COMPUTING THE CUP PRODUCT STRUCTURE FOR COMPLEMENTS OF COMPLEX AFFINE VARIETIES

ULI WALTHER
MATHEMATICAL SCIENCES RESEARCH INSTITUTE

ABSTRACT. Let $X = \mathbb{C}^n$. In this paper we present an algorithm that computes the cup product structure for the de Rham cohomology ring $H^\bullet_{dR}(U;\mathbb{C})$ where U is the complement of an arbitrary Zariski-closed set Y in X.

Our method relies on the fact that Tor is a balanced functor, a property which we make algorithmic, as well as a technique to extract explicit representatives of cohomology classes in a restriction or integration complex. We also present an alternative approach to computing V-strict resolutions of complexes that is seemingly much more efficient than the algorithm presented in [16].

All presented algorithms are based on Gröbner basis computations in the Weyl algebra.

CONTENTS

1. INTRODUCTION AND THE V-FILTRATION

1.1. The main purpose of this paper is to give an algorithm that determines the multiplicative structure of the singular cohomology ring $H^\bullet(U;\mathbb{C})$ on the complement U of a complex variety Y in affine space $X = \mathbb{C}^n$. We shall utilize the fact that singular and algebraic de Rham cohomology $H^\bullet_{dR}(-;\mathbb{C})$ coincide and compute the latter. The computations will be done via D-module theory, in particular by using the *V-filtration* (see Subsection 1.7).

The fundamental algorithmic techniques for D-module theoretic *restriction* and *integration* (see Subsections 1.6, 1.7) based on the V-filtration were introduced in the landmark papers [10, 11]. They were used for example to compute local cohomology modules [11, 15], the dimensions of the de Rham cohomology groups of complements of hypersurfaces [12] and general affine varieties [16]. In short, these latter two papers consider the algebraic de

Rham complex on U as a complex for D-module theoretic restriction to the origin via the Fourier transform (see Subsection 1.3). In the hypersurface case the restriction procedure (see Algorithm 2.4) is applied to the Fourier image of the ring of sections $\Gamma(U, \mathcal{O}_U)$ on U, while in the general setting the (reduced) Čech complex $\check{C}^\bullet(Y)$ of Y takes this place (Subsection 1.2). An algorithm to compute $\check{C}^\bullet(Y)$ as complex of finitely generated modules over the Weyl algebra was given in [15].

The structure of the paper is as follows. In the remainder of this section we give basic definitions and principles. Section 2 is devoted to the hypersurface case. There we give an algorithm to compute the ring structure of $H^\bullet_{dR}(X \setminus Y; \mathbb{C})$ if $Y = \mathrm{Var}(f)$ for $f \in \Gamma(X, \mathcal{O}_X)$. In Section 3 we introduce the reader to a new technique of computing free V-strict resolutions of complexes. It generalizes the module case in a different way than Cartan-Eilenberg resolutions do and constructs apparently much smaller resolutions than the method from [16].

In Section 4 finally we give an algorithm to compute the multiplicative structure of $H^\bullet_{dR}(X \setminus Y; \mathbb{C})$ for general Zariski closed Y in X.

The algorithmic main components of this paper are Corollary 3.3 for the computation of a free V-strict complex quasi-isomorphic to a given one, and the *Transfer Theorem* (Theorem 2.5), which explicitly relates the cohomology classes of the de Rham complex on U to the cohomology classes of the complex computed in [11, 12, 16].

1.2. We need to fix some notation. D_n will denote the n-th Weyl algebra $\mathbb{C}\langle x_1, \partial_1, \ldots, x_n, \partial_n\rangle$ where $[\partial_i, x_j] = 0$ if $i \neq j$, $[\partial_i, x_i] = 1$, $[\partial_i, \partial_j] = [x_i, x_j] = 0$. ∂_i represents the operator $\frac{\partial}{\partial x_i}$ on $R_n = \mathbb{C}[x_1, \ldots, x_n]$. We shall write $\cdot$ for the product of elements in D_n or R_n, while $\bullet$ denotes the action of D_n on R_n. So $\partial_i \cdot x_i = \partial_i x_i = x_i \partial_i + 1 \in D_n$ while $\partial_i \bullet x_i = 1 \in R_n$. We use parentheses to indicate the order of operations so that for example $\partial_i \bullet (x_j) \cdot x_i = 0 \cdot x_i = 0$. We use multi-index notation: $x^\alpha = {x_1}^{\alpha_1} \cdot \ldots \cdot {x_n}^{\alpha_n}$, $\partial^\beta = {\partial_1}^{\beta_1} \cdot \ldots \cdot {\partial_n}^{\beta_n}$.

