}_A, \mathfrak{m}_B, \mathfrak{m}_D, \mathfrak{m}_F$ to make both sequences V_d-strict simultaneously.

Example 3.13. Returning to our running example we now perform Algorithm 3.11 in order to make the two sequences obtained in Example 3.10 V_3-strict. We start with $0 \to B^1 \to Z^1 \to H^1 \to 0$. One checks that $\{\partial_y, \partial_z, x\partial_x\}$ is a V_3-strict Gröbner basis for $\mathrm{im}(f^0) + \mathrm{im}(I^1)$ and thus

$$H^1 = \mathrm{coker}\begin{pmatrix} \partial_y \\ \partial_z \\ x\partial_x \end{pmatrix}.$$

So we have in the notation of Algorithm 3.11 $J_C = \begin{pmatrix} \partial_y \\ \partial_z \\ x\partial_x \end{pmatrix}$, $J_A = \mathrm{div}(f^0, I^1)$, $I_B = I^1$, $f_{AB} = f^0$ and $f_{BC} = \mathrm{id}_{1\times 1}$.

Now we want to replace I_B by a suitable matrix, according to Algorithm 3.11. To this end we need a lift for the identity map $D_3 \to D_3$, which we take to be the identity. Then according to Step 3 of Algorithm 3.11, J_B will be a Gröbner basis for the kernel of the map

$${D_3}^2 \oplus D_3 \xrightarrow{\left(\frac{f_{AB}}{\mathrm{id}_{1\times 1}}\right)} D_3 \to D_3/I_B,$$

which is given by $\operatorname{div}(\begin{pmatrix} f_{AB} \\ \hline \mathrm{id}_{1\times 1}\end{pmatrix}, I_B)$. With Kan we compute J_B to be generated by the rows of the matrix

$$\begin{pmatrix} 1 & 0 & -\partial_z \\ 0 & 1 & \partial_y \\ 0 & 0 & -x\partial_x \\ -z & 0 & 0 \\ 0 & -y & 0 \\ \partial_y & \partial_y & 0 \\ y\partial_y & 0 & 0 \\ -x\partial_x & 0 & 0 \\ 0 & -x\partial_x & 0 \end{pmatrix}.$$

In order to make the sequence V_3-strict we need to find the $\{\rho_i\}_1^3$ of Step 4 in the algorithm, i.e. preimages of the rows of J_C inside $\operatorname{im}(J_B)$ for the projection ${D_3}^2 \oplus D_3 \twoheadrightarrow D_3$. These are given by the three top rows of J_B. Since ∂_z and ∂_y have degree 1 in $D_3[0]$, the shift vector for ${D_3}^2 \oplus D_3$ is $[1, 1, 0]$.

Setting $I^0_{B,1} := J_A$, $I^0_{C,1} := J_B$, $I^0_{H,1} := J_C$ we obtain the new V_3-strict short exact sequence

$$0 \to \operatorname{coker}(I^0_{B,1})[1,1] \to \operatorname{coker}(I^0_{C,1})[1,1,0] \to \operatorname{coker}(I^0_{H,1})[0] \to 0.$$

Now we make the sequence $0 \to Z^0 \to C^0 \to B^1 \to 0$ V_3-strict. In the new situation we put $I_C = J_C = \operatorname{div}(f^0, I^1)$, $I_B = I^0$, $I_A = J_A = \operatorname{div}(\operatorname{div}(f^0, I^1), I^0)$, $f_{AB} = \operatorname{div}(f^0, I^1)$ and $f_{BC} = \mathrm{id}_{6\times 6}$. Proceeding in a similar way as before we see that the new number of generators for C^0 will be 2+6=8, given by the images of the generators for Z^0 and preimages for the generators of B^1. Thus $J_B = \operatorname{div}(\begin{pmatrix} \operatorname{div}(f^0, I^1) \\ \hline \mathrm{id}_{2\times 2}\end{pmatrix}, I_A)$

which turns out to be

$$\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & \partial_y & \partial_z \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & y \\
0 & 0 & 1 & 0 & 0 & 0 & z & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -z & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -y \\
0 & 0 & 0 & 0 & 0 & 0 & -x\partial_x & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -y\partial_y & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -x\partial_x \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
z & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-y & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
-x\partial_x & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}.$$

As before we need preimages of the rows of J_C inside J_A. They are given by rows 1, 5, 4, 6, 8, 7 in that sequence. We conclude that the shift on ${D_3}^6$ must be $[2,0,0,1,1,1]$ (here the last three shifts are arbitrary).

Thus the V_3-strict sequence is (with $I^0_{C,0} := J_B$, $I^0_{Z,0} = J_A$)

$$\left[I^0_{Z,0}[2,0,0,1,1,1] \longrightarrow I^0_{C,0}[2,0,0,1,1,1,1,1] \longrightarrow I^0_{B,1}[1,1]\right].$$

3.5. Now we come to an algorithm that shows how to perform "one step" in the resolution of a short exact V_d-strict sequence, a preparatory result for Step 3 of Algorithm 3.8.

Algorithm 3.14. (From a V_d-strict short exact sequence to a V_d-strict 2×3 complex.)

INPUT: $\left[I_A[\mathfrak{m}_A] \xrightarrow{f_{AB}} I_B[\mathfrak{m}_B] \xrightarrow{f_{BC}} I_C[\mathfrak{m}_C]\right]$ such that

- I_A and I_C are V_d-strict Gröbner bases for $\operatorname{im}(I_A)[\mathfrak{m}_A]$ and $\operatorname{im}(I_C)[\mathfrak{m}_C]$ respectively,
- $\operatorname{cols}(I_A) + \operatorname{cols}(I_C) = \operatorname{cols}(I_B)$,
- $f_{AB} = \left(\ \mathrm{id}_{\operatorname{cols}(I_A)\times\operatorname{cols}(I_A)} \ \middle| \ 0_{\operatorname{cols}(I_A)\times\operatorname{cols}(I_C)} \ \right)$,
  $f_{BC} = \left(\dfrac{0_{\operatorname{cols}(I_A)\times\operatorname{cols}(I_C)}}{\mathrm{id}_{\operatorname{cols}(I_C)\times\operatorname{cols}(I_C)}}\right)$,
- $\mathfrak{m}_B = (\mathfrak{m}_A, \mathfrak{m}_C)$.

OUTPUT: A 2×3 double complex, a one step free V_d-strict resolution of the input.

Begin

1. Set $\mathfrak{n}_C(i) = V_d \deg((I_C)_{[i]}[\mathfrak{m}_C])$, $\mathfrak{n}_A(i) = V_d \deg((I_A)_{[i]}[\mathfrak{m}_A])$ and $\mathfrak{n}_B = (\mathfrak{n}_A, \mathfrak{n}_C)$. This defines shift vectors on the domains of I_A, I_B, I_C.
2. Set J_A a V_d-strict Gröbner basis for $\mathrm{im}(\mathrm{syz}(I_A)) \subseteq {D_n}^{\mathrm{rows}(I_A)}[\mathfrak{n}_A]$, let J_C be a V_d-strict Gröbner basis for $\mathrm{im}(\mathrm{syz}(I_C)) \subseteq {D_n}^{\mathrm{rows}(I_C)}[\mathfrak{n}_C]$.
3. Let
$$\begin{aligned} f_{CB} &= \mathrm{sol}(f_{BC}, I_C)_{[\bullet]}, \\ f_{ACB} &= \left(\frac{f_{AB}}{f_{CB}}\right), \\ J_B &= \mathrm{div}(f_{ACB}, I_B). \end{aligned}$$
4. Set
$$\begin{aligned} g_{AB} &= \left(\mathrm{id}_{\mathrm{rows}(I_A)\times\mathrm{rows}(I_A)} \,\middle|\, 0_{\mathrm{rows}(I_A)\times\mathrm{rows}(I_C)} \right), \\ g_{BC} &= \left(\frac{0_{\mathrm{rows}(I_A)\times\mathrm{rows}(I_C)}}{\mathrm{id}_{\mathrm{rows}(I_C)\times\mathrm{rows}(I_C)}}\right). \end{aligned}$$
5. Return $\left[J_A[\mathfrak{n}_A] \xrightarrow{g_{AB}} J_B[\mathfrak{n}_A, \mathfrak{n}_C] \xrightarrow{g_{BC}} J_C[\mathfrak{n}_C]\right]$.

End.

The output of Algorithm 3.14 can be put into the following picture

$$\begin{array}{ccccc}
\mathrm{coker}(I_A) & \longrightarrow & \mathrm{coker}(I_B) & \longrightarrow & \mathrm{coker}(I_C) \\
\uparrow & & \uparrow & & \uparrow \\
{D_n}^{\mathrm{cols}(I_A)}[\mathfrak{m}_A] & \xrightarrow{f_{AB}} & {D_n}^{\mathrm{cols}(I_B)}[\mathfrak{m}_B] & \xrightarrow{f_{BC}} & {D_n}^{\mathrm{cols}(I_C)}[\mathfrak{m}_C] \\
I_A \uparrow & & I_B \uparrow & & I_C \uparrow \\
{D_n}^{\mathrm{cols}(J_A)}[\mathfrak{n}_A] & \xrightarrow{g_{AB}} & {D_n}^{\mathrm{cols}(J_B)}[\mathfrak{n}_B] & \xrightarrow{g_{BC}} & {D_n}^{\mathrm{cols}(J_C)}[\mathfrak{n}_C] \\
J_A \uparrow & & J_B \uparrow & & J_C \uparrow \\
{D_n}^{\mathrm{rows}(J_A)} & \longrightarrow & {D_n}^{\mathrm{rows}(J_B)} & \longrightarrow & {D_n}^{\mathrm{rows}(J_C)}
\end{array}$$

which has exact and V_d-strict rows and columns.

It is important to notice that $\left[J_A[\mathfrak{n}_A] \xrightarrow{g_{AB}} J_B[\mathfrak{n}_A, \mathfrak{n}_C] \xrightarrow{g_{BC}} J_C[\mathfrak{n}_C]\right]$ satisfies the conditions on the input of Algorithm 3.14. This facilitates repetition of the algorithm which leads to a Cartan-Eilenberg resolution for the sequence $\left[I_A[\mathfrak{m}_A] \xrightarrow{f_{AB}} I_B[\mathfrak{m}_B] \xrightarrow{f_{BC}} I_C[\mathfrak{m}_C]\right]$ with V_d-strict rows and columns:

Algorithm 3.15. (From a short exact sequence to a V_d-strict Cartan-Eilenberg resolution of length L.)

INPUT: $\left[I_A \xrightarrow{f_{AB}} I_B \xrightarrow{f_{BC}} I_C\right]$.
OUTPUT: a free double complex $I_A^\bullet[\mathfrak{m}_{A,\bullet}] \to I_B^\bullet[\mathfrak{m}_{B,\bullet}] \to I_C^\bullet[\mathfrak{m}_{C,\bullet}]$ such that the rows are exact, the columns resolutions, and the rows and columns are V_d-strict.
Begin

1. Apply Algorithm 3.11 with input $\left[I_A \xrightarrow{f_{AB}} I_B \xrightarrow{f_{BC}} I_C\right]$ and output $\left[I_A^0[\mathfrak{m}_{A,0}] \xrightarrow{f_{AB}^0} I_B^0[\mathfrak{m}_{B,0}] \xrightarrow{f_{BC}^0} I_C^0[\mathfrak{m}_{C,0}]\right]$ to render the sequence V_d-strict.
2. For $l = 0, \dots, L$ repeat
   (a) INPUT(Algorithm 3.14):= $\left[I_A^l[\mathfrak{m}_{A,l}] \xrightarrow{f_{AB}^l} I_B^l[\mathfrak{m}_{B,l}] \xrightarrow{f_{BC}^l} I_C^l[\mathfrak{m}_{C,l}]\right]$,
   (b) OUTPUT(Algorithm 3.14)=: $\left[I_A^{l+1}[\mathfrak{m}_{A,l+1}] \xrightarrow{f_{AB}^{l+1}} I_B^{l+1}[\mathfrak{m}_{B,l+1}] \xrightarrow{f_{BC}^{l+1}} I_C^{l+1}[\mathfrak{m}_{C,l+1}]\right]$.
3. Return the double complex of free modules with vertical maps $\delta_X^{l+1} : P_X^{l+1} = {D_n}^{\mathrm{cols}(I_X^l)} \to {D_n}^{\mathrm{rows}(I_X^l)} = P_X^l$ equal to right multiplication by I_X^l (where $X \in \{A, B, C\}$ and horizontal maps $(-1)^l f_{AB}^l : P_A^l \to P_B^l$, $(-1)^l f_{BC}^l : P_B^l \to P_C^l$.

End.

Remark 3.16.
3.16.1. The modules in the resolution for B in the middle are the direct sum of the outer two. Moreover, the map from the left column to the middle one is an inclusion of resolutions while the map from the middle to the right column is a projection of resolutions.
3.16.2. We do not need to build resolutions of length greater than $n+r$ since the global homological dimension of D_n is n and we are interested in computing hyper-Tor of a complex of length r.
3.16.3. We shall write $\left[I_A^\bullet[\mathfrak{m}_{A,\bullet}] \xrightarrow{f_{AB}^\bullet} I_B^\bullet[\mathfrak{m}_{B,\bullet}] \xrightarrow{f_{BC}^\bullet} I_C^\bullet[\mathfrak{m}_{C,\bullet}]\right]$ for the double complex returned by Algorithm 3.15.

Example 3.17. Now we come to the computation of resolutions in our running example.

We first compute resolutions for $I_{Z,0}[2,0,0,1,1,1]$, $I_{B,1}[1,1]$ and $I_{H,1}[0]$. These are as follows.

For Z^0:

$$D_3[0] \xrightarrow{\delta_{Z,0}^3} {D_3}^3[0,1,1] \xrightarrow{\delta_{Z,0}^2} {D_3}^8[0,0,1,1,1,1,1,2] \xrightarrow{\delta_{Z,0}^1} {D_3}^6[2,0,0,1,1,1] \twoheadrightarrow Z^0$$

where

$$\begin{aligned}
\delta^1_{Z,0} &= \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ y & 0 & 0 & 0 & 0 & 0 \\ -z & 0 & 0 & 0 & 0 & 0 \\ x\partial_x & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \\
\delta^2_{Z,0} &= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & -z & -y & 0 \\ 0 & 0 & 0 & 0 & 0 & x\partial_x & 0 & -y \\ 0 & 0 & 0 & 0 & 0 & 0 & x\partial_x & z \end{pmatrix} \\
\text{and} \quad \delta^3_{Z,0} &= \begin{pmatrix} x\partial_x & z & y \end{pmatrix}.
\end{aligned}$$

For B^1:

$$D_3{}^2[2,1] \xrightarrow{\delta^3_{B,1}} D_3{}^6[2,1,1,2,1,1] \xrightarrow{\delta^2_{B,1}} D_3{}^6[2,1,1,1,1,1] \xrightarrow{\delta^1_{B,1}} D_3{}^2[1,1] \twoheadrightarrow B^1$$

where

$$\begin{aligned}
\delta^1_{B,1} &= \begin{pmatrix} -\partial_y & -\partial_z \\ 0 & -y \\ -z & 0 \\ -x\partial_x & 0 \\ 0 & x\partial_x \\ y\partial_y & 0 \end{pmatrix}, \\
\delta^2_{B,1} &= \begin{pmatrix} -y & \partial_z & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & y\partial_y & 0 & x\partial_x \\ 0 & 0 & y\partial_y & 0 & 0 & z \\ -x\partial_x & 0 & 0 & \partial_y & -\partial_z & 0 \\ 0 & 0 & -x\partial_x & z & 0 & 0 \\ 0 & x\partial_x & 0 & 0 & y & 0 \end{pmatrix} \\
\text{and} \quad \delta^3_{B,1} &= \begin{pmatrix} -x\partial_x & -1 & 0 & y & 0 & \partial_z \\ 0 & z & -x\partial_x & 0 & -y\partial_y & 0 \end{pmatrix}.
\end{aligned}$$

For H^1:

$$D_3[2] \xrightarrow{\delta^3_{H,1}} D_3{}^3[2,1,1] \xrightarrow{\delta^2_{H,1}} D_3{}^3[1,1,0] \xrightarrow{\delta^1_{H,1}} D_3[0] \twoheadrightarrow H^1$$

where $\delta^1_{H,1} = \begin{pmatrix} \partial_y \\ \partial_z \\ x\partial_x \end{pmatrix}$, $\delta^2_{H,1} = \begin{pmatrix} \partial_z & -\partial_y & 0 \\ x\partial_x & 0 & -\partial_y \\ 0 & x\partial_x & -\partial_z \end{pmatrix}$ and $\delta^3_{H,1} = \begin{pmatrix} x\partial_x & -\partial_z & \partial_y \end{pmatrix}$.