If $(C^\bullet, \delta^\bullet)$ is a complex and $c \in \ker(\delta^i)$ then $\overline{c}$ denotes the cohomology class generated by c. If $(A^\bullet, \alpha^\bullet)$ is another complex and $\phi : A^\bullet \to C^\bullet$ is a chain map then we write $\phi : A^\bullet \xrightarrow{\cong} C^\bullet$ provided that ϕ induces an isomorphism on cohomology, in which case we call ϕ a *quasi-isomorphism.*

Let $f_0, \ldots, f_r \in R_n$, set $X = \mathbb{C}^n$, $Y = \mathrm{Var}(f_0, \ldots, f_r) \subseteq X$ and $U = X \setminus Y$. If I is a subset of $\{0, \ldots, r\}$ then we will write $|I|$ for its cardinality and F_I for the product $\prod_{\iota \in I} f_\iota$. Throughout, let $\check{C}^\bullet = \check{C}^\bullet(f_0, \ldots, f_r)$ denote the *reduced Čech complex* (cf. [4], Section A4.1) which in [16] is called the *Mayer-Vietoris complex*:

$$0 \to \underbrace{\bigoplus_{|I|=1} R_n[{F_I}^{-1}]}_{\text{degree } 0} \to \ldots \to \underbrace{\bigoplus_{|I|=r+1} R_n[{F_I}^{-1}]}_{\text{degree } r} \to 0. \tag{1.1}$$

If $U = X$, then we set $\check{C}^\bullet$ to be the complex concentrated in degree zero whose entry $\check{C}^0$ is R_n.

The action $\bullet$ extends to an action of D_n on the field of fractions of R_n in a natural way. For all $f \in R_n$ there is an operator $P_f(s) \in D_n[s]$ and a polynomial $b_f(s) \in \mathbb{Q}[s]$ such that there is an identity

$$P_f(s) \bullet f^{s+1} = b_f(s) f^s. \tag{1.2}$$

Here, the $\bullet$ represents the symbol of the action of $D_n[s]$ on the free $R_n[s, f^{-1}]$-module $\mathcal{M}_f = R_n[s, f^{-1}] \otimes f^s$ given by

$$\begin{aligned} x_i \bullet \left(\frac{g(x,s)}{f^k} \otimes f^s \right) &= \frac{x_i \cdot g(x,s)}{f^k} \otimes f^s, \\ \partial_i \bullet \left(\frac{g(x,s)}{f^k} \otimes f^s \right) &= \left[\partial_i \bullet \left(\frac{g(x,s)}{f^k} \right) + s\partial_i \bullet (f) \cdot \frac{g(x,s)}{f^{k+1}} \right] \otimes f^s, \\ s \bullet \left(\frac{g(x,s)}{f^k} \right) &= \frac{s \cdot g(x,s)}{f^k} \otimes f^s. \end{aligned}$$

The monic $b_f(s)$ of smallest degree satisfying an equation of type (1.2) is called the *Bernstein-Sato polynomial of* f and a corresponding $P_f(s)$ we shall call a *Bernstein operator* [1]. $P_f(s)$ is, as opposed to $b_f(s)$, not unique and only determined up to elements in $J^\Delta(f^s)$, the annihilator ideal inside $D_n[s]$ of $1 \otimes f^s \in \mathcal{M}_f$. For ways of computing $b_f(s)$ and $J^\Delta(f^s)$ see [10, 15].

1.3. Let $\Omega_U^\bullet = \Omega^\bullet(\check{C}^\bullet)$ stand for the *algebraic Čech-de Rham complex* on U. This is defined iteratively by $\check{C}_0^\bullet = \check{C}^\bullet$, setting $\check{C}_{i+1}^\bullet$ equal to the total complex of $\check{C}_i^\bullet \to \check{C}_i^\bullet \wedge dx_{i+1}$ where the map $\check{C}_i^j \to \check{C}_i^j \wedge dx_{i+1}$ is given by $(-1)^j \partial_{i+1} \bullet (-)$, and $\Omega^\bullet(\check{C}^\bullet) = \check{C}_n^\bullet$.

The origin of $\Omega_U^\bullet$ is as follows. Define on X a complex $\mathcal{DR}_X^\bullet$ of sheaves by $\mathcal{DR}_0^\bullet = \mathcal{O}_X$, $\mathcal{DR}_{i+1}^\bullet = \operatorname{Tot}^\bullet(\mathcal{DR}_i^\bullet \xrightarrow{\partial_{i+1}\bullet} \mathcal{DR}_i^\bullet \wedge dx_{i+1})$ and $\mathcal{DR}_X^\bullet = \mathcal{DR}_n^\bullet$. $\mathcal{DR}_X^\bullet$ is a resolution of the constant sheaf $\mathbb{C}$, and if X^{an} denotes the associated analytic space then

$$\mathcal{DR}_{X^{an}}^\bullet = \mathcal{DR}_X^\bullet \otimes_{\mathcal{O}_X} \mathcal{O}_{X^{an}}$$

induces a resolution of the constant sheaf $\mathbb{C}$ on all $U^{an} \subseteq X^{an}$.