We now construct the resolution for $C^1 = Z^1$. From the above resolutions for B^1 and H^1 we see that the modules in the resolution will be

$$\begin{aligned} P^0_{C,1} &= {D_3}^2[1,1] \oplus D_3[0], \\ P^1_{C,1} &= {D_3}^6[2,1,1,1,1,1] \oplus {D_3}^3[1,1,0], \\ P^2_{C,1} &= {D_3}^6[2,1,1,2,1,1] \oplus {D_3}^3[2,1,1] \text{ and} \\ P^3_{C,1} &= {D_3}^2[2,1] \oplus D_3[2]. \end{aligned}$$

In order to produce the differentials in the complex $P^\bullet_{C,1}$ we need to find elements in $\ker(\delta^{\bullet-1}_{C,1}) = I^{\bullet-1}_{C,1}$ that project onto the rows of the matrices in the resolution for H^1 under the projection.

For example, to find $\delta^1_{C,1}$ we need to find three 3-vectors in $I^0_{C,1}$ whose last (third) components are ∂_y, ∂_z and $x\partial_x$ respectively, and who do have the same V_3-degree inside $P^1_{C,1}$ as their projections into $P^1_{H,1} = D_3[0]$. Those are for example $(0,1,\partial_y)$, $(-1,0,\partial_z)$ and $(0,0,x\partial_x)$. Thus, $\delta^1_{C,1} = \left(\begin{array}{cc|c} \multicolumn{2}{c|}{\delta^1_{B,1}} & 0_{6\times 1} \\ \hline 0 & 1 & \partial_y \\ -1 & 0 & \partial_z \\ 0 & 0 & x\partial_x \end{array}\right)$. In order to find $\delta^2_{C,1}$ we need to lift the rows of $\delta^2_{H,1}$ into $\ker(\delta^1_{C,1}) \subseteq P^1_{C,1}$ without increasing V_3-degree. One can see that one way of accomplishing this is

$$\delta^2_{C,1} = \left(\begin{array}{cccccc|ccc} \multicolumn{6}{c|}{\delta^2_{B,1}} & \multicolumn{3}{c}{0_{6\times 3}} \\ \hline 1 & 0 & 0 & 0 & 0 & 0 & \partial_z & -\partial_y & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & x\partial_x & 0 & -\partial_y \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & x\partial_x & -\partial_z \end{array}\right).$$

Finally one finds

$$\delta^3_{C,1} = \left(\begin{array}{cccccc|ccc} \multicolumn{6}{c|}{\delta^3_{B,1}} & \multicolumn{3}{c}{0_{2\times 3}} \\ \hline 0 & 0 & 0 & 1 & 0 & 0 & x\partial_x & -\partial_z & -\partial_y \end{array}\right).$$

For the first sequence, $0 \to Z^0 \to C^0 \to B^1 \to 0$ one computes

$$\begin{aligned} P^0_{C,0} &= {D_3}^6 \oplus {D_3}^2[2,0,0,1,1,1,1,1], \\ P^1_{C,0} &= {D_3}^8 \oplus {D_3}^6[0,0,1,1,1,1,1,2,2,1,1,1,1,1], \\ P^2_{C,0} &= {D_3}^3 \oplus {D_3}^6[0,1,12,1,1,2,1,1], \\ P^3_{C,0} &= D_3 \oplus {D_3}^2[0,2,1]. \end{aligned}$$

Moreover, we may take

$$
\delta^1_{C,0} = \left(\begin{array}{cccccc|cc}
\multicolumn{6}{c|}{\delta^1_{Z,0}} & \multicolumn{2}{c}{0_{8\times 2}} \\ \hline
-1 & 0 & 0 & 0 & 0 & 0 & -\partial_y & -\partial_z \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -y \\
0 & 0 & 0 & 0 & 0 & 0 & -z & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -x\partial_x & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & x\partial_x \\
0 & 0 & 0 & 0 & 0 & 0 & y\partial_y & 0
\end{array}\right),
$$

$$
\delta^2_{C,0} = \left(\begin{array}{cccccccc|cccccc}
\multicolumn{8}{c|}{\delta^2_{Z,0}} & \multicolumn{6}{c}{0_{3\times 6}} \\ \hline
0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -y & \partial_z & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & y\partial_y & 0 & x\partial_x \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & y\partial_y & 0 & 0 & z \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -x\partial_x & 0 & 0 & \partial_y & -\partial_z & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -x\partial_x & z & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x\partial_x & 0 & 0 & y & 0
\end{array}\right),
$$

$$
\delta^3_{C,0} = \left(\begin{array}{ccc|cccccc}
\multicolumn{3}{c|}{\delta^3_{Z,0}} & \multicolumn{6}{c}{0_{1\times 6}} \\ \hline
0 & -1 & 0 & -x\partial_x & -1 & 0 & y & 0 & \partial_z \\
0 & 0 & 0 & 0 & z & -x\partial_x & 0 & -y\partial_y & 0
\end{array}\right).
$$

Algorithm 3.15 shows how to compute V_d-strict Cartan-Eilenberg resolutions of short exact sequences. Remark 3.12 shows how to find a V_d-strict presentation of all the short exact sequences $0 \to B^i \to Z^i \to H^i \to 0$ and $0 \to Z^i \to C^i \to B^{i+1} \to 0$ in a compatible way. Lemma 4.2 in the next chapter explains how to modify Algorithm 3.15 in order to find simultaneously V_d-strict resolutions for the short exact sequences above in such a way that the two resolutions B^i (resp. Z^i) are identical.

Assuming we have constructed such resolutions, we can now find a V_d-strict Cartan-Eilenberg resolution for $C^\bullet$ as follows.

Algorithm 3.18. (Assembling double complexes.)
INPUT: A complex $C^\bullet$, broken into short exact sequences $0 \to B^i \to Z^i \to H^i \to 0$, $0 \to Z^i \to C^i \to B^{i+1} \to 0$, and V_d-strict Cartan-Eilenberg resolutions $\left[I^\bullet_{B,i} \stackrel{f_{B^i,Z^i}}{\longrightarrow} I^\bullet_{Z,i} \stackrel{f_{Z^i,H^i}}{\longrightarrow} I^\bullet_{H,i}\right]$ and $\left[I^\bullet_{Z,i} \stackrel{f_{Z^i,C^i}}{\longrightarrow} I^\bullet_{C,i} \stackrel{f_{C^i,B^{i+1}}}{\longrightarrow} I^\bullet_{B,i+1}\right]$ for those sequences such that the two resolutions for Z^i, B^i are the same for all i.
OUTPUT: A double complex

$$
\begin{array}{ccc}
P^j_{C,i}[\mathfrak{m}_{C,i,j}] & \longrightarrow & P^j_{C,i+1}[\mathfrak{m}_{C,i+1,j}] \\
\uparrow & & \uparrow \\
P^{j+1}_{C,i}[\mathfrak{m}_{C,i,j+1}] & \longrightarrow & P^{j+1}_{C,i+1}[\mathfrak{m}_{C,i+1,j+1}]
\end{array}
$$

such that

- $P^\bullet_{C,i}[\mathfrak{m}_{C,i,\bullet}]$ is a V_d-strict resolution of C^i,
- the rows are exact and V_d-strict,
- the total complex $Tot^\bullet(P^\bullet_{C,\bullet}[\mathfrak{m}_{C,\bullet,\bullet}])$ is V_d-strict and quasi-isomorphic to $C^\bullet$.

Begin

- Set $P^l_{C,i}[\mathfrak{m}_{C,i,l}] = P^l_{B^i} \oplus P^l_{Z^i} \oplus P^l_{B^{i+1}}[\mathfrak{m}_{B,i,l}, \mathfrak{m}_{Z,i,l}, \mathfrak{m}_{B,i+1,l}]$,
- and take $P^l_{C,i} \to P^l_{C,i+1}$ to be the canonical map $P^l_{C,i} \twoheadrightarrow P^l_{B,i+1} \hookrightarrow P^l_{C,i+1}$ multiplied by $(-1)^l$.

End.

The correctness of this algorithm (i.e., the V_n-strictness of the associated total complex) is the topic of Proposition 4.5.

Example 3.19. In the running example we see that the complex $C^\bullet$ is quasi-isomorphic to the V_3-strict complex $A^\bullet[\mathfrak{m}_\bullet]$ given by the total complex of

(3.5)
$$\begin{tikzcd}
D_6{}^6 \oplus D_3{}^2[2,0,0,1,1,1,1,1] \arrow[r, "\left(\begin{array}{c|c} \multicolumn{2}{c}{0_{6\times 3}} \\ \hline \mathrm{id}_{2\times 2} & 0_{2\times 1}\end{array}\right)"] & D_3{}^2 \oplus D_3[1,1,0] \\
D_3{}^8 \oplus D_3{}^6[0,0,1,1,1,1,1,2,2,1,1,1,1,1] \arrow[u] \arrow[r, "\left(\begin{array}{c|c} \multicolumn{2}{c}{0_{8\times 9}} \\ \hline -\mathrm{id}_{6\times 6} & 0_{6\times 2}\end{array}\right)"] & D_3{}^6 \oplus D_3{}^3[2,1,1,1,1,1,1,1,0] \arrow[u] \\
D_3{}^3 \oplus D_3{}^6[0,1,1,2,1,1,2,1,1] \arrow[u] \arrow[r, "\left(\begin{array}{c|c} \multicolumn{2}{c}{0_{3\times 9}} \\ \hline \mathrm{id}_{6\times 6} & 0_{6\times 3}\end{array}\right)"] & D_3{}^6 \oplus D_3{}^3[2,1,1,2,1,1,2,1,1] \arrow[u] \\
D_3{}^1 \oplus D_3{}^2[0,2,1] \arrow[u] \arrow[r, "\left(\begin{array}{c|c} \multicolumn{2}{c}{0_{1\times 3}} \\ \hline -\mathrm{id}_{2\times 2} & 0_{2\times 1}\end{array}\right)"] & D_3{}^2 \oplus D_3{}^1[2,1,2] \arrow[u]
\end{tikzcd}$$

with vertical maps given as in Example 3.17, horizontal maps as given in Algorithm 3.18 and where the terminal module $D_2{}^2 \oplus D_3[1,1,0]$ is positioned in cohomological degree 1.

Remark 3.20. Set $P^l_{Z,i} = P^l_{B,i} \oplus P^l_{H,i}$. Since we are dealing with Cartan-Eilenberg resolutions, then $P^\bullet_{Z,i}[\mathfrak{m}_{B,i,\bullet}, \mathfrak{m}_{H,i,\bullet}]$ is a V_d-strict resolution of Z^i sitting inside $P^\bullet_{C,i}[\mathfrak{m}_{B,i,\bullet}, \mathfrak{m}_{H,i,\bullet}, \mathfrak{m}_{B,i+1,\bullet}]$.

Example 3.21. This example will show that our algorithm does not provide minimal resolutions, far from it. Let us find a V_2-strict complex quasi-isomorphic to

$$0 \to \underbrace{D_2}_{\text{degree } 1} \xrightarrow{f_1} \underbrace{D_2{}^2}_{\text{degree } 2} \xrightarrow{f_2} \underbrace{D_2}_{\text{degree } 3} \to 0 \tag{3.6}$$

where

$$f_1 = (-xy, x + x^2 + y^2),\ f_2 = \begin{pmatrix} x + x^2 + y^2 \\ xy \end{pmatrix}.$$

We note that this complex is quasi-isomorphic to

$$0 \to \underbrace{D_2/D_2 \cdot (x + x^2 + y^2, xy)}_{\text{degree } 3} \to 0,$$

which is a complex with single support, so it is possible to transform it into a V_d-strict free complex by the method given in [11].

Let us execute the various steps of Algorithm 3.8. Consider the complex (3.6). Then $I^1 = I^2 = I^3 = I^4 = 0$. We split it into short exact sequences as we explained in subsection 3.3. The last short exact sequence is

$$0 \to B^3 \longrightarrow Z^3 \longrightarrow H^3 \to 0.$$

Here we have the following presentations of the modules.

$$\begin{aligned} (3.7)\quad B^3 &= \operatorname{coker}(\operatorname{div}(f^2, I^3)) = \operatorname{coker}(\operatorname{div}(f^2, 0)) \\ &=: \operatorname{coker}(I_A), \\ (3.8)\quad Z^3 &= \operatorname{coker}(\operatorname{div}(\operatorname{div}(f^3, I^4), I^3)) = \operatorname{coker}(\operatorname{syz}(\operatorname{syz}(D_2 \to 0))) \\ &= \operatorname{coker}(0_{1\times 1}). \end{aligned}$$

Thus, with $H^3 = Z^3/B^3 =: \operatorname{coker}(I_C)$ we can express the above sequence with the data

$$I_A = (xy, -x - x^2 - y^2), \quad I_B = (0), \quad I_C = \begin{pmatrix} x + x^2 + y^2 \\ xy \end{pmatrix},$$

$$f_{AB} = \begin{pmatrix} x + x^2 + y^2 \\ xy \end{pmatrix}, \quad f_{BC} = (1).$$

We transform it into a V_2-strict complex with the starting shift vector $\mathfrak{m}_C = [0]$ applying Algorithm 3.11. In Step 1, we obtain

$$J_C = \begin{pmatrix} x + x^2 + y^2 \\ xy \\ y^3 \end{pmatrix}$$

and $J_A = I_A$. In Steps 2 and 3, we determine J_{AC}. We note that $f_{BC} = (1)$ and $f_{CB} = (1)$ whence $f_{ACB} = \begin{pmatrix} x + x^2 + y^2 \\ xy \\ 1 \end{pmatrix}$. So J_{AC} is the syzygy module on $x + x^2 + y^2$, xy, 1 and consequently

$$J_{AC} = \begin{pmatrix} 1 & 0 & -x - x^2 - y^2 \\ 0 & 1 & -xy \end{pmatrix}.$$

From this we see that $\rho_1 = (J_{AC})_{[1]}$, $\rho_2 = (J_{AC})_{[2]}$. Then $\mathfrak{m}_A(1) = \min(-1-0) = -1$ and $\mathfrak{m}_A(2) = \min(-2-0) = -2$. Hence the output is

(3.9)

$$0 \to {D_2}^2/J_A[-1,-2] \xrightarrow{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}} ({D_2}^2 \oplus D_2)/J_{AC}[-1,-2,0] \xrightarrow{\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}} D_2/J_C[0] \to 0.$$

Now apply Algorithm 3.15 to (3.9). We obtain a double complex over (3.9), which is a V_2-strict free complex. It is given below. To save space, we put $p = x + x^2 + y^2$ and $(M) = \begin{pmatrix} y^3 & -xy^2 & -x-y^2 \\ 0 & y^2 & -x \\ y & -1-x & -1 \end{pmatrix}$.