By [9], sheaf cohomology with coefficients in the constant sheaf $\mathbb{C}$ coincides with singular cohomology with complex coefficients. By standard homological algebra, $H^\bullet(U; \mathbb{C})$ can therefore be computed as the hypercohomology of $(\mathcal{DR}_{X^{an}}^\bullet)|_{U^{an}}$. By algebraic-analytic comparison theorems we may instead compute the hypercohomology of $(\mathcal{DR}_X^\bullet)|_U =: \mathcal{DR}_U^\bullet$. By [5], Theorème 5.9.2, this can be done with Čech cohomology on a suitable cover. We choose the cover $U = \bigcup_{\iota=0}^r (X \setminus \operatorname{Var}(f_\iota))$ of affine sets. $\Omega^\bullet(\check{C}^\bullet)$ is the resulting complex for the computation of Čech cohomology.

1.4. If U is affine, then $H^i(U^{an}; \mathcal{DR}^j_{U^{an}}) = 0$ for all $i > 0$ because $\mathcal{DR}^j_{U^{an}}$ is a finite sum of copies of $\mathcal{O}_{U^{an}}$. Hence each C^∞-cohomology class has a holomorphic representative ([7], pp. 448-449) and by the comparison theorems the holomorphic representative can be picked to be algebraic ([3], Theorem 6.1.21). So in this case the multiplicative structure of $H^\bullet(U;\mathbb{C})$ is equivalent to the wedge-product within $\Gamma(U;\mathcal{DR}^\bullet_U)$.

For non-affine U, the corresponding multiplication of Čech cocycles in $\Omega^\bullet(\check{C}^\bullet)$ is described in Theorem 4.1.

1.5. Denote by $\mathcal{F}(M)$ the Fourier transform of the D_n-module M (see Subsection 1.6) and let $\Omega = D_m/(\partial_1, \ldots, \partial_n) \cdot D_n$. The techniques used in [12] and [16] to compute de Rham cohomology make it somewhat complicated to understand the results of the algorithms in terms of differential forms. This is because these papers compute the cohomology of a complex that is a subquotient of $\mathcal{F}(\Omega) \otimes_{D_n} A^\bullet$ where $A^\bullet$ is a D_n-free resolution of $\mathcal{F}(R_n[f^{-1}])$ (or free and quasi-isomorphic to $\mathcal{F}(\check{C}^\bullet)$ if $r > 0$). Thus one needs to trace several quasi-isomorphisms in order to multiply two classes. In this paper we make these quasi-isomorphisms algorithmic.

Routines for the computation of de Rham cohomology of U have been implemented in *Macaulay 2* by A. Leykin and H. Tsai. Some of these routines are still under construction and so some steps in our algorithms do at this moment have to be done by hand. Currently there are procedures available that compute localizations, local cohomology modules, and integration and restriction modules of holonomic modules. In particular de Rham cohomology is implemented. Besides that there are many other objects defined over rings of differential operators that can be computed.

Macaulay 2 can be obtained from `www.math.uiuc.edu/Macaulay2`. The routines needed to do D-module computations as well as an online documentation are available under `www.math.berkeley.edu/~htsai/`.

1.6. We need to define some special objects in the category of D_n-modules. $\Omega = D_n/(\partial_1, \ldots, \partial_n) \cdot D_n$ is a right D_n-module and is as an R_n-module non-canonically isomorphic to R_n. Let $\mathfrak{S}_k$ be the set of strictly increasing sequences of length k of integers strictly between 0 and $n+1$. Let $|S|$ be the length of such a sequence and for $S = \{s_1, \ldots, s_k\}$ set $dx_S = dx_{s_1} \cdot \ldots \cdot d_{s_k}$. We write $(\Omega^\bullet, \varepsilon^\bullet)$ for the Koszul cocomplex of right D_n-modules on D_n induced by the sequence $\partial_1, \ldots, \partial_n$. That means

$$\Omega^k = \bigoplus_{S \in \mathfrak{S}_k} D_n \cdot dx_S$$

and $\Omega^k \to \Omega^{k+1}$ is the sum of the maps

$$\Omega^k \twoheadrightarrow D_n \cdot dx_S \xrightarrow{(-1)^{\mathrm{sgn}(S,t)}\partial_t \bullet} D_n \cdot dx_{S \cup \{t\}} \hookrightarrow \Omega^{k+1}$$

where $\mathrm{sgn}(S,t)$ is the number of elementary permutations needed to order the sequence (S,t) (and $\mathrm{sgn}(S,t) = 0$ if $t \in S$). Then $\Omega = H^n(\Omega^\bullet)$ is the only nonzero cohomology group of $\Omega^\bullet$.

If M is a module then we write $M(b)$ for the complex that consists of a single module, M, placed in cohomological degree b. We shall call π the quasi-isomorphism $\Omega^\bullet \twoheadrightarrow \Omega(n)$. For $\mathfrak{S} \ni S = (s_1, \dots, s_k)$ write $dx_{\wedge S} := dx_{s_1} \wedge \cdots \wedge dx_{s_k}$. Then there is an isomorphism $\Omega^\bullet \otimes_{D_n} R_n \cong \Omega^\bullet_X$ identifying $dx_{s_1} \dots dx_{s_k} \otimes g$ with $g dx_{s_1} \wedge \dots \wedge dx_{s_k}$. More generally, there is an isomorphism of $\mathrm{Tot}^\bullet(\Omega^\bullet \otimes_{D_n} \check{C}^\bullet)$ with $\Omega^\bullet(\check{C}^\bullet)$.

By way of rephrasing the Grothendieck-Deligne comparison theorem it has been observed in [12, 16] that thus

$$H^i_{dR}(U; \mathbb{C}) \cong H^i(\Omega(n) \otimes^L_{D_n} \check{C}^\bullet(Y)),$$

the L denoting the left (hyper-)derived functor of $(-) \otimes_{D_n} (-)$. The functor $\Omega \otimes^L_{D_n} (-)$ is called *integration.*

1.7. Let us now introduce some basic concepts related to the $\tilde{V}$-filtration which is the mirror image of the V-filtration (cf. [11, 16]) under the Fourier transform.

If $\alpha \in \mathbb{Z}^n$ we write $|\alpha|$ for $\sum_{i=1}^n \alpha_i$. We let $\tilde{F}^k(D_n)$ be the subgroup

$$\tilde{F}^k(D_n) = \sum_{|\alpha-\beta| \leq k} \mathbb{C} \cdot x^\alpha \partial^\beta$$

of D_n using multi-index notation, and for a given operator $P \in D_n$ we define

$$\tilde{V}_n \deg(P) = \min\{k | P \in \tilde{F}^k(D_n)\}.$$

So $\tilde{F}^k(D_n) = \{P \in D_n | \tilde{V}_n \deg(P) \leq k\}$. This gives an increasing $\tilde{V}_n$-*filtration* on D_n which we generalize to free modules $\bigoplus_{i=1}^t D_n \cdot e_i[\mathfrak{m}]$ by

$$\tilde{F}^k(\bigoplus_1^t D_n \cdot e_i[\mathfrak{m}]) = \sum_{i=1}^t \sum_{|\alpha-\beta|+\mathfrak{m}(i) \leq k} \mathbb{C} \cdot x^\alpha \partial^\beta \cdot e_i$$

and

$$\tilde{V}_n \deg(P[\mathfrak{m}]) = \min\{k | P \in \tilde{F}^k({D_n}^t[\mathfrak{m}])\}$$

for fixed $\mathfrak{m} \in \mathbb{Z}^t$. $\mathfrak{m}$ is called the *shift vector*. Submodules and quotient modules inherit a filtration by taking intersections and images of $\tilde{F}^k$ on free modules respectively.

In order to make $\Omega^\bullet$ a $\tilde{V}_n$-strict (in fact, graded) resolution of Ω, we shift Ω^{n-k} by $-k$.

There is an equivalence of the category of left D_n-modules with itself induced by the algebra automorphism of D_n given by the *Fourier transform* $\mathcal{F}$,

$$\mathcal{F}(x_i) = \partial_i, \qquad \mathcal{F}(\partial_i) = -x_i.$$

The minus sign is required to keep the Leibniz relation $[\partial_i, x_i] = 1$ intact. We shall often write $\tilde{M}$ instead of $\mathcal{F}(M)$ in order to keep notation at a minimum. Since $\mathcal{F}(\Omega) = \tilde{\Omega} = D_n/(x_1, \dots, x_n) \cdot D_n$, the Fourier transform turns the computation of $\Omega \otimes^L_{D_n} (-)$ into a *restriction*, the derived functor of $\tilde{\Omega} \otimes_{D_n} (-)$.

If one defines

$$F^k(D_n) = \sum_{|-\alpha+\beta|\leq k} \mathbb{C} \cdot x^\alpha \partial^\beta$$

on D_n then one obtains the *V-filtration*. The V-filtration is the mode of writing in [10, 11, 12, 13, 16], but not convenient for dealing with de Rham cohomology. By the symmetry $\tilde{F}^k(D_n) = \mathcal{F}(F^k(D_n))$ every statement about the V-filtration proved in those papers has a corresponding companion for the $\tilde{V}$-filtration. In the sequel we will state facts about the $\tilde{V}$-filtration proved in the above papers about the V-filtration.