Horizontal boundary maps are either of the form $\left(\begin{array}{c|c} \mathrm{id} & 0 \end{array} \right)$ or $\left(\dfrac{0}{\mathrm{id}} \right)$.

$$\begin{tikzcd}
{D_2}^2/I_A \arrow[r] & ({D_2}^2 \oplus D_2)/I_{AC} \arrow[r] & D_2/I_C \\
{D_2}^2[-1,-2] \arrow[r] \arrow[u] & {D_2}^2 \oplus D_2[-1,-2,0] \arrow[r] \arrow[u] & D_2[0] \arrow[u] \\
D_2[-3] \arrow[r] \arrow[u, "\cdot(xy,-p)"'] & D_2 \oplus {D_2}^3[-3,-1,-2,-3] \arrow[r] \arrow[u, "\cdot\begin{pmatrix} xy & -p & 0 \\ 0 & 0 & p \\ 0 & 0 & xy \\ 0 & 0 & y^3 \end{pmatrix}"'] & {D_2}^3[-1,-2,-3] \arrow[u, "\cdot\begin{pmatrix} p \\ xy \\ y^3 \end{pmatrix}"'] \\
 & {D_2}^3[-4,-4,-2] \arrow[r] \arrow[u, "\cdot\left(\begin{array}{c|c} \begin{matrix} 0 \\ 0 \\ 0 \end{matrix} & M \end{array}\right)"'] & {D_2}^3[-4,-4,-2] \arrow[u, "\cdot(M)"'] \\
 & D_2[-4] \arrow[r] \arrow[u] & D_2[-4] \arrow[u, "\cdot(1,-1,-y^2)"']
\end{tikzcd}$$

The total complex of this double complex is substantially bigger than the V_2-strict resolution for $D_2/D_2 \cdot (p, xy)$ one computes with Kan directly and which agrees essentially with the rightmost column of the diagram.

4. Correctness of the Algorithms in Section 3

In this section we justify the algorithms from the previous section. Let $0 \to C^0 \to \cdots \to C^r \to 0$ be the given complex. For the remainder of the paper, if X is a module, P_X will denote a free module projecting onto X and I_X the kernel of that projection.

4.1. We first prove the correctness of Algorithm 3.11.

Lemma 4.1. *Let $0 \to P_A/I_A \to P_B/I_B \to P_C/I_C \to 0$ be exact and assume that on P_C there is given a shift vector $\mathfrak{m}_C$.*

Then P_B/I_B can be replaced by a certain other quotient of a free module isomorphic to P_B/I_B, such that there exist shift vectors $\mathfrak{m}_A, \mathfrak{m}_B$ making the sequence V_d-strict.

Proof. We remark that making the sequence V_d-adapted is trivial (but not good enough).

Set $Q_B = P_A \oplus P_C$. Pick a D_n-morphism ψ from P_C to P_B/I_B that lifts $P_B/I_B \to P_C/I_C$. We then define $Q_B \to P_B/I_B$ as $(Q_B \twoheadrightarrow P_A \twoheadrightarrow P_A/I_A \hookrightarrow P_B/I_B) + (Q_B \twoheadrightarrow P_C \xrightarrow{\psi} P_B/I_B)$. This map is surjective.

Let $I_{A,C} = \ker(Q_B \twoheadrightarrow P_B/I_B)$. $I_{A,C}$ contains $I_A \oplus 0$, corresponding to the natural inclusion $P_A/I_A \hookrightarrow P_B/I_B = Q_B/I_{A,C}$.

We prove now that this sequence can be given shift vectors to make it V_d-strict. Define the shift on Q_B by taking the given shift from P_C on the second component, and for the generators of P_A take an arbitrary shift for the moment.

It is clear that the resulting short exact sequence

$$0 \to P_A[\mathfrak{m}_A]/I_A \to Q_B[\mathfrak{m}_A, \mathfrak{m}_C]/I_{A,C} \to P_C[\mathfrak{m}_C]/I_C \to 0$$

is V_d-adapted. It is just as clear that it is V_d-strict at P_C/I_C and P_A/I_A. Let $b = \sum \alpha_i e_{A,i} + \sum \gamma_j e_{C,j}$ be an element of Q_B that is sent to zero in P_C/I_C. (Here, $\{e_{A,i}\}$ and $\{e_{C,j}\}$ represent a basis for P_A and P_C respectively, while α_i and γ_j are in D_n.) That means that $\sum \gamma_j c_j \in I_C$. Since $I_{A,C}$ contains for all elements $c \in I_C$ an element (a_c, c) (after all, modulo the first component of Q_B, elements of $0 \oplus I_C$ must be zero in $Q_B/I_{A,C}$!) we can transform our element b modulo $I_{A,C}$ into an element of $P_A \oplus 0$. We would like this element to have V_d-degree at most $V_d \deg(b)$.

Remember that we can still choose the shift for P_A. Take a V_d-strict Gröbner basis $\{c_i\}$ for I_C relative to the given shift $\mathfrak{m}_C$. For each c_i find a a_i in P_A such that $(a_i, c_i) \in I_{A,C}$. Now define the shift on P_A in such a way that a_i has V_d-degree at most equal to the V_d-degree of c_i for all i. We claim that with this shift the sequence is V_d-strict. To see that, return to $b = \sum \alpha_i e_{A,i} + \sum \gamma_j e_{C,j}$. The V_d-degree of b is the maximum of the degrees of the two sums. Since $\sum \gamma_j e_{C,j}$ is in I_C, we can write it as a sum $\sum \gamma'_k c_k$, where the V_d-degree of the sum, let's call it e, is the V_d-degree of the largest summand in the sum, because the c_i form a V_d-strict Gröbner basis. Modulo $I_{A,C}$, this is the same as the sum $-\sum \gamma_i' a_i$, which has lower or equal V_d-degree, by construction of the shift on P_A. Then $\sum \alpha_i e_{A,i} - \sum \gamma_k' a_k \in P_A$ is an operator that maps onto b, modulo $I_{A,C}$, and has V_d-degree at most equal to e. Strictness at $Q_B/I_{A,C}$ follows.

This ends the proof and shows that Algorithm 3.11 is correct. □

We obtain a commutative diagram with exact and V_d-strict rows and columns

$$\begin{tikzcd}
& 0 & 0 & 0 & \\
0 \arrow[r] & P_A/I_A \arrow[u] \arrow[r] & P_B/I_B \arrow[u] \arrow[r] & P_C/I_C \arrow[u] \arrow[r] & 0 \\
0 \arrow[r] & P_A[\mathfrak{m}_A] \arrow[u] \arrow[r] & Q_B[\mathfrak{m}_A, \mathfrak{m}_C] \arrow[u] \arrow[r] & P_C[\mathfrak{m}_C] \arrow[u] \arrow[r] \arrow[ul, "\psi"'] & 0 \\
0 \arrow[r] & I_A \arrow[u] \arrow[r] & I_{A,C} \arrow[u] \arrow[r] & I_C \arrow[u] \arrow[r] & 0 \\
& 0 \arrow[u] & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}$$

and ψ is V_d-adapted. ψ corresponds to the map f_{CB} in Algorithm 3.11. We will have need of a slight improvement of Lemma 4.1, because in Algorithm 3.8 we have to simultaneously make two interrelated short exact sequences V_d-strict.

Lemma 4.2. *Suppose we have two short exact sequences*

$$0 \to P_A/I_A \to P_B/I_B \to P_C/I_C \to 0$$

and

$$0 \to P_D/I_D \to P_A/I_A \to P_F/I_F \to 0$$

and assume that on P_C is given a shift vector $\mathfrak{m}_C$. Then one can rewrite P_A/I_A as $Q_A/I_{D,F}$ and P_B/I_B as $Q_B/I_{D,F,C}$ and find shift vectors $\mathfrak{m}_A, \mathfrak{m}_B, \mathfrak{m}_D, \mathfrak{m}_F$ such that the resulting two sequences are exact and V_d-strict.

Proof. First use the first half of the proof of the previous lemma to write P_A/I_A as $Q_A/I_{D,F}$ and then with that presentation of P_A/I_A rewrite P_B/I_B as $Q_B/I_{D,F,C}$. So in particular, $Q_B = Q_A \oplus P_C = P_D \oplus P_F \oplus P_C$.

In order to find the proper shift vectors, proceed as follows:

1. Take a V_d-strict Gröbner basis $\{c_i\}$ for I_C relative to the given shift $\mathfrak{m}_C$. For all i find $a_i = (d_i, f_i)$ such that $(d_i, f_i, c_i) \in I_{D,F,C}$.
2. Pick a shift $\mathfrak{m}_F$ on F such that $V_d \deg(f_i[\mathfrak{m}_F]) \leq V_d \deg(c_i[\mathfrak{m}_C])$ for all i.

3. Compute a V_d-strict Gröbner basis $\{f_{i'}\}$ for I_F, using the shift $\mathfrak{m}_F$. For all i' find $d'_{i'}$ with $(d'_{i'}, f'_{i'}) \in I_{D,F}$.
4. Pick a shift $\mathfrak{m}_D$ on P_D such that $V_d \deg(d_i[\mathfrak{m}_D]) \leq V_d \deg(c_i[\mathfrak{m}_C])$ for all i and $V_d \deg(d'_{i'}[\mathfrak{m}_D]) \leq V_d \deg(f'_{i'}[\mathfrak{m}_F])$ for all i.

By arguments similar to those that prove Lemma 4.1, the sequences are V_d-strict. □

Lemma 4.1 and 4.2 providing the basis for the construction, the following result is the inductive step for Cartan-Eilenberg resolutions, showing correctness of Algorithm 3.14.

Lemma 4.3. *Let I_A, I_B, I_C be three submodules of the free modules $F_A[\mathfrak{m}_A]$, $F_B[\mathfrak{m}_B]$, $F_C[\mathfrak{m}_C]$. Assume that $0 \to I_A[\mathfrak{m}_A] \xrightarrow{\phi_A} I_B[\mathfrak{m}_B] \xrightarrow{\phi_B} I_C[\mathfrak{m}_C] \to 0$ is exact and V_d-strict.*

Then one can construct a diagram

$$\begin{tikzcd}
 & 0 & 0 & 0 & \\
0 \arrow[r] & I_A \arrow[u] \arrow[r, "\phi_A"] & I_B \arrow[u] \arrow[r, "\phi_B"] & I_C \arrow[u] \arrow[r] & 0 \\
0 \arrow[r] & P_A \arrow[u] \arrow[r] & P_B \arrow[u] \arrow[r] & P_C \arrow[u] \arrow[r] & 0 \\
0 \arrow[r] & J_A \arrow[u] \arrow[r] & J_B \arrow[u] \arrow[r] & J_C \arrow[u] \arrow[r] & 0 \\
 & 0 \arrow[u] & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd} \tag{4.1}$$

such that

- *all P_X are free,*
- *all rows and columns are exact,*
- *there are shift vectors $\mathfrak{n}_A, \mathfrak{n}_B, \mathfrak{n}_C$ such that if P_A, P_B, P_C are shifted accordingly, all rows and columns become V_d-strict.*

In fact, we may prescribe $0 \to J_A \to P_A[\mathfrak{n}_A] \to I_A[\mathfrak{m}_A] \to 0$, provided it is V_d-strict.

Proof. Let $\{a_i\}$ and $\{c_j\}$ be V_d-strict Gröbner bases for I_A and I_C inside $F_A[\mathfrak{m}_A]$ and $F_C[\mathfrak{m}_C]$ respectively.

Let P_A be the free module on the symbols $\{e_{a_i}\}$, define the projection from P_A to I_A by $e_{a_i} \to a_i$. Let P_C be a free module on the symbols $\{e_{c_j}\}$, define the projection $P_C \to I_C$ by $e_{c_j} \to c_j$. Define degree shifts on P_A, P_C by $\mathfrak{n}_A(i) = V_d \deg(a_i[\mathfrak{m}_A])$, $\mathfrak{n}_C(i) = V_d \deg(c_i[\mathfrak{m}_C])$.

Set $P_B = P_A \oplus P_C$. For the generators of P_B corresponding to $\{c_i\}$ use a lift $\psi : P_C \to B$ for ϕ_B (which exists as P_C is free) that satisfies: $V_d \deg(\psi(e_{c_i})[\mathfrak{m}_B]) \leq V_d \deg(c_i[\mathfrak{m}_C])$ (which exists because $0 \to I_A[\mathfrak{m}_A] \to I_B[\mathfrak{m}_B] \to I_C[\mathfrak{m}_C] \to 0$ is V_d-strict). Then define

$P_B \twoheadrightarrow I_B$ by $(P_B \twoheadrightarrow P_A \twoheadrightarrow I_A \hookrightarrow I_B) + (P_B \twoheadrightarrow P_C \xrightarrow{\psi} I_B)$. Set J_A, J_B, J_C to be the kernels of the corresponding natural projections. This gives the required diagram with exact rows and columns.

It is clear that $0 \to P_A[\mathfrak{n}_A] \to P_B[\mathfrak{n}_B]$ and $P_B[\mathfrak{n}_B] \to P_C[\mathfrak{n}_C] \to 0$ are V_d-strict. If an element of V_d-degree e is in the kernel of $P_B \to P_C$, then its second component (the one in P_C) is zero, so the V_d-degree came from the P_A-component. Hence the second row is V_d-strict. Then automatically the third row is too.

By [12], the remarks after Proposition 3.11, the outer columns are V_d-strict. By the lemma below, $\{\phi_A(a_i)\} \cup \{\psi(e_{c_j})\}$ is a V_d-strict Gröbner basis for $I_B[\mathfrak{m}_B]$. It follows that $P_B \twoheadrightarrow I_B$ is V_d-strict, and hence the entire column. $\square$

Lemma 4.4. *Let I_A, I_B, I_C be three submodules of the free modules $F_A[\mathfrak{m}_A]$, $F_B[\mathfrak{m}_B]$, $F_C[\mathfrak{m}_C]$. Assume that $0 \to I_A[\mathfrak{m}_A] \xrightarrow{\phi_A} I_B[\mathfrak{m}_B] \xrightarrow{\phi_B} I_C[\mathfrak{m}_C] \to 0$ is exact and V_d-strict.*

Let $\{a_i\}$ and $\{c_j\}$ be V_d-strict Gröbner bases for I_A and I_C inside $F_A[\mathfrak{m}_A]$ and $F_C[\mathfrak{m}_C]$ respectively.