A complex of D_n-modules $\{A^\bullet[\mathfrak{m}_\bullet], \alpha^\bullet\}$ is *$\tilde{V}_n$-adapted* if

$$\alpha^i\left(\tilde{F}^k(A^i[\mathfrak{m}_i])\right) \subseteq \tilde{F}^k(A^{i+1}[\mathfrak{m}_{i+1}])$$

for all i, k, and we call the complex *$\tilde{V}_n$-strict* if in addition

$$\alpha^{i-1}\left(\tilde{F}^k(A^{i-1}[\mathfrak{m}_{i-1})])\right) = \alpha^{i-1}(A^{i-1}) \cap \tilde{F}^k(A^i[\mathfrak{m}_i])$$

for all i, k. For a $\tilde{V}_n$-adapted complex one can compute the associated graded complex $\tilde{\operatorname{gr}}^\bullet(A^\bullet[\mathfrak{m}_\bullet])$. $\tilde{V}_n$-strict complexes are those in which taking cohomology commutes with the formation of graded objects. In [16], Algorithm 3.8. it is shown how for a given right bounded complex of finitely generated left D_n-modules one can construct a quasi-isomorphism onto it from a V_n-strict D_n-free complex which can easily be modified to construct $\tilde{V}_n$-strict complexes instead. The purpose of Section 3 is to improve this algorithm, i.e., to give an algorithm that computes smaller complexes $A^\bullet[\mathfrak{m}_\bullet]$.

1.8. Recall that a finitely generated D_n-module M is called *holonomic* if $\operatorname{Ext}^i_{D_n}(M, D_n) = 0$ unless $i = n$. The holonomic D_n-modules form a full Abelian subcategory of the left D_n-modules. If M/N is a holonomic module described by the left submodules $M \supseteq N$ of $A[\mathfrak{m}]$ where A is D_n-free, then there exists a nonzero polynomial $b(s)$ satisfying

$$b(-\partial_1 x_1 - \ldots - \partial_n x_n + j) \cdot \tilde{F}^j(M[\mathfrak{m}]) \subseteq \tilde{F}^{j-1}(M[\mathfrak{m}]) + N \tag{1.3}$$

for all $j \in \mathbb{Z}$. The monic polynomial of least degree with such a property is called the *b-function for integration of $(M/N)[\mathfrak{m}]$ along $\partial_1, \ldots, \partial_n$*. The b-function for integration of a $\tilde{V}_n$-strict complex $A^\bullet[\mathfrak{m}_\bullet]$ is the nonzero polynomial $b(s)$ of minimal degree that satisfies

(1.4)
$$b(-\partial_1 x_1 - \ldots - \partial_n x_n + j) \cdot \tilde{F}^j(H^i(A^\bullet[\mathfrak{m}_\bullet])) \quad \subseteq \quad \tilde{F}^{j-1}(H^i(A^\bullet[\mathfrak{m}_\bullet]))$$

for all i.

We remind the reader that for all $I \subseteq \{0, \ldots, r\}$ the module $R_n[{F_I}^{-1}]$ is holonomic [1] and thus the complex $\check{C}^\bullet(Y)$ has holonomic cohomology.

1.9. If $A[\mathfrak{m}]$ is a free D_n-module with shift vector $\mathfrak{m}$ then $\Omega \otimes_{D_n} A[\mathfrak{m}]$ is filtered by the $\mathbb{C}$-vector spaces

$$\tilde{F}^k(\Omega \otimes_{D_n} A[\mathfrak{m}]) := \mathbb{C} \cdot \{\overline{P} \otimes_{D_n} Q \,|\, \tilde{V}_n \deg(P) + \tilde{V}_n \deg(Q[\mathfrak{m}]) \leq k\}.$$

Since $\Omega \cong \mathbb{C}[x_1, \dots, x_n]$ as right D_n-modules, $\tilde{F}^k(\Omega \otimes_{D_n} A[\mathfrak{m}])$ equals the free $\mathbb{C}$-module on the symbols

$$\{P_{j,\beta}\} = \{(P_1, \dots, P_{\mathrm{rk}_{D_n}(A)}) | P_j = x^\alpha \in \mathbb{C}[x_1, \dots, x_n], \\ |\alpha| \leq k - \mathfrak{m}(j), P_l = 0 \forall l \neq j\}. \tag{1.5}$$

If $A^\bullet[\mathfrak{m}_\bullet]$ is a $\tilde{V}_d$-strict complex, we denote by $\tilde{F}^k(\Omega \otimes A^\bullet[\mathfrak{m}_\bullet])$ the complex whose modules are the $\tilde{F}^k(\Omega \otimes A^i[\mathfrak{m}_i])$ as defined above, and the maps are induced from $A^\bullet$.

Unless specified otherwise, all tensor products in the sequel will be over D_n.

2. The case of a hypersurface

In this section we assume that $r = 0$ and hence that U is affine. We set $f = f_0$, $M = R_n[f^{-1}]$ and we assume that M is generated by $1/f$ as D_n-module (which can always be arranged by replacing f by a suitable power of f). Let us first consider an explicit

Example 2.1. Let $X = \mathbb{C}^2$, $R_2 = \mathbb{C}[x, y]$, $f = xy$, $Y = \mathrm{Var}(f)$ and $U = X \setminus Y$. The algebraic de Rham complex $\Omega_U^\bullet$ on U takes the form

$$(0 \to R_2[f^{-1}] \xrightarrow{d^0} R_2[f^{-1}]\, dx \oplus R_2[f^{-1}]\, dy \xrightarrow{d^1} R_2[f^{-1}]\, dx \wedge dy \to 0) \tag{2.1}$$

with maps given by $d^0 = \begin{pmatrix} \partial_x \bullet \\ \partial_y \bullet \end{pmatrix}$ and $d^1 = (\partial_y \bullet, -\partial_x \bullet)$.

It is not hard to see that the cohomology groups of this complex are $H^0 \cong \mathbb{C}$, $H^1 \cong \mathbb{C}^2$ and $H^2 \cong \mathbb{C}$, generated by the classes of the forms 1, $\frac{dx}{x} \oplus 0$ and $0 \oplus \frac{dy}{y}$, and $\frac{1}{xy} dx \wedge dy$. Let us compute the cup product of the two classes in degree 1. To do this we recall the following theorem.

Theorem 2.2 (cf. [2], p.174). *Let $U \subseteq \mathbb{C}^n$ be an open subset. Assume that $\overline{c}$ and $\overline{c'}$ are two cohomology classes in $H^i(U; \mathbb{C})$ and $H^j(U; \mathbb{C})$ respectively, represented by the cycles*

$$c = \sum_{S \in \mathfrak{S}_i} g_S \, dx_{\wedge S}$$

and

$$c' = \sum_{S' \in \mathfrak{S}_j} g'_{S'} \, dx_{\wedge S'}$$

where g_S and $g'_{S'}$ are in $\Gamma(U, \mathcal{O}_U)$. Then the cup product $\overline{c} \cup \overline{c'}$ is represented by the cycle

$$\sum_{S \in \mathfrak{S}_i} \sum_{S' \in \mathfrak{S}_j} g_S \cdot g'_{S'}\, dx_{\wedge S} \wedge dx_{\wedge S'}.$$

$\square$

Remark 2.3. The theorem in [2] really deals with C^∞-differential forms, but is in particular applicable if c and c' are represented by algebraic cycles, in which case the product is algebraic as well. As pointed out in the introduction, the de Rham cohomology ring can be modelled with algebraic data.

It follows that in our example $(\frac{dx}{x} \oplus 0) \cup (0 \oplus \frac{dy}{y}) = \frac{1}{xy} dx \wedge dy$.

In general, it is not easy to see what the cohomology groups of $\Omega^\bullet_U$ are. Even if one knows the cohomology, it is very unlikely that the product of any two of the chosen generators gives exactly another one. Moreover, if the dimensions of the cohomology groups were computed by means of the algorithms given in [12] and [16] then it is a priori rather unclear what the meaning of their generators is in terms of the cohomology groups of $\Omega^\bullet_U = \Omega^\bullet(\check{C}^\bullet)$. In order to understand this relationship better, let us review Algorithm 2.1 in [12] (rephrased in terms of integration rather than restriction):

Algorithm 2.4.
INPUT: $f \in R_n$; $i \in \mathbb{N}$.
OUTPUT: $\dim_{\mathbb{C}}(H^i_{dR}(U; \mathbb{C}))$ where $U = \mathbb{C}^n \setminus \mathrm{Var}(f)$.
Begin

1. Find a $\tilde{V}_n$-strict resolution $A^\bullet[\mathfrak{m}_\bullet]$ of the D_n-module $R_n[f^{-1}]$ by finitely generated free D_n-modules where $R_n[f^{-1}]$ is positioned in cohomological degree 0.
2. Replace each D_n by the right D_n-module $\Omega = D_n/(\partial_1, \ldots, \partial_n) \cdot D_n \cong \mathbb{C}[x_1, \ldots, x_n]$ in that resolution.
3. Find the b-function for the integration of $R_n[f^{-1}]$ along $\partial_1, \ldots, \partial_n$; let $k_0, k_1 \in \mathbb{Z}$ with $(b(k) = 0, k \in \mathbb{Z}) \Rightarrow (k_0 \le k \le k_1)$.
4. Truncate the output of Step 2 to the complex of finite dimensional $\mathbb{C}$-vector spaces $\frac{\tilde{F}^{k_1}(\Omega(n) \otimes A^\bullet[\mathfrak{m}_\bullet])}{\tilde{F}^{k_0-1}(\Omega(n) \otimes A^\bullet[\mathfrak{m}_\bullet])}$ (note the shift in cohomological degree).
5. Take the i-th cohomology and return its dimension.

End.

The resolution in Step 1 is computed as follows. First write $R_n[f^{-1}]$ as A^0/I_0 where $A^0 = D_n[0]$. Compute a $\tilde{V}_n$-strict Gröbner basis (see [10, 11, 14]) for I_0, let $A^{-1}[\mathfrak{m}_{-1}]$ be a free module surjecting its generators onto this Gröbner basis while preserving $\tilde{V}_n$-degree, let I_{-1} be the kernel and iterate this procedure to obtain a resolution of desired length.