Assume $\psi(c_j) \in I_B$ satisfies $\phi_B(\psi(c_j)) = c_j$ and $V_d \deg(\psi(c_j)[\mathfrak{m}_B]) = V_d \deg(c_j[\mathfrak{m}_C])$ for all j. Then $\{\psi(c_j)\} \cup \{\phi_A(a_i)\}$ is a V_d-strict Gröbner basis for $I_B[\mathfrak{m}_B]$.

Proof. Let $b \in I_B[\mathfrak{m}_B]$ be of V_d-degree e. Then $\phi_B(b) = \sum \gamma_j c_j$ with

$$V_d \deg(\gamma_j c_j[\mathfrak{m}_B]) \leq V_d \deg(\phi_B(b)[\mathfrak{m}_C]) \leq V_d \deg(b[\mathfrak{m}_B])$$

for all j as $\{c_j\}$ is a V_d-strict Gröbner basis. Therefore

$$b - \sum \gamma_j \psi(c_j) \in \ker(\phi_B) \cap F_H^{V_d \deg(b[\mathfrak{m}_B])}(I_B[\mathfrak{m}_B]).$$

By exactness and V_d-strictness of the sequence, $b - \sum \gamma_j \psi(c_j) = \sum \alpha_i a_i$ with $V_d \deg(\alpha_i a_i[\mathfrak{m}_A]) \leq V_d \deg(b[\mathfrak{m}_B])$. $\square$

This lemma shows in particular that we can assume $\mathrm{rows}(J_B) = \mathrm{rows}(J_A) + \mathrm{rows}(J_C)$ in the previous lemma if we want.

4.2. Here is an explanation what Algorithm 3.8 does. We assume we have completed Step 1, which is explained in the algorithm itself.

Invoke Lemma 4.2 with $P_C/I_C = B^{r+1} = 0$ to find a presentation for Z^r, H^r, B^r and C^r together with shift vectors $\mathfrak{m}_{Z,r}, \mathfrak{m}_{C,r}, \mathfrak{m}_{H,r}$ and

$\mathfrak{m}_{B,r}$ such that there are commutative diagrams

$$\begin{tikzcd}
 & 0 & 0 & 0 & \\
0 \arrow[r] & Z^r \arrow[r] \arrow[u] & C^r \arrow[r] \arrow[u] & B^{r+1} = 0 \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & P^0_{Z,r}[\mathfrak{m}_{Z,r,0}] \arrow[r] \arrow[u] & P^0_{C,r}[\mathfrak{m}_{C,r,0}] \arrow[r] \arrow[u] & P^0_{B,r+1}[\mathfrak{m}_{B,r+1,0}] = 0 \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & I^0_{Z,r}[\mathfrak{m}_{Z,r,0}] \arrow[r] \arrow[u] & I^0_{C,r}[\mathfrak{m}_{C,r,0}] \arrow[r] \arrow[u] & I^0_{B,r+1}[\mathfrak{m}_{B,r+1,0}] = 0 \arrow[r] \arrow[u] & 0 \\
 & 0 \arrow[u] & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}$$

and

$$\begin{tikzcd}
 & 0 & 0 & 0 & \\
0 \arrow[r] & B^r \arrow[r] \arrow[u] & Z^r \arrow[r] \arrow[u] & H^r \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & P^0_{B,r}[\mathfrak{m}_{B,r,0}] \arrow[r] \arrow[u] & P^0_{Z,r}[\mathfrak{m}_{Z,r,0}] \arrow[r] \arrow[u] & P^0_{H,r}[\mathfrak{m}_{H,r,0}] \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & I^0_{B,r}[\mathfrak{m}_{B,r,0}] \arrow[r] \arrow[u] & I^0_{Z,r}[\mathfrak{m}_{Z,r,0}] \arrow[r] \arrow[u] & I^0_{H,r}[\mathfrak{m}_{H,r,0}] \arrow[r] \arrow[u] & 0 \\
 & 0 \arrow[u] & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}$$

where the P^0_X are free and the rows and columns are exact and V_d-strict. Then invoke the lemma again, this time starting with the shift just obtained on B^r and construct presentations for $Z^{r-1}, C^{r-1}, H^{r-1}, B^{r-1}$ and shifts on $P^0_{Z,r-1}, P^0_{C,r-1}, P^0_{H,r-1}, P^0_{B,r-1}$. Repeating this process r times leads to V_d-strict commutative diagrams with exact rows and columns

$$\begin{tikzcd}
 & 0 & 0 & 0 & \\
0 \arrow[r] & Z^i \arrow[r] \arrow[u] & C^i \arrow[r] \arrow[u] & B^{i+1} \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & P^0_{Z,i}[\mathfrak{m}_{Z,i,0}] \arrow[r] \arrow[u] & P^0_{C,i}[\mathfrak{m}_{C,i,0}] \arrow[r] \arrow[u] & P^0_{B,i+1}[\mathfrak{m}_{B,i+1,0}] \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & I^0_{Z,i}[\mathfrak{m}_{Z,i,0}] \arrow[r] \arrow[u] & I^0_{C,i}[\mathfrak{m}_{C,i,0}] \arrow[r] \arrow[u] & I^0_{B,i+1}[\mathfrak{m}_{B,i+1,0}] \arrow[r] \arrow[u] & 0 \\
 & 0 \arrow[u] & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}$$

and

$$\begin{tikzcd}
 & 0 & 0 & 0 & \\
0 \arrow[r] & B^i \arrow[r] \arrow[u] & Z^i \arrow[r] \arrow[u] & H^i \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & P^0_{B,i}[\mathfrak{m}_{B,i,0}] \arrow[r] \arrow[u] & P^0_{Z,i}[\mathfrak{m}_{Z,i,0}] \arrow[r] \arrow[u] & P^0_{H,i}[\mathfrak{m}_{H,i,0}] \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & I^0_{B,i}[\mathfrak{m}_{B,i,0}] \arrow[r] \arrow[u] & I^0_{Z,i}[\mathfrak{m}_{Z,i,0}] \arrow[r] \arrow[u] & I^0_{H,i}[\mathfrak{m}_{H,i,0}] \arrow[r] \arrow[u] & 0 \\
 & 0 \arrow[u] & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}$$

for $0 \le i \le r$. The point of this procedure is the creation of a presentation of C^i as $P^0_{C,i}/I^0_{C,i}$ with V_d-strict maps between these modules.

Now we assemble a V_d-strict Cartan-Eilenberg resolution for $C^\bullet$. First find a V_d-strict resolution for B^0 using the method of [12]. With Lemma 4.3 find step by step a resolution for Z^0 and H^0 over $0 \to B^0 \to Z^0 \to H^0 \to 0$. Then, using the resolution for Z^0, construct resolutions for C^0 and B^1 over $0 \to Z^0 \to C^0 \to B^1 \to 0$. Now use the resolution obtained for B^1 to find V_d-strict resolutions for Z^1, H^1 over $0 \to B^1 \to Z^1 \to H^1 \to 0$. In this way construct resolutions for Z^i, H^i, C^i and B^{i+1} for $0 \le i \le r$ which fit into appropriate V_d-strict and exact commutative diagrams.

We denote the l-th module of the resolution for X^i (where X is Z, C, B or H) by $P^l_{X,i}$, and the l-th map $P^l_{X,i} \to P^{l-1}_{X,i}$ by $\delta^l_{X,i}$. Then by construction $P^l_{Z,i} = P^l_{B,i} \oplus P^l_{H,i}$ and $P^l_{C,i} = P^l_{Z,i} \oplus P^l_{B,i+1}$. We define a map $\varepsilon^l_{C,i}$ from $P^l_{C,i}$ to $P^l_{C,i+1}$ as the combined map $P^l_{C,i} \twoheadrightarrow P^l_{B,i+1} \hookrightarrow P^l_{Z,i+1} \hookrightarrow P^l_{C,i+1}$, multiplied by $(-1)^l$. So up to sign $\varepsilon^l_{C,i}$ is $P^l_{C,i} = P^l_{B,i} \oplus P^l_{H,i} \oplus P^l_{B,i+1} \twoheadrightarrow P^l_{B,i+1} \hookrightarrow P^l_{B,i+1} \oplus P^l_{H,i+1} \oplus P^l_{B,i+2}$. Clearly $P^l_{C,i} \to P^l_{C,i+1} \to P^l_{C,i+2}$ is the zero map.

We have created a free double complex $\{P^\bullet_{C,\bullet}, \delta^\bullet_{C,\bullet}, \varepsilon^\bullet_{C,\bullet}\}$ over D_n such that $\{P^\bullet_{C,i}, \delta^\bullet_{C,i}\}$ is V_d-strict for fixed i and $\{P^l_{C,\bullet}, \varepsilon^l_{C,\bullet}\}$ is V_d-strict for fixed l. Moreover, the associated total complex $Tot^\bullet(P^\bullet_{C,\bullet})$ is quasi-isomorphic to $C^\bullet$ and by construction V_d-adapted. We shall write $\delta_T(-)$ for the differential of $Tot^\bullet(P^\bullet_{C,\bullet})$.

Proposition 4.5. *$Tot^\bullet(P^\bullet_{C,\bullet})$ is in fact V_d-strict.*

Proof. To that end assume that the element $p = p^0_i \oplus p^1_{i+1} \oplus \cdots \oplus p^{r-i}_r \in Tot^i(P^\bullet_{C,\bullet}) = P^0_{C,i} \oplus P^1_{C,i+1} \oplus \cdots \oplus P^{r-i}_{C,r}$ is in the image of the total differential δ_T, and that the V_d-degree of p under the shift vectors is e. We have to show that p is the image of an element of V_d-degree at most

e. We need to take a closer look at the maps and modules in front of us.

$P^l_{C,i}$ is by construction $P^l_{B,i}\oplus P^l_{H,i}\oplus P^l_{B,i+1}$. The map $\delta^{l+1}_{C,i}: P^{l+1}_{C,i} = P^{l+1}_{B,i}\oplus P^{l+1}_{H,i}\oplus P^{l+1}_{B,i+1}\to P^l_{C,i} = P^l_{B,i}\oplus P^l_{H,i}\oplus P^l_{B,i+1}$ is on the first component the differential from the resolution $P^\bullet_{B,i}$ while the map from $P^{l+1}_{H,i}\oplus P^{l+1}_{B,i+1}\to P^l_{C,i}$ is defined using certain lifts, obtained using Lemma 4.3. Inspection shows that the matrix of boundary maps which represents $P^{l+1}_{C,i}\to P^l_{C,i}$ looks like this:

$$\begin{pmatrix} \delta^{l+1}_{B,i} & \phi^{l+1}_{1,i} & \psi^{l+1}_{1,i} \\ 0 & \delta^{l+1}_{H,i} & \psi^{l+1}_{2,i} \\ 0 & 0 & \delta^{l+1}_{B,i+1} \end{pmatrix}$$

where $\phi^{l+1}_{1,i}: P^{l+1}_{H,i}\to P^l_{B,i}, \psi^{l+1}_{1,i}: P^{l+1}_{B,i+1}\to P^l_{B,i}, \psi^{l+1}_{2,i}: P^{l+1}_{B,i+1}\to P^l_{H,i}$ are the maps that are used to produce the mentioned lifts. $\phi^{l+1}_{1,i}$, $\psi^{l+1}_{1,i}$,$\psi^{l+1}_{2,i}$ are all V_d-adapted by construction. (Nota bene: if we wrote down the matrices of operators that represent this map by right multiplication, we would obtain a lower triangular block matrix.)

We shall argue by decreasing induction on the variable s, starting with $s = r+1$, that the components p^{s-i}_s of p may be assumed to be zero modulo images of V_d-degree no greater than e under the total differential. We will at the same time show that we may assume that the third component of p^{s-i-1}_{s-1} is zero. For $s\le r$ this will follow from the induction. For $s = r+1$ it follows from the fact that $B^{r+1} = 0$.

So assume that $s\le r$, that p has only zero components beyond the s-th component and that the third piece (to $P^{s-i}_{B,s+1}$) of the s-th component of p is zero.

The following lemma will essentially show that our p is then in fact the image of an element in $Tot^{i-1}(P^\bullet_{C,\bullet})$ with zero component in $P^{s-i+1}_{B,s+1}$ and only zeros in all columns beyond the s-th.

Lemma 4.6. *Let $(a,b,0)\in P^{s-i}_{C,s}$ and assume $(a,b,0) = \delta^{s-i+1}_{C,s}(\alpha,\beta,\gamma)$ with $(\alpha,\beta,\gamma)\in P^{s-i+1}_{C,s}$. Then $(a,b,0) = \delta^{s-i+1}_{C,s}(\alpha',\beta',0)$ for some $(\alpha',\beta',0)$ in $P^{s-i+1}_{C,s}$ where $V_d\deg(\alpha',\beta',0)\le V_d\deg(a,b,0)$.*

Proof. By construction, $(\psi^{s-i+1}_{1,s}(\gamma),\psi^{s-i+1}_{2,s}(\gamma))$ is in $\ker(P^{s-i}_{B,s}\oplus P^{s-i}_{H,s}\to P^{s-i-1}_{B,s}\oplus P^{s-i-1}_{H,s})$. Since this kernel is exactly $\delta^{s-i+1}_{Z,s}(P^{s-i+1}_{B,s}\oplus P^{s-i+1}_{H,s})$, it follows that

$$(\psi^{s-i+1}_{1,s}(\gamma),\psi^{s-i+1}_{2,s}(\gamma)) = \delta^{s-i+1}_{Z,s}(\alpha'',\beta'')$$

where we can pick α'' and β'' to be of V_d-degree at most $V_d\deg(\gamma)$. Thus

$$\begin{aligned}\delta_{C,s}^{s-i+1}(\alpha,\beta,\gamma) &= (\delta_{B,s}^{s-i+1}(\alpha)+\psi_{1,s}^{s-i+1}(\gamma)+\phi_{1,s}^{s-i+1}(\beta),\\ &\qquad \delta_{H,s}^{s-i+1}(\beta)+\psi_{2,s}^{s-i+1}(\gamma),0)\\ &= \delta_{C,s}^{s-i+1}(\alpha+\alpha'',\beta+\beta'',0)\end{aligned}$$

Since $P^\bullet_{Z,\bullet}$ is a V_d-strict resolution, we can find $(\alpha',\beta',0)$ such that $\delta_{C,s}^{s-i+1}(\alpha',\beta',0)=\delta_{C,s}^{s-i+1}(\alpha+\alpha'',\beta+\beta'',\gamma)$ and $V_d\deg(\alpha',\beta',0)\leq V_d\deg(a,b,0)$. $\square$

Since by assumption $p_s^{s-i}=(a,b,0)$, the lemma tells us that b equals $\delta_H(b_1)$ for some $b_1\in P_{H,s}^{s-i+1}$, of V_d-degree at most e because $P^\bullet_{H,s}$ is V_d-strict. We replace p by $p:=p-\delta_T(b_1)$. This erases a nonzero entry in p, keeps it in the image of δ_T, leaves components beyond the s-th invariant and is a modification by the image of an element of V_d-degree at most $V_d\deg(p)$.

Lemma 4.7. *Let $(a,0,0)\in P_{C,s}^{s-i}$ be the image of $(\alpha,\beta,\gamma)\in P_{C,s}^{s-i+1}$ under $\delta_{C,s}^{s-i+1}$. Then there is $\alpha'\in P_{B,s}^{s-i+1}$ with $\delta_{C,s}^{s-i+1}(\alpha',0,0)=(a,0,0)$ and α' can be chosen to be of V_d-degree no bigger than $V_d\deg(a)$.*

Proof. By the previous lemma, we can assume that $(a,0,0)$ is the image of $(\alpha,\beta,0)\in P_{C,s}^{s-i+1}$. By construction, $\phi_{1,s}^{s-i+1}(\beta)\in\ker(P_{B,s}^{s-i}\to P_{B,s}^{s-i-1})=\operatorname{im}(P_{B,s}^{s-i+1}\to P_{B,s}^{s-i})$. As $P^\bullet_{B,s}$ is a resolution , $\phi_{1,s}^{s-i+1}(\beta)=\delta_{B,s}^{s-i+1}(\alpha'')$ for some $\alpha''\in P_{B,s}^{s-i+1}$. Hence

$$\begin{aligned}\delta_{C,s}^{s-i+1}(\alpha,\beta,0) &= (\delta_{B,s}^{s-i+1}(\alpha)+\phi_{1,s}^{s-i+1}(\beta),0,0)\\ &= \delta_{C,s}^{s-i+1}(\alpha,0,0)+\delta_{C,s}^{s-i+1}(\alpha'',0,0).\end{aligned}$$

This proves that $(a,0,0)$ is in fact the image of an element $(\alpha',0,0)$. Since $P^\bullet_{B,s}$ is a V_d-strict resolution, we can choose α' to be of V_d-degree at most $V_d\deg(a)$. $\square$

Before Lemma 4.7 we reduced the component of p that corresponds to $P_{C,s}^{s-i}$ to the form $(a,0,0)$. Lemma 4.7 tells us that since p is an image under δ_T, $a=\delta_{B,s}^{s-i+1}(\alpha')+(-1)^{s-1}\alpha$ where α' belongs to $P_{B,s}^{s-i+1}$ (the first component of $P_{C,s}^{s-i+1}$) and $\alpha\in P_{B,s}^{s-i}$ (the third component of $P_{C,s-1}^{s-i}$).

Then the third component of p in $P_{C,s-1}^{s-i-1}$ must be exactly $\delta_{B,s}^{s-i}(\alpha)=(-1)^{s-1}\delta_{B,s}^{s-i}(a)$. Replace p by $p-\delta_T((-1)^{s-1}(0,0,a))$, where $(0,0,a)\in P_{C,s-1}^{s-i-1}$.

The result is a p with zero component for $P^{s-i}_{C,s}$ and zero component in the third component of $P^{s-i-1}_{C,s-1}$. Furthermore, it differs from the original p by the δ_T-boundary of the element $(0,0,a) \in P^{s-i-1}_{C,s-1}$ of V_d-degree at most e. This completes the induction step and the proof of Proposition 4.5. □

We have proved

Theorem 4.8. *If $C^\bullet$ is a bounded complex of left D_n-modules and presentations of all C^i in terms of generators and relations over D_n are given, then one can produce a V_d-strict complex of free D_n-modules that is quasi-isomorphic to $C^\bullet$.* □

Remark 4.9. It is not true that the total complex associated to any double complex with V_d-strict rows and columns is V_d-strict. This would be equivalent to saying that all finite subsets of a free D_n-module form a Gröbner basis for any order refining V_d-degree. Consider for example the diagram

$$\begin{array}{ccll} \overbrace{D_1[1]}^{\text{degree } 0} & \xrightarrow{\cdot\,\partial_1} & \overbrace{D_1[0]}^{\text{degree } 1} & \}\ \text{degree } 1 \\ \uparrow & & \uparrow \cdot(\partial_1 - 1) & . \\ 0 & \longrightarrow & D_1[1] & \}\ \text{degree } 0 \end{array}$$

Here, $1 = 1 \cdot \partial_1 - 1 \cdot (\partial_1 - 1) \in \operatorname{im}(F^1_H(Tot^1[1,1]))$ but it is not in $\operatorname{im}(F^0_H(Tot^1[1,1])) \subseteq F^0_H((Tot^2[0])))$, although it is of V_1-degree 0.

5. The Restriction of a Complex to a Subspace

For this section let $A^\bullet[\mathfrak{m}_\bullet]$ be a given bounded V_d-strict complex of finitely generated D_n-free modules. We shall assume further that the cohomology modules of $A^\bullet[\mathfrak{m}_\bullet]$ are specializable to $H = \operatorname{Var}(x_1, \ldots, x_d)$ (cf. Definition 3.2).

Definition 5.1. The *restriction of the complex $A^\bullet[\mathfrak{m}_\bullet]$ to the subspace H* is the complex $\tilde{\Omega}_d \otimes^L_{D_n} A^\bullet[\mathfrak{m}_\bullet]$ considered as a complex in the category of $\mathbb{C}\langle x_{d+1}, \partial_{d+1}, \ldots, x_n, \partial_n\rangle$-modules.

The main purpose of this section is to determine a finite dimensional truncation of $\tilde{\Omega} \otimes_{D_n} A^\bullet$ such that the cohomology of $\tilde{\Omega} \otimes_{D_n} A^\bullet$ is captured by the cohomology of that truncation.

Computation of the truncation accomplishes then to the computation of (a complex that is homotopy equivalent to) $\Omega \otimes^L_{D_n} A^\bullet$. We shall exhibit a method to compute more generally $\Omega_d \otimes^L_{D_n} A^\bullet$ represented by a bounded complex of free $\mathbb{C}\langle x_{d+1}, \partial_{d+1}, \ldots, x_n, \partial_n\rangle$-modules of finite rank.

Recall that in [11] $A^\bullet$ is a V_n-strict resolution of a specializable module M and the truncation is determined by considering roots of the b-function $b(M)$ corresponding to restriction to the origin ([11], Algorithm 5.4). Our approach generalizes this method.

5.1. Let H as before be the subspace defined by $x_1 = \ldots = x_d = 0$.

Definition 5.2. Let $\kappa \in Z^i = \ker(A^i \to A^{i+1})$. Recall that $\theta_j = x_1\partial_1 + \ldots + x_j\partial_j$ for $1 \leq j \leq n$ and $\theta_0 = 0$. We call $0 \neq b_\kappa(s) \in K[s]$ the *b-function for restriction of $\kappa[\mathfrak{m}_i]$ to the subspace $x_1 = \ldots = x_d = 0$* provided that

$$b_\kappa(\theta + k + V_d \deg(\kappa)) \cdot F^k(D_n) \cdot \kappa \subseteq F_H^{k-1}(D_n) \cdot \kappa + \mathrm{im}(A^{i-1} \to A^i)$$

for all k, and $b_\kappa(s)$ is of minimal degree. The least common multiple $b_{A^\bullet[\mathfrak{m}_\bullet]}(s)$ of all $b_\kappa(s)$ (κ varies over Z^i, i over all integers) is called the *b-function for restriction of $A^\bullet[\mathfrak{m}_\bullet]$ to $x_1 = \ldots = x_d = 0$*.

We need to insure that this is a meaningful definition (i.e., that $b_{A^\bullet[\mathfrak{m}_\bullet]}(s)$ is nonzero), and that we can compute $b_{A^\bullet[\mathfrak{m}_\bullet]}(s)$. Pick generators $\kappa_{i,l}$ for the modules Z^i. To each of the generators is associated a degree in the V_d-filtration on $A^i[\mathfrak{m}_i]$ which we shall call $\lambda_{i,l}$.

Let a bar denote cosets of elements of Z^i in $H^i = H^i(A^\bullet) = Z^i/B^i$ where $B^i = \mathrm{im}(A^{i-1} \to A^i)$. Since $D_n \cdot \bar{\kappa}_{i,l}$ is an H-specializable D_n-module (see Remark 3.3), there is a b-function $b_{i,l}(\theta_d)$ associated to it which corresponds to the restriction of $D_n \cdot \bar{\kappa}_{i,l}[\mathfrak{m}_i]$ to $x_1 = \ldots = x_d = 0$. Therefore,

$$b_{i,l}(\theta_d) \cdot \kappa_{i,l} \in F_H^{-1}(D_n) \cdot \kappa_{i,l} + \mathrm{im}(A^{i-1} \to A^i).$$

Let $b(\theta_d)$ be the least common multiple of all $b_{i,l}(\theta_d - \lambda_{i,l})$.

Assume that for each i the $\kappa_{i,l}$ form a V_d-strict Gröbner basis for $Z^i[\mathfrak{m}_i]$ and consider the associated complex of graded $\mathrm{gr}_H^\bullet(D_n)$-modules $\oplus F_H^k(A^\bullet[\mathfrak{m}_\bullet])/F_H^{k-1}(A^\bullet[\mathfrak{m}_\bullet])$, the sum being taken over k.

For all $\zeta \in Z^i$ we can write $\zeta = \sum \alpha_{i,l}(\zeta)\kappa_{i,l}$ with $V_d \deg(\alpha_{i,l}(\zeta)\kappa_{i,l}[\mathfrak{m}_i]) \leq V_d \deg(\zeta[\mathfrak{m}_i])$. Hence the $\kappa_{i,l}$ are generators for $\mathrm{gr}_H^\bullet(H^i[\mathfrak{m}_i])$ and moreover $\mathrm{gr}_H^k(Z^i[\mathfrak{m}_i]) = \sum \mathrm{gr}_H^{k-\lambda_{i,l}}(D_n)\kappa_{i,l}$. Since

$$\mathrm{im}(F_H^k(A^{i-1}[\mathfrak{m}_{i-1}]) \to F_H^k(A^i[\mathfrak{m}_i])) = F_H^k(A^i[\mathfrak{m}_i]) \cap \mathrm{im}(A^{i-1} \to A^i)$$

we have $\mathrm{gr}_H^k(H^i(A^\bullet[\mathfrak{m}_i])) = \sum_l \mathrm{gr}^{k-\lambda_{i,l}}(D_n)\bar{\kappa}_{i,l}$.

Then observe the following:

$$\begin{aligned} b(\theta_d + k) \cdot \mathrm{gr}_H^k(H^i(A^\bullet[\mathfrak{m}_\bullet])) &= b(\theta_d + k) \cdot \sum \mathrm{gr}_H^{k-\lambda_{i,l}}(D_n) \cdot \bar{\kappa}_{i,l} \\ &= \sum \mathrm{gr}_H^{k-\lambda_{i,l}}(D_n) \cdot b(\theta_d + \lambda_{i,l}) \cdot \bar{\kappa}_{i,l} \qquad (5.1) \\ &= 0 \end{aligned}$$

because $b(\theta_d+\lambda_{i,l})$ sends $\kappa_{i,l}$ into $F_H^{-1}(D_n)\cdot\kappa_{i,l}+\operatorname{im}(A^{i-1}\to A^i)$, which is zero in $\operatorname{gr}_H^{\lambda_{i,l}}[\mathfrak{m}_i](H^i(A^\bullet))$.

$b(\theta_d)$ is thus a multiple of the b-function for restriction of $A^\bullet[\mathfrak{m}_\bullet]$ to H with respect to the given shift vectors and by construction cannot be a proper multiple.

Hence we have found an algorithm to compute $b_{A^\bullet[\mathfrak{m}_\bullet]}(s)$ and shown that it is nonzero.

Example 5.3. We now proceed to calculate the b-function for restriction of the complex (3.5) under the given shift. For this, set $H=\operatorname{Var}(x,y,z)$, the origin. We have to find the b-functions for all generators of the cohomology objects, and then take the least common multiple.

The cohomology of (3.5) comes exclusively from the top row of the double complex, and its generators are (by construction) the right summands in the right column and the left summands of the left column.

Thus there is exactly one $\kappa_{1,l}$, namely $\kappa_{1,1}=1\in D_3[0]$. Since the entries of (the V_3-strict map) $\delta^1_{H,1}$ are ∂_y,∂_z and $x\partial_x$, $\operatorname{gr}_H(H^1[0])=D_3/D_3\cdot(\partial_y,\partial_z,x\partial_x)$. Clearly $x\partial_x+y\partial_y+z\partial_z=0$ in $\operatorname{gr}_H(H^1)$ and thus $b_{1,1}(\theta)=\theta$.

For H^0 there are 6 generators. We observe that

$$\begin{aligned}\operatorname{gr}_H(H^0[2,0,0,1,1,1]) &= D_3{}^6[2,0,0,1,1,1]/\operatorname{im}(\delta^1_{Z,0})\\ &\cong D_3[2]/D_3\cdot(y,z,x\partial_x).\end{aligned}$$

Hence $(\partial_y y+\partial_z z+x\partial_x)\cdot\operatorname{gr}^0_H(H^0[2,0,0,1,1,1])=0$ and thus $b_{0,1}(\theta)=\theta+2$ while $b_{0,l}(\theta)=1$ for $2\le l\le 6$.

We conclude that the least common multiple of all $b_{i,l}(\theta-V_3\deg(\kappa_{i,l}))$ is $b(\theta)=\theta$.

5.2. We have paved the way for a result generalizing Proposition 5.2 in [11]. The proof is very similar to the one given there.

We need to introduce a number of Koszul complexes. Let $\mathcal{L}$ be a $\operatorname{gr}^\bullet_H(D_n)$-module and let $\mathcal{L}_k$, $k\in\mathbb{Z}$, be subgroups of $\mathcal{L}$ such that

$$x_i\mathcal{L}_k\subseteq\mathcal{L}_{k-1}\text{ and }\partial_i\mathcal{L}_k\subseteq\mathcal{L}_{k+1}$$

for $1\le i\le d$, $\mathcal{L}_k\cap\mathcal{L}_{k'}=0$ for all $k\ne k'$, and $\mathcal{L}=\bigcup_k\mathcal{L}_k$. In that case we will say that the $\mathcal{L}_k$ give an *H-grading* for $\mathcal{L}$. For any integer k let $\mathcal{K}^\bullet(\mathcal{L}_\bullet,x_1,\dots,x_d)[k]$ be the Koszul complex

$$0\to\mathcal{L}_{k+d}\otimes_{\mathbb{Z}}\bigwedge^0\mathbb{Z}^d\to\mathcal{L}_{k+d-1}\otimes_{\mathbb{Z}}\bigwedge^1\mathbb{Z}^d\to\cdots\to\mathcal{L}_k\otimes_{\mathbb{Z}}\bigwedge^d\mathbb{Z}^d\to 0$$

equipped with the usual Koszul maps $\delta(u\otimes e_{i_1}\wedge\cdots\wedge e_{i_j})=\sum_l x_l u\otimes e_l\wedge e_{i_1}\wedge\cdots\wedge e_{i_j}$.

Unifying all the graded pieces, let $\mathcal{K}^\bullet(\mathcal{L}_\bullet, x_1, \dots, x_d)$ be the usual Koszul complex of $\mathcal{L}$ relative to $x_1, \dots, x_d$.