2.1. Let us consider the effects of Algorithm 2.4 on Example 2.1. First we need to write $M = \mathbb{C}[x, y, (xy)^{-1}]$ as a D_2-module: $M = D_2/D_2 \cdot (\partial_x x, \partial_y y)$ generated by $\frac{1}{xy} \in M$. M has a $\tilde{V}_2$-strict resolution $(A^\bullet[\mathfrak{m}_\bullet], \alpha^\bullet)$ as follows:

(2.2)

$$0 \to \underbrace{D_2[0]}_{A^{-2}[0]} \xrightarrow{\overbrace{\cdot(\partial_y y, -\partial_x x)}^{\alpha^{-2}}} \underbrace{(D_2 \oplus D_2)[0,0]}_{A^{-1}[0,0]} \xrightarrow{\overbrace{\cdot \begin{pmatrix} \partial_x x \\ \partial_y y \end{pmatrix}}^{\alpha^{-1}}} \underbrace{D_2[0]}_{A^0[0]} \longrightarrow M \to 0.$$

Since

$$\begin{aligned} (-\partial_x x - \partial_y y) \cdot \tilde{F}^0(D_2) &\subseteq \tilde{F}^{-1}(D_2) + \tilde{F}^0(D_2) \cdot (-\partial_x x - \partial_y y) \\ &\subseteq \tilde{F}^{-1}(D_2) + D_2 \cdot (\partial_x x, \partial_y y), \end{aligned}$$

the b-function associated to the integration of M is $b(s) = s$.

Thus, replacing D_2 by $\Omega(2)$ in the resolution (2.2) and truncating the complex to forms of $\tilde{V}_2$-degree at most zero (modulo those of negative $\tilde{V}_2$-degree) one obtains the complex

$$0 \to \underbrace{\mathbb{C} \cdot \overline{1}}_{\text{degree } 0} \to \underbrace{\mathbb{C} \cdot \overline{(1,0)} \oplus \mathbb{C} \cdot \overline{(0,1)}}_{\text{degree } 1} \to \underbrace{\mathbb{C} \cdot \overline{1}}_{\text{degree } 2} \to 0,$$

in which all differentials are zero.

2.2. If the cohomology groups are obtained in this way, the cup product is considerably more opaque. Continuing our example, we do not have an obvious identification of the classes $\overline{(1,0)}$ and $\overline{(0,1)}$ with forms in the de Rham complex on U. (Note in particular, that it is completely coincidental that the number of direct summands in degree j in (2.1) and (2.2) agree for all j: one is $\binom{r+1}{j}$, the other depends on Gröbner basis computations.) Thus, we need to understand better the relation between what Algorithm 2.4 computes and the de Rham complex of U.

The basic reason for the output of Algorithm 2.4 to be the same as $H^\bullet_{dR}(U; \mathbb{C})$ is that both compute $\operatorname{Tor}^{D_n}_\bullet(\Omega(n), M)$ as was pointed out in [12]. The following theorem allows to construct an explicit algorithmic correspondence between the cohomology groups of $\Omega^\bullet_U$ and those of the complex $\Omega(n) \otimes A^\bullet$ appearing in Algorithm 2.4. In other words, it translates the output of Algorithm 2.4 into the language of differential forms. This is the most technical piece of the paper, and for the purpose of later reference, we state the theorem somewhat more formally. Recall that $B(m)$ denotes the complex that consists of the single module B located in cohomological degree m.

Theorem 2.5 (Transfer Theorem). *Let $(B^\bullet, \beta^\bullet)^m_0 \xrightarrow{\pi} B(m)$ be a free resolution for the right D_n-module $B(m)$. Let $(A^\bullet, \alpha^\bullet)^l_k \xrightarrow{\pi'} (C^\bullet, \delta^\bullet)^l_k$ be a quasi-isomorphism from a complex of free left D_n-modules to a complex of* *finitely generated D_n-modules, constructed either as in our Section 3 or* [16], *Section 3. There exists a natural $\mathbb{C}$-module isomorphism*

$$H^i(\mathrm{Tot}^\bullet(B(m)\otimes A^\bullet)) \ni \overline{c} \leftrightarrow \tau(\overline{c}) \in H^i(\mathrm{Tot}^\bullet(B^\bullet \otimes C^\bullet)),$$

as is well known. We shall call τ the Transfer map.