More generally, for a complex of H-graded $\mathrm{gr}_H^\bullet(D_n)$-modules $(\mathcal{L}^\bullet, \delta^\bullet)$ with a differential $\delta^\bullet$ that respects the H-grading we define inductively $\mathcal{K}^\bullet(\mathcal{L}_\bullet^\bullet, x_1, \dots, x_d)[k]$ as the total complex of the double complex

$$\begin{array}{c} \mathcal{K}^\bullet(\mathcal{L}_\bullet^\bullet, x_1, \dots, x_{d-1})[k] \\ (-1)^i x_d \cdot \Big\uparrow \\ \mathcal{K}^\bullet(\mathcal{L}_\bullet^\bullet, x_1, \dots, x_{d-1})[k+1] \end{array}$$

where $\mathcal{K}^\bullet(\mathcal{L}_\bullet^\bullet, \emptyset)[k] = \mathcal{L}_\bullet^\bullet[k] = (\cdots \to \mathcal{L}_k^i \to \mathcal{L}_k^{i+1} \to \cdots)$ is the k-th piece of the H-grading of the original complex. Thus $\mathcal{K}^\bullet(\mathcal{L}_\bullet^\bullet, x_1, \dots, x_d)[k]$ is the component of the usual Koszul complex $\mathcal{K}^\bullet(\mathcal{L}^\bullet, x_1, \dots, x_d)$ associated to $\mathcal{L}^\bullet$ and $x_1, \dots, x_d$ that "ends" in the k-th piece of the grading.

In the following theorem, which is also interesting in its own light, we will explain which H-graded pieces of the complex $\mathcal{K}^\bullet(\mathcal{L}^\bullet, x_1, \dots, x_d)$ are responsible for nontrivial cohomology.

Theorem 5.4. *Let $(\mathcal{L}^\bullet, \delta^\bullet)$ be a complex of H-graded $\mathrm{gr}_H^\bullet(D_n)$-modules where the maps $\delta^i : \mathcal{L}^i \to \mathcal{L}^{i+1}$ preserve the grading. Assume that there is a polynomial $b(\theta)$ in $\mathbb{C}[\theta]$ that satisfies*

$$b(\theta_d + k) \cdot \ker(\mathcal{L}_k^i \to \mathcal{L}_k^{i+1}) \subseteq \mathrm{im}(\mathcal{L}_k^{i-1} \to \mathcal{L}_k^i)$$

for all k and all i. Let k_0 be an integer for which $b(k_0) \neq 0$. Then $\mathcal{K}^\bullet(\mathcal{L}_\bullet^\bullet, x_1, \dots, x_d)[k_0]$ is exact.

Proof. Let $H' = \mathrm{Var}(x_1, \dots, x_{d-1})$. The complex $\mathcal{K}^\bullet(\mathcal{L}_\bullet^\bullet, x_d)$ inherits an H'-grading via the given H-grading. Here is the essential idea of the argument:

Lemma 5.5. *If $b(\theta_d + k)$ kills the cohomology of $\mathcal{K}^\bullet(\mathcal{L}_\bullet^\bullet, \emptyset)[k]$ then $b^2(\theta_{d-1} + k)$ kills the cohomology of $\mathcal{K}^\bullet(\mathcal{L}_\bullet^\bullet, x_d)[k]$.*

Proof. We may assume that $b(\theta_d)$ is not a constant since otherwise $\mathcal{L}^\bullet$ is exact and then a spectral sequence argument shows that $\mathcal{K}^\bullet(\mathcal{L}^\bullet, x_d)$ is exact as well.

So assume that $(u_{k+1}^{i+1}, u_k^i) \in \mathcal{L}_{k+1}^{i+1} \oplus \mathcal{L}_k^i$ is in $\mathcal{K}^i(\mathcal{L}_\bullet^\bullet, x_d)[k]$ and suppose this element is in the kernel of the differential in $\mathcal{K}^\bullet(\mathcal{L}^\bullet, x_d)$. For the convenience of the reader we illustrate the situation with the following diagram.

$$\begin{tikzcd}
\mathcal{L}_k^{i-1} \arrow[r, "\delta^{i-1}"] & \mathcal{L}_k^i \arrow[r, "\delta^i"] & \mathcal{L}_k^{i+1} \arrow[r, "\delta^{i+1}"] & \mathcal{L}_k^{i+2} \\
\mathcal{L}_{k+1}^{i-1} \arrow[r, "\delta^{i-1}"'] & \mathcal{L}_{k+1}^i \arrow[r, "\delta^i"'] \arrow[u, "(-1)^i x_d \cdot"] & \mathcal{L}_{k+1}^{i+1} \arrow[r, "\delta^{i+1}"'] \arrow[u, "(-1)^{i+1} x_d \cdot"'] & \mathcal{L}_{k+1}^{i+2}
\end{tikzcd}$$

Then we must have

$$\delta^{i+1}(u^{i+1}_{k+1}) = 0, \tag{5.2}$$

$$(-1)^{i+1}x_d u^{i+1}_{k+1} + \delta^i(u^i_k) = 0. \tag{5.3}$$

By hypothesis on b, $b(\theta_d + k + 1)u^{i+1}_{k+1} = \delta^i(u^i_{k+1})$ for some $u^i_{k+1} \in \mathcal{L}^i_{k+1}$. So $b(\theta_{d-1} + k + \partial_d x_d)u^{i+1}_{k+1} = \delta^i(u^i_{k+1})$ and therefore $b(\theta_{d-1} + k)u^{i+1}_{k+1} + \partial_d P x_d u^{i+1}_{k+1} = \delta^i(u^i_{k+1})$ for some V_d-homogeneous $P \in F^0_H(D_n)\setminus F^{-1}_H(D_n)$. Hence $b(\theta_{d-1}+k)u^{i+1}_{k+1} = \delta^i(u^i_{k+1} + (-1)^i\partial_d P u^i_k)$ using relation (5.3). Let us write this as

$$b(\theta_{d-1} + k) \cdot u^{i+1}_{k+1} = \delta^i(a^i_{k+1}), \tag{5.4}$$

$a^i_{k+1} \in (\mathcal{L}^i_{k+1} \oplus 0) \subset \mathcal{K}^{i-1}(\mathcal{L}^\bullet_\bullet, x_d)[k+1]$.

This implies that if (u^{i+1}_{k+1}, u^i_k) is in the kernel of δ^i_T, the differential on the total complex $\mathcal{K}^\bullet(\mathcal{L}^\bullet, x_d)$, then $b(\theta_{d-1} + k)(u^{i+1}_{k+1}, u^i_k)$ is, modulo the image of δ^{i-1}_T, congruent to an element $(0, v^i_k)$. Namely, $(0, v^i_k) = b(\theta_{d-1} + k) \cdot (u^{i+1}_{k+1}, u^i_k) - \delta^i_T(a^i_{k+1}, 0)$ which of course is also in the kernel of δ^i_T. So it suffices to show that any such kernel element $(0, v^i_k)$ satisfies $b(\theta_{d-1} + k)(0, v^i_k) \in \operatorname{im} \delta^{i-1}_T$.

Since $\delta^i_T(0, v^i_k) = 0$, we must have $\delta^i(v^i_k) = 0$. Hence $b(\theta_d + k)v^i_k = \delta^{i-1}(a^{i-1}_k)$ for some $a^{i-1}_k \in \mathcal{L}^{i-1}_k$ by assumption on $b(\theta)$. Now $b(\theta_d + k)v^i_k = b(\theta_{d-1} + k)v^i_k + x_d Q\partial_d v^i_k$ for some V_d-homogeneous operator $Q \in F^0_H(D_n)\setminus F^{-1}_H(D_n)$. Therefore $b(\theta_{d-1}+k)v^i_k = \delta^{i-1}(a^{i-1}_k) - x_d Q\partial_d v^i_k$.

The element $Q\partial_d v^i_k \in \mathcal{L}^i_{k+1}$ satisfies $\delta^i(Q\partial_d v^i_k) = Q\partial_d\delta^i(v^i_k) = 0$. It follows that $b(\theta_{d-1}+k)(0, v^i_k) = \delta^{i-1}_T((-1)^i Q\partial_d v^i_k, a^{i-1}_k)$. This concludes the proof of the lemma. □

Now recall the inductive definition of $\mathcal{K}^\bullet(\mathcal{L}^\bullet, x_1, \ldots, x_d)$, which together with the lemma shows that if the cohomology of $\mathcal{K}^\bullet(\mathcal{L}^\bullet_\bullet, \emptyset)[k_0]$ is killed by $b(\theta_d + k_0)$ then the cohomology of $\mathcal{K}^\bullet(\mathcal{L}^\bullet_\bullet, x_1, \ldots, x_d)[k_0]$ is killed by $b^{(2^d)}(k_0)$.

Since $b(k_0) \neq 0$ and $K \subseteq \mathbb{C}$ is a domain the theorem follows. □

We come now to the final result of this section. The purpose is to exhibit an algorithm that computes the cohomology groups of the restriction of a complex.

We need to make a convention about the V_d-filtration on tensor products over D_n with $\tilde\Omega_d = D_n/(x_1, \ldots, x_d) \cdot D_n$.

Definition 5.6. If $A[\mathfrak{m}]$ is a free H-graded D_n-module with shift vector $\mathfrak{m}$ then $\tilde\Omega_d \otimes_{D_n} A[\mathfrak{m}]$ is filtered by $F^k_H(\tilde\Omega_d \otimes_{D_n} A[\mathfrak{m}]) :=$ the K-span of $\{\bar P \otimes_{D_n} Q | V_d \deg(P) + V_d \deg(Q[\mathfrak{m}]) \leq k\}$. Note that as $\tilde\Omega_d$ equals $\mathbb{C}[\partial_1, \ldots, \partial_d]\langle x_{d+1}, \partial_{d+1}, \ldots, x_n, \partial_n\rangle$ as right D_n-module, $F^k_H(\tilde\Omega_d \otimes_{D_n}$

$A[\mathfrak{m}]$) equals the free $\mathbb{C}\langle x_{d+1}, \partial_{d+1}, \dots, x_n, \partial_n\rangle$-module on the symbols $\{(P_1, \dots, P_{\mathrm{rk}_{D_n}(A)}) | P_j \in \mathbb{C}[\partial_1, \dots, \partial_d], \deg_\partial(P_j) \le k - \mathfrak{m}(j)\ \forall j\}$.

If $A^\bullet[\mathfrak{m}_\bullet]$ is a V_d-strict complex, we denote by $F^k(\tilde{\Omega}_d \otimes A^\bullet[\mathfrak{m}_\bullet])$ the complex whose modules are the $F^k(\tilde{\Omega}_d \otimes A^i[\mathfrak{m}_i])$ as defined above, and the maps are induced from $A^\bullet$.

Before we state our theorem we point out that if $A^\bullet[\mathfrak{m}_\bullet]$ is a V_d-strict complex of D_n-modules then the associated graded complex $\mathrm{gr}_H^\bullet(A^\bullet[\mathfrak{m}_\bullet])$ is an H-graded complex.

Theorem 5.7. *Let $(A^\bullet[\mathfrak{m}_\bullet], \delta^\bullet)$ be a V_d-strict complex of free D_n-modules. The restriction of $A^\bullet[\mathfrak{m}_\bullet]$ to $H = \mathrm{Var}(x_1, \dots, x_d)$, interpreted as a complex of modules over $\mathbb{C}\langle x_{d+1}, \partial_{d+1}, \dots, x_n, \partial_n\rangle$, can be computed as follows:*

1. *Compute the b-function $b_{A^\bullet[\mathfrak{m}_\bullet]}(s)$ for restriction of $A^\bullet[\mathfrak{m}_\bullet]$ to H.*
2. *Find integers k_0, k_1 with $(b_{A^\bullet[\mathfrak{m}_\bullet]}(k) = 0, k \in \mathbb{Z}) \Rightarrow (k_0 \le k \le k_1)$.*
3. *$\tilde{\Omega}_d \otimes^L_{D_n} A^\bullet$ is quasi-isomorphic to the complex*

$$\cdots \to \frac{F_H^{k_1}(\tilde{\Omega}_d \otimes_{D_n} A^i[\mathfrak{m}_i])}{F_H^{k_0-1}(\tilde{\Omega}_d \otimes_{D_n} A^i[\mathfrak{m}_i])} \to \frac{F_H^{k_1}(\tilde{\Omega}_d \otimes_{D_n} A^{i+1}[\mathfrak{m}_{i+1}])}{F_H^{k_0-1}(\tilde{\Omega}_d \otimes_{D_n} A^{i+1}[\mathfrak{m}_{i+1}])} \to \cdots \tag{5.5}$$

This is a complex of free finitely generated $\mathbb{C}\langle x_{d+1}, \partial_{d+1}, \dots, x_n, \partial_n\rangle$-modules.

Proof. Set $b(s) := b_{A^\bullet[\mathfrak{m}_\bullet]}(s)$. Let us consider first the complex $\mathrm{gr}_H^\bullet(A^\bullet[\mathfrak{m}_\bullet])$ with differential induced from $\delta^\bullet$. The hypotheses on $b(s)$ imply by theorem 5.4 that $\mathcal{K}^\bullet(\mathrm{gr}_H^\bullet(A^\bullet[\mathfrak{m}_\bullet]), x_1, \dots, x_d)[k]$ is exact for $k \notin [k_0, k_1] \cap \mathbb{Z}$. Let us define inductively (on d) $\mathcal{K}^\bullet(A^\bullet[\mathfrak{m}_\bullet], x_1, \dots, x_d)\{k\}$ to be the complex

$$\cdots \to F^k(A^i[\mathfrak{m}_i]) \to F^k(A^{i+1}[\mathfrak{m}_{i+1}]) \to \cdots$$

if $d = 0$ and to be the total complex of the double complex

$$\begin{array}{c} \mathcal{K}^\bullet(A^\bullet[\mathfrak{m}_\bullet], x_1, \dots, x_{d-1})\{k\} \\ (-1)^i x_d \cdot \big\uparrow \\ \mathcal{K}^\bullet(A^\bullet[\mathfrak{m}_\bullet], x_1, \dots, x_{d-1})\{k+1\} \end{array}$$

for $d > 0$. We also write $\mathcal{K}^\bullet(A^\bullet[\mathfrak{m}_\bullet], x_1, \dots, x_d)\{k/k'\}$ for the complex whose j-th entry is the j-th entry of $\mathcal{K}^\bullet(A^\bullet[\mathfrak{m}_\bullet], x_1, \dots, x_d)\{k\}$ modulo the j-th entry of the complex $\mathcal{K}^\bullet(A^\bullet[\mathfrak{m}_\bullet], x_1, \dots, x_d)\{k'\}$ and whose maps are induced from $\mathcal{K}^\bullet(A^\bullet[\mathfrak{m}_\bullet], x_1, \dots, x_d)\{k\}$.

Lemma 5.8. *$\mathcal{K}^\bullet(A^\bullet[\mathfrak{m}_\bullet], x_1, \dots, x_d)\{k\}$ is exact for $k < k_0$.*

Proof. We observe first that for $k \ll 0$ this is a consequence of the fact that operators of negative V_d-degree in $D_n[0]$ are necessarily right multiples of x_j for $1 \leq j \leq d$ and hence in small V_d-degrees left multiplication by such x_j is a quasi-isomorphism between $F^{k+1}(A^\bullet[\mathfrak{m}_\bullet])$ and $F^k(A^\bullet[\mathfrak{m}_\bullet])$, forcing the corresponding Koszul complex to be exact. Let $\bar{\alpha} \in H^i(\mathcal{K}^\bullet(A^\bullet[\mathfrak{m}_\bullet], x_1, \ldots, x_d)\{k\})$ with $V_d \deg(\alpha) \leq k$. Then α represents a cohomology class in $\mathcal{K}^\bullet(A^\bullet[\mathfrak{m}_\bullet], x_1, \ldots, x_d)\{k/k-1\} = \mathcal{K}^\bullet(\mathrm{gr}^\bullet_H(A^\bullet[\mathfrak{m}_\bullet]), x_1, \ldots, x_d)[k]$. Since that latter complex is acyclic by Theorem 5.4, $\alpha = \alpha_0 + \delta_T^{i-1}(\alpha_1)$ where $V_d \deg(\alpha_0) \leq k-1$. As $A^\bullet[\mathfrak{m}_\bullet]$ is V_d-strict, α_1 can be chosen to be of V_d-degree at most k which proves that $\bar{\alpha} = \bar{\alpha}_0$ in $\mathcal{K}^\bullet(A^\bullet, x_1, \ldots, x_d)\{k\}$. Since $A^\bullet[\mathfrak{m}_\bullet]$ is V_d-adapted, α_0 represents also a cohomology class in $\mathcal{K}^\bullet(A^\bullet[\mathfrak{m}_\bullet], x_1, \ldots, x_d)\{k-1\}$. By induction on k we can assume that this cohomology class is zero, proving that α_0 is in fact an image of an element of V_d-degree at most $k-1$. This proves the lemma. $\square$

Lemma 5.9. *If $\mathcal{K}^\bullet(A^\bullet[\mathfrak{m}_\bullet], x_1, \ldots, x_d)\{\infty\}$ stands for the Koszul complex $\mathcal{K}^\bullet(A^\bullet[\mathfrak{m}_\bullet], x_1, \ldots, x_d)$, then $\mathcal{K}^\bullet(A^\bullet[\mathfrak{m}_\bullet], x_1, \ldots, x_d)\{\infty/k\}$ is exact for $k \geq k_1$.*

Proof. It is sufficient to prove this for $k = k_1$. We show first that $\mathcal{K}^\bullet(A^\bullet[\mathfrak{m}_\bullet], x_1, \ldots, x_d)\{k/k_1\}$ is exact for $k \geq k_1$.

The complex $\mathcal{K}^\bullet(A^\bullet[\mathfrak{m}_\bullet], x_1, \ldots, x_d)\{k/k_1\}$ is filtered by the complexes $\mathcal{K}^\bullet(A^\bullet[\mathfrak{m}_\bullet], x_1, \ldots, x_d)\{k'/k_1\}$ with $k_1 \leq k' \leq k$. Since each quotient $\mathcal{K}^\bullet(A^\bullet[\mathfrak{m}_\bullet], x_1, \ldots, x_d)\{k'/k'-1\}$ is exact by Theorem 5.4, so must be the complex $\mathcal{K}^\bullet(A^\bullet[\mathfrak{m}_\bullet], x_1, \ldots, x_d)\{k/k_1\}$.

If now $\bar{\alpha}$ is a cohomology class of $\mathcal{K}^\bullet(A^\bullet[\mathfrak{m}_\bullet], x_1, \ldots, x_d)\{\infty/k_1\}$ and represented by the element α of V_d-degree k, then α also represents a class in $\mathcal{K}^\bullet(A^\bullet[\mathfrak{m}_\bullet], x_1, \ldots, x_d)\{k/k_1\}$. Since the latter complex is exact, $\bar{\alpha}$ is the class of an image and hence is the zero class in $\mathcal{K}^\bullet(A^\bullet[\mathfrak{m}_\bullet], x_1, \ldots, x_d)\{\infty/k_1\}$. $\square$

Now we turn to the proof of the theorem. Consider the exact sequence of complexes

$$\begin{array}{c}
\mathcal{K}^\bullet(A^\bullet[\mathfrak{m}_\bullet], x_1, \ldots, x_d)\{\infty/k_1\} \\
\uparrow \\
\mathcal{K}^\bullet(A^\bullet[\mathfrak{m}_\bullet], x_1, \ldots, x_d)\{\infty\} \\
\uparrow \\
\mathcal{K}^\bullet(A^\bullet[\mathfrak{m}_\bullet], x_1, \ldots, x_d)\{k_1\}.
\end{array}$$

Lemma 5.9 shows that the middle and the bottom complex are quasi-isomorphic under the inclusion. Similarly,

$$\begin{array}{c}\mathcal{K}^\bullet(A^\bullet[\mathfrak{m}_\bullet], x_1, \ldots, x_d)\{k_1/k_0 - 1\} \\ \uparrow \\ \mathcal{K}^\bullet(A^\bullet[\mathfrak{m}_\bullet], x_1, \ldots, x_d)\{k_1\} \\ \uparrow \\ \mathcal{K}^\bullet(A^\bullet[\mathfrak{m}_\bullet], x_1, \ldots, x_d)\{k_0 - 1\}\end{array}$$

is exact and Lemma 5.8 shows that the top two complexes are quasi-isomorphic under the projection. Therefore, both $\mathcal{K}^\bullet(A^\bullet[\mathfrak{m}_\bullet], x_1, \ldots, x_d)$ and $\mathcal{K}^\bullet(A^\bullet[\mathfrak{m}_\bullet], x_1, \ldots, x_d)\{k_1/k_0 - 1\}$ represent the same object in the derived category.

We finish the proof of Theorem 5.7 with the following

Lemma 5.10. *The cohomology of the complex (5.5) is precisely the cohomology of the complex* $\mathcal{K}^\bullet(A^\bullet[\mathfrak{m}_\bullet], x_1, \ldots, x_d)\{k_1/k_0 - 1\}$.

Proof. We show that $\mathcal{K}^\bullet(A^\bullet[\mathfrak{m}_\bullet], x_1, \ldots, x_d)\{k_1/k_0-1\}$ is quasi-isomorphic to $\frac{F_H^{k_1}(\tilde{\Omega}\otimes A^\bullet[\mathfrak{m}_\bullet])}{F_H^{k_0-1}(\tilde{\Omega}\otimes A^\bullet[\mathfrak{m}_\bullet])}$. To this end consider the sequence of complexes

$$0 \to F^{k+1}(A^\bullet[\mathfrak{m}_\bullet]) \xrightarrow{x_1\cdot} F^k(A^\bullet[\mathfrak{m}_\bullet]) \to \frac{F^k(A^\bullet[\mathfrak{m}_\bullet])}{x_1 \cdot F^{k+1}(A^\bullet[\mathfrak{m}_\bullet])} \to 0.$$

By definition, the complex on the right is $F^k(\tilde{\Omega}_1 \otimes A^\bullet[\mathfrak{m}_\bullet])$. The mapping cone $L^\bullet$ of the inclusion on the left is by Exercise II.5.3 of [9] quasi-isomorphic to the quotient on the right. However, inspecting the mapping cone definition we find that $L^\bullet = \mathcal{K}^\bullet(A^\bullet[\mathfrak{m}_\bullet], x_1)\{k\}$. This proves that $\mathcal{K}^\bullet(A^\bullet[\mathfrak{m}_\bullet], x_1)\{k\}$ and $F^k(\tilde{\Omega}_1 \otimes A^\bullet[\mathfrak{m}_\bullet])$ are quasi-isomorphic.

By the inductive definition of $\mathcal{K}^\bullet(A^\bullet[\mathfrak{m}_\bullet], x_1, \ldots, x_d)\{k\}$ and the fact that $\tilde{\Omega}_d \otimes_{D_n} (-)$ can be interpreted as killing x_j with $1 \le j \le d$ one at a time, $\mathcal{K}^\bullet(A^\bullet[\mathfrak{m}_\bullet], x_1, \ldots, x_d)\{k\}$ and $F^k(\tilde{\Omega}_d \otimes A^\bullet[\mathfrak{m}_\bullet])$ are quasi-isomorphic.

It is not hard to see that inclusions $\mathcal{K}^\bullet(A^\bullet[\mathfrak{m}_\bullet], x_1, \ldots, x_d)\{k\} \hookrightarrow \mathcal{K}^\bullet(A^\bullet[\mathfrak{m}_\bullet], x_1, \ldots, x_d)\{k'\}$ for $k < k'$ are carried into inclusions $F^k(\tilde{\Omega}_d \otimes A^\bullet[\mathfrak{m}_\bullet]) \hookrightarrow F^{k'}(\tilde{\Omega}_d \otimes A^\bullet[\mathfrak{m}_\bullet])$ under this family of quasi-isomorphisms. The Lemma follows. □

□

Remark 5.11. The complex (5.5) is a complex in the category of finitely generated left modules over the Weyl algebra in the $n - d$ variables $x_{d+1}, \ldots, x_n$, and the i-th module of that complex is the free $\mathbb{C}\langle x_{d+1}, \partial_{d+1}, \ldots, x_n, \partial_n\rangle$-module on the symbols $\{(P_1, \ldots, P_{\mathrm{rk}_{D_n}(A^i)})|P_j \in$

$\mathbb{C}[\partial_1, \ldots, \partial_d], k_0 - \mathfrak{m}_i(j) \leq \deg_\partial(P_j) \leq k_1 - \mathfrak{m}_i(j)\}$. Thus, the cohomology objects of (5.5) are finitely generated modules over $\mathbb{C}\langle x_{d+1}, \partial_{d+1}, \ldots, x_n, \partial_n\rangle$ and computable by standard Gröbner basis procedures. In particular, if $d = n$ then the resulting complex is a complex of finite-dimensional $\mathbb{C}$-vector spaces.

6. Computation of de Rham Cohomology

Recall that $X = \mathbb{C}^n, Y = \mathrm{Var}(f_0, \ldots, f_r), U = X \setminus Y$. In this section we apply the results of the previous sections to obtain an algorithm that computes $H^\bullet_{dR}(U, \mathbb{C})$. We proved in Section 2 that

$$H^i_{dR}(U, \mathbb{C}) = H^{i-n}(\Omega \otimes^L_{D_n} MV^\bullet) = H^{i-n}(\mathcal{K}^\bullet(MV^\bullet, \partial_1, \ldots, \partial_n)).$$

In order to cope with the problem that we would like to compute the cohomology of $\mathcal{K}^\bullet(MV^\bullet, \partial_1, \ldots, \partial_n)$ and not $\mathcal{K}^\bullet(MV^\bullet, x_1 \ldots, x_n)$, we shall make use of the Fourier transform.

The n-th Fourier transform $\mathcal{F} = \mathcal{F}_n$ sends Ω to $\tilde{\Omega} = D_n/(x_1, \ldots, x_n)\cdot D_n$. Thus, $H^i(\mathcal{K}^\bullet(MV^\bullet, \partial_1, \ldots, \partial_n))$ is isomorphic as a vector space to $H^i(\mathcal{K}^\bullet(\tilde{MV}^\bullet, x_1, \ldots, x_n))$, where $\tilde{MV}^\bullet$ is the image of the complex $MV^\bullet$ under the n-th Fourier transform.

Then $\tilde{MV}^\bullet$ may be replaced by a free V_n-strict complex $A^\bullet[\mathfrak{m}_\bullet]$ as constructed in Section 3 and the cohomology of $\tilde{MV}^\bullet$ is holonomic and therefore specializable to the origin [8].

Let $b(s) \in K[s]$ be the polynomial found in Subsection 5.1 for $d = n$, the b-function for the complex $A^\bullet[\mathfrak{m}_\bullet]$ for restriction to the origin. Then $b(\theta_d+k)\,\mathrm{gr}^k_H(H^i(A^\bullet[\mathfrak{m}_\bullet])) = b(\theta_d+k)\cdot H^i(\mathrm{gr}^k_H(A^\bullet[\mathfrak{m}_\bullet]))$ is zero according to (5.1). Therefore by Theorem 5.4, $b(k)$ kills the degree k pieces of the cohomology of $\mathcal{K}^\bullet(\mathrm{gr}^\bullet_H(A^\bullet[\mathfrak{m}_\bullet]), x_1, \ldots, x_n)$. In other words, if k_0, k_1 are integers with $\{b(s) = 0, s \in \mathbb{Z}\} \Rightarrow \{s \in [k_0, k_1] \cap \mathbb{Z}\}$, then $\mathcal{K}^\bullet(\mathrm{gr}^\bullet_H(A^\bullet[\mathfrak{m}_\bullet]), x_1, \ldots, x_n)[k]$ is exact if $k \notin [k_0, k_1] \cap \mathbb{Z}$.

Theorem 6.1. *Let $X = \mathbb{C}^n, Y = \mathrm{Var}(f_0, \ldots, f_r), U = X \setminus Y$. Then $H^i_{dR}(U, \mathbb{C}) = H^{i-n}(\Omega \otimes^L_{D_n} MV^\bullet)$. Moreover, the cohomology of $\Omega \otimes^L_{D_n} MV^\bullet$ can be computed as follows:*

1. *Compute $MV^\bullet = 0 \to \check{C}^1 \to \ldots \to \check{C}^{r+1} \to 0$ as in* [14], *Algorithm 5.1 as complex of finitely generated D_n-modules and place $\check{C}^1$ in homological degree 0.*
2. *Compute a V_n-strict D_n-free complex*

$$\cdots \to A^{r-1}[\mathfrak{m}_{r-1}] \to A^r[\mathfrak{m}_r] \to 0$$

quasi-isomorphic to $\tilde{MV}^\bullet$, where $\tilde{MV}^\bullet$ denotes the image of $MV^\bullet$ under the n-th Fourier automorphism $\mathcal{F}_n$.

3. *Compute the b-function* $b(s) = b_{A^\bullet[\mathfrak{m}_\bullet]}(s)$ *for the restriction of* $A^\bullet[\mathfrak{m}_\bullet]$ *to the origin.*
4. *Find integers* k_0, k_1 *with* $(b(k) = 0, k \in \mathbb{Z}) \Rightarrow (k_0 \leq k \leq k_1)$.
5. $\Omega \otimes^L_{D_n} MV^\bullet$ *is quasi-isomorphic to the complex*

$$(6.1) \qquad \cdots \to \frac{F_H^{k_1}(\tilde\Omega \otimes_{D_n} A^i[\mathfrak{m}_i])}{F_H^{k_0-1}(\tilde\Omega \otimes_{D_n} A^i[\mathfrak{m}_i])} \to \frac{F_H^{k_1}(\tilde\Omega \otimes_{D_n} A^{i+1}[\mathfrak{m}_{i+1}])}{F_H^{k_0-1}(\tilde\Omega \otimes_{D_n} A^{i+1}[\mathfrak{m}_{i+1}])} \to \cdots$$

which is a complex of finite dimensional $\mathbb{C}$*-vector spaces.*

Proof. We already remarked that $H^i(\mathcal{K}^\bullet(MV^\bullet, \partial_1, \ldots, \partial_n))$ is isomorphic to $H^i(\mathcal{K}^\bullet(\tilde{MV}^\bullet, x_1, \ldots, x_n))$, and so we only need to show that the latter can be computed from (6.1).

By Theorem 5.7 the complex (6.1) is quasi-isomorphic to $\tilde\Omega \otimes_{D_n} A^\bullet[\mathfrak{m}_\bullet]$. Since $\tilde\Omega \otimes_{D_n} A^\bullet[\mathfrak{m}_\bullet]$ is quasi-isomorphic to $\tilde\Omega \otimes_{D_n} MV^\bullet$, the conclusion follows. □

Remark 6.2.