τ can be computed with Gröbner basis techniques in the following sense. If $\overline{c}$ is given by means of the representative $c \in B\otimes A^{i-m}$ with $(B\otimes\alpha^{i-m})(c)=0$ then one can construct an element

$$\sum a^{m-j,i-m+j} \in \ker(\mathrm{Tot}^i(B^\bullet\otimes A^\bullet)\to \mathrm{Tot}^{i+1}(B^\bullet\otimes A^\bullet))$$

with $\overline{(\pi\otimes A^\bullet)(\sum a^{m-j,i-m+j})} = \overline{c}$ and $\overline{(B^\bullet\otimes\pi')(\sum a^{m-j,i-m+j})} = \tau(\overline{c})$. Similarly, if $\overline{c'} \in H^i(\mathrm{Tot}^\bullet(B^\bullet\otimes C^\bullet))$ is given by

$$\sum c^{m-j,i-m+j} \in \ker(\mathrm{Tot}^i(B^\bullet\otimes C^\bullet)\to \mathrm{Tot}^{i+1}(B^\bullet\otimes C^\bullet))$$

then one can compute an element

$$\sum a^{m-j,i-m+j} \in \ker(\mathrm{Tot}^i(B^\bullet\otimes A^\bullet)\to \mathrm{Tot}^{i+1}(B^\bullet\otimes A^\bullet))$$

with $(B^\bullet\otimes\pi')(a^{m-j,i-m+j}) = c^{m-j,i-m+j}$ and $\overline{(\pi\otimes A^\bullet)(\sum a^{m-j,i-m+j})} = \tau^{-1}(\overline{c})$.

The necessary computations are exclusively Gröbner basis computations in free D_n-modules.

Proof. Consider the double complex

$$\begin{tikzcd}
B\otimes A^0 \arrow[r] & \cdots \arrow[r] & B\otimes A^m \\
B^m\otimes A^k \arrow[u] \arrow[r, "\alpha^{m,k}"] & \cdots \arrow[r, "\alpha^{m,l-1}"] & B^m\otimes A^l \arrow[u] \\
\vdots \arrow[u, "\beta^{m-1,k}"] & & \vdots \arrow[u, "\beta^{m-1,l}"] \\
B^0\otimes A^k \arrow[u, "\beta^{0,k}"] \arrow[r, "\alpha^{0,k}"] & \cdots \arrow[r, "\alpha^{0,l-1}"] & B^0\otimes A^l \arrow[u, "\beta^{0,l}"]
\end{tikzcd} \tag{2.3}$$

together with its unique row of non-vanishing cohomology. Here we write $\alpha^{i,j} := B^i\otimes\alpha^j$, $\beta^{i,j} := \beta^i\otimes A^j$ and $\delta^{i,j} = B^i\otimes\delta^j$. The total complex $\mathrm{Tot}^\bullet(B^\bullet\otimes A^\bullet)$ is quasi-isomorphic to both $\mathrm{Tot}^\bullet(B^\bullet\otimes C^\bullet)$ and $\mathrm{Tot}^\bullet(B(m)\otimes A^\bullet)$ via the projections $\pi : B^\bullet\to B(m)$ and $\pi' : A^\bullet\to C^\bullet$. We call these induced chain maps π and π' as well.

Let $\overline{c}\in H^i(\mathrm{Tot}^\bullet(B(m)\otimes A^\bullet))$ be represented by $c\in B\otimes A^{i-m}$. Since $B^m \twoheadrightarrow B(m)$ and A^{i-m} is free, we can pick $a^{m,i-m}\in B^m\otimes A^{i-m}$ with $\pi(a^{m,i-m}) = c$. So $\alpha^{m,i-m}(a^{m,i-m}) \in \ker(\pi) = \mathrm{im}(\beta^{m-1,i-m+1})$ since $B^\bullet$ is a resolution and A^{i-m+1} is flat. Thus, there is $a^{m-1,i-m+1}\in B^{m-1}\otimes A^{i-m+1}$ with $\alpha^{m,i-m}(a^{m,i-m}) = \beta^{m-1,i-m+1}(a^{m-1,i-m+1})$.

Then $\alpha^{m-1,i-m+1}(a^{m-i,i-m+1}) \in \ker(\beta^{m-1,i-m+2}) = \mathrm{im}(\beta^{m-2,i-m+2})$. We can pick $a^{m-2,i-m+2}\in B^{m-2}\otimes A^{i-m+2}$ with $\alpha^{m-1,i-m+1}(a^{m-1,i-m+1}) =$

$\beta^{m-2,i-m+2}(a^{m-2,i-m+2})$. Progressing in this way we obtain successively $a^{m-j,i-m+j} \in B^{m-j} \otimes A^{i-m+j}$ for $0 \leq j \leq m$ with

$$\alpha^{m-j,i-m+j}(a^{m-j,i-m+j}) = \beta^{m-j-1,i-m+j+1}(a^{m-j-1,i-m+j+1})$$

for $0 \leq j < m$.

$\sum(-1)^j a^{m-j,i-m+j}$ represents a cohomology class in $H^i(\mathrm{Tot}^\bullet(B^\bullet \otimes A^\bullet))$, this cohomology class projects onto $\overline{c}$ under