6.2.1. The quotient $\frac{F_H^{k_1}(\tilde\Omega \otimes_{D_n} A^i[\mathfrak{m}_i])}{F_H^{k_0-1}(\tilde\Omega \otimes_{D_n} A^i[\mathfrak{m}_i])}$ should be thought of as vectors of polynomials in $\partial_1, \ldots, \partial_n$ of degrees bounded between $k_0 - \mathfrak{m}_i(j)$ and $k_1 - \mathfrak{m}_i(j)$.

6.2.2. Since Gröbner bases do not change under field extensions, we may interpret (6.1) as a complex of K-vector spaces and evaluate the dimension of its $(i-n)$-th cohomology group which will equal the complex dimension of $H^i_{dR}(U, \mathbb{C})$.

6.2.3. Since $\mathcal{K}^{r-i}(\mathrm{gr}^\bullet_H(A^\bullet[\mathfrak{m}_\bullet]), x_1, \ldots, x_n)$ involves only terms from $A^r, \ldots, A^{r-i}$, the following statement can be made: if $MV^\bullet$ is exact in degree $r-i$ and beyond, then $\Omega \otimes^L_{D_n} MV^\bullet$ is exact in degree $j \geq n+r-i$. That follows by considering $b(s) \cong 1$, which kills the last i cohomology terms in $MV^\bullet$; inspecting the proof of Theorem 5.4 one sees that then 1 also kills the last i cohomology terms in $\mathcal{K}^\bullet(A^\bullet[\mathfrak{m}_\bullet], x_1, \ldots, x_n)$. As a byproduct we obtain the well-known estimate

$$H^i_{dR}(U, \mathbb{C}) = 0 \text{ if } i \geq n + \mathrm{cd}(f_0, \ldots, f_r).$$

Example 6.3. In this example we compute the de Rham cohomology of $U = X \setminus Y$ where $X = C^3$ and $Y = \mathrm{Var}(xy, xz)$. In this case we get a Mayer-Vietoris complex of the form

$$MV^\bullet = \left(R_3[(xy)^{-1}] \oplus R_3[(xz)^{-1}] \to R_3[(xyz)^{-1}]\right),$$

$\mathrm{cd}(xy, xz) = 2$ and hence de Rham cohomology may be nonzero up to degree $2 + 3 - 1 = 4$. The image of this complex under $\mathcal{F}_3$ is the complex given in Example 3.7. In Example 3.19 we computed a V_3-strict D_3-free complex quasi-isomorphic to $\tilde{MV}^\bullet$. Now we compute the cohomology of $\tilde\Omega \otimes A^\bullet[\mathfrak{m}_\bullet]$.

In Example 5.3 we computed that the b-function for restriction of $\tilde{MV}^{\bullet}$ to the origin in $\mathbb{C}^3$ is $b(s) = s$. Since $b(s)$ has the unique root $s = 0$, Theorem 6.1 proves that the de Rham cohomology of U is captured by the complex $\frac{F_H^0(\tilde{\Omega}\otimes_{D_3} A^{\bullet}[\mathfrak{m}_\bullet])}{F_H^{-1}(\tilde{\Omega}\otimes_{D_3} A^{\bullet}[\mathfrak{m}_\bullet])}$ which is given below.

$$\begin{tikzcd}
\mathbb{C}\cdot 1 \oplus \mathbb{C}\cdot 1 \arrow[r, "\cdot\begin{pmatrix} 0 \\ 0 \end{pmatrix}"] & \mathbb{C}\cdot 1 \\
\mathbb{C}\cdot 1 \oplus \mathbb{C}\cdot 1 \arrow[r, "\cdot\begin{pmatrix} 0 \\ 0 \end{pmatrix}"] \arrow[u, "\cdot\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}"'] & \mathbb{C}\cdot 1 \arrow[u, "\cdot(0)"'] \\
\mathbb{C}\cdot 1 \arrow[r] \arrow[u, "\cdot(0,0)"'] & 0 \arrow[u] \\
\mathbb{C}\cdot 1 \arrow[r] \arrow[u, "\cdot(0)"'] & 0 \arrow[u]
\end{tikzcd}$$

From this diagram we read off the cohomology groups

$$\begin{aligned}
H^1(\tilde{\Omega}\otimes A^{\bullet}) &\cong \mathbb{C}, \\
H^0(\tilde{\Omega}\otimes A^{\bullet}) &\cong \mathbb{C}, \\
H^{-1}(\tilde{\Omega}\otimes A^{\bullet}) &\cong 0, \\
H^{-2}(\tilde{\Omega}\otimes A^{\bullet}) &\cong \mathbb{C}, \\
H^{-3}(\tilde{\Omega}\otimes A^{\bullet}) &\cong \mathbb{C}, \\
H^{k}(\tilde{\Omega}\otimes A^{\bullet}) &\cong 0
\end{aligned}$$

for all other indices. Hence the de Rham cohomology of U is $H^0(U,\mathbb{C}) = H^1(U,\mathbb{C}) = H^3(U,\mathbb{C}) = H^4(U,\mathbb{C}) = \mathbb{C}$ and zero otherwise.

7. De Rham Cohomology with Supports

Let Y, Z be two Zariski-closed subsets of X. In this section we are concerned with finding an algorithm that computes the de Rham cohomology groups $H^{\bullet}_{dR,Z}(U,\mathbb{C})$ of $U = X \setminus Y$ with coefficients in $\mathbb{C}$ and supports in Z.

$H^{\bullet}_{dR,Z}(U,\mathbb{C})$ is defined as follows. Recall the de Rham complex $\Omega^{\bullet}(U)$ on U. The usual de Rham cohomology is defined as the hypercohomology of $\Omega^{\bullet}(U)$. De Rham cohomology with supports is defined as the hypercohomology with supports in Z of $\Omega^{\bullet}(U)$. In other words, $H^{\bullet}_{dR,Z}(U,\mathbb{C}) = H^{\bullet}(R\Gamma_Z(U,\Omega^{\bullet}(U)))$.

As was pointed out by Hartshorne, there is a natural exact sequence

$$\cdots \to H^i_{dR,Z}(U,\mathbb{C}) \to H^i_{dR}(U,\mathbb{C}) \to H^i_{dR}(U\setminus Z,\mathbb{C}) \to H^{i+1}_{dR,Z}(U,\mathbb{C}) \to \cdots \tag{7.1}$$

which indicates that $H^\bullet_{dR,Z}(U,\mathbb{C})$ measures the change in cohomology due to the removal of $Z\cap U$ from U.

Notation 7.1. For the entire section we assume that $F=(f_0,\dots,f_r)$, $Y=\operatorname{Var}(F)$ and $G=(g_0,\dots,g_s)$, $Z=\operatorname{Var}(G)$. Write $F\cdot G=\{f_i\cdot g_j \mid 0\le i\le r, 0\le j\le s\}$. As before we will write F_I for $\prod_{i\in I} f_i$ and G_J for $\prod_{j\in J} g_j$.

There is a natural map of Mayer-Vietoris complexes

$$MV^\bullet(F\cdot G,F)\to MV^\bullet(F\cdot G)$$

given by the natural projection

$$\bigoplus_{|I|+|J|+|K|=l} R_n[(F_I\cdot G_J)^{-1}]\otimes_{R_n} R_n[F_K{}^{-1}] \to \bigoplus_{|I|+|J|=l} R_n[(F_I\cdot G_J)^{-1}]$$

sending each summand with $|K|>0$ to zero. This map corresponds to the embedding $X\setminus(Y\cup Z)\hookrightarrow X\setminus Y$. Clearly the map is surjective and the kernel is the subcomplex of $MV^\bullet(F\cdot G,F)$ consisting of those pieces which contain at least one factor from F. It is not hard to check that this kernel is exactly $MV^\bullet(F)\otimes_{R_n}\check{C}^\bullet(F\cdot G)$, where $\check{C}^\bullet(F\cdot G)$ is the Čech complex to $F\cdot G$ given by $\bigotimes_{i,j}(0\to R_n \xrightarrow{nat} R_n[(f_i\cdot g_j)^{-1}]\to 0)$.

Notice that the sequences

$$0\to \big(MV^\bullet(F)\otimes_{R_n}\check{C}^\bullet(F\cdot G)\big)^i \to MV^i(F\cdot G,F)\to MV^i(F\cdot G)\to 0$$

are all split exact. Then there is a short exact sequence of complexes

$$\begin{aligned}\Omega^\bullet\otimes_{D_n}\big(MV^\bullet(F)\otimes_{R_n}\check{C}^\bullet(F\cdot G)\big)\to\\ \to \Omega^\bullet\otimes_{D_n} MV^\bullet(F\cdot G,F)\to \Omega^\bullet\otimes_{D_n} MV^\bullet(F\cdot G)\end{aligned} \tag{7.2}$$

with split exact rows. In other words, we have a short exact sequence of complexes.

As was explained in previous sections, the cohomology of $\Omega^\bullet\otimes_{D_n} MV^\bullet(F\cdot G,F)$ is $H^i_{dR}(X\setminus Y,\mathbb{C})$ while the cohomology of $\Omega^\bullet\otimes_{D_n} MV^\bullet(F\cdot G)$ is $H^i_{dR}(X\setminus(Y\cup Z),\mathbb{C})$ and the map on cohomology is induced by the natural inclusion.

Comparison of the long exact sequence (7.1) with the long exact sequence that results from the short exact sequence of complexes (7.2) shows that the cohomology of $\Omega^\bullet\otimes_{D_n}(MV^\bullet(F)\otimes_{R_n}\check{C}^\bullet(F\cdot G))$ is exactly $H^i_{dR,Z}(X\setminus Y,\mathbb{C})$.

Computationally this situation is very bad: de Rham cohomology of $X \setminus Y$ and $X \setminus Z$ come from the Mayer-Vietoris complex of F and G respectively while here we have (approximately) the Mayer-Vietoris complex of $F \cup F \cdot G$. We shall try to improve this situation now. As a first step in that direction we point out that the long exact sequence (7.1) shows that for complements of affine closed varieties $H^i_{dR,Z}(X \setminus Y, \mathbb{C})$ is in fact nothing but the relative cohomology group $H^i(X \setminus Y, X \setminus (Y \cup Z); \mathbb{C})$.

Consider the space $X \setminus (Y \cap Z)$ and its open covering by the two sets $X \setminus Y$ and $X \setminus Z$. It follows from [4], Example 17.1, that this is an exact triad for homology with integer coefficients, and from [3], Theorem 11.4, that the same holds for cohomology with coefficients in $\mathbb{C}$. This means that the natural inclusion of pairs

$$(X \setminus Y, X \setminus (Y \cup Z)) \hookrightarrow (X \setminus (Y \cap Z), X \setminus Z)$$

induces a natural isomorphism between the groups $H^i(X \setminus Y, X \setminus (Y \cup Z); \mathbb{C})$ and $H^i(X \setminus (Y \cap Z), X \setminus Z; \mathbb{C})$. This in turn implies that instead of $H^i_{dR,Z}(X \setminus Y, \mathbb{C})$ we may calculate $H^i_{dR,Z}(X \setminus (Y \cap Z), \mathbb{C})$.

Now consider the natural projection of complexes

$$MV^\bullet(F, G) \to MV^\bullet(G)$$

given by $\bigoplus_{|I|+|J|=l} R_n[{F_I}^{-1}] \otimes_{R_n} R_n[{G_J}^{-1}] \to \bigoplus_{|I|=l} R_n[{F_I}^{-1}]$ induced by the inclusion $X \setminus Z \hookrightarrow X \setminus (Y \cap Z)$. As before, this induces a short exact sequence of complexes

$$0 \to MV^\bullet(F) \otimes_{R_n} \check{C}^\bullet(G) \to MV^\bullet(F, G) \to MV^\bullet(G) \to 0.$$

Applying $\Omega^\bullet \otimes_{D_n} (-)$ we discover that the cohomology of $\Omega^\bullet \otimes_{D_n} (MV^\bullet(F) \otimes_{R_n} \check{C}^\bullet(G))$ is $H^\bullet_{dR,Z}(X \setminus (Y \cap Z)) \cong H^\bullet_{dR,Z}(X \setminus Y)$. Now the complexity is down to the level of computing $H^\bullet_{dR}(X \setminus (Y \cap Z))$. So we have

Algorithm 7.2.
INPUT: Polynomials $F = \{f_0, \ldots, f_r\}$ defining Y, and $G = \{g_0, \ldots, g_s\}$ defining Z; $i \in \mathbb{N}$.
OUTPUT: The de Rham cohomology groups of $U = X \setminus Y$ with supports in Z, $H^i_{dR,Z}(X \setminus Y, \mathbb{C})$, which equal the relative cohomology groups $H^i(X \setminus Y, X \setminus (Y \cup Z); \mathbb{C})$.

Begin

1. Compute the complex $MV^\bullet(F) \otimes_{R_n} \check{C}^\bullet(G)$ as a complex of left D_n-modules as in [14], Algorithm 5.1.
2. Compute a free V_n-strict complex $A^\bullet[\mathfrak{m}_\bullet]$ that is quasi-isomorphic to the image of $MV^\bullet(F) \otimes_{R_n} \check{C}^\bullet(G)$ under the n-th Fourier automorphism as in Section 3, Theorem 4.8.

3. Find the b-function $b(s)$ for restriction of $A^\bullet[\mathfrak{m}_\bullet]$ to the origin. Let $k_0, k_1 \in \mathbb{Z}$ satisfy $\{b(k) = 0, k \in \mathbb{Z}\} \Rightarrow \{k_0 \leq k \leq k_1\}$.
4. Replace each D_n in $A^\bullet$ by $k[\partial_1, \ldots, \partial_n] = \tilde{\Omega}$ and restrict the complex to the components of V_n-degree k with $k_0 \leq k \leq k_1$.
5. Take the $(i-n)$-th cohomology of the resulting complex of vector spaces and return its dimension d_i.
6. $H^i_{dR,Z}(X \setminus Y, \mathbb{C}) \cong \mathbb{C}^{d_i}$.

End.

Example 7.3. Let us compute $H^\bullet_{dR,\mathrm{Var}(xz)}(X \setminus \mathrm{Var}(xy, xz))$ in $\mathbb{C}^3$. Here, $F = \{xy, xz\}$ and $G = \{xz\}$.

The relative de Rham complex is the tensor product of Ω and

$$\left(R_3[(xy)^{-1}] \oplus R_3[(xz)^{-1}] \to R_3[(xyz)^{-1}]\right) \bigotimes \left(R_3 \to R_3[(xz)^{-1}]\right).$$

This tensor product has cohomology $H^2 \cong H^4 \cong \mathbb{C}$ and $H^3 \cong \mathbb{C}^2$ while all other cohomology groups are zero.

Note that the de Rham cohomology of $X \setminus \mathrm{Var}(xz)$ is 1-dimensional in degrees 0 and 2 and 2-dimensional in degree 1, and that these groups fit perfectly into the long exact sequence (7.1) and that from this sequence we can for example infer that the elements of $H^i_{dR}(X \setminus Y, \mathbb{C})$ for $i = 3, 4$ are supported on $X \setminus Z$ while those of $H^1_{dR}(X \setminus Y, \mathbb{C})$ are not.

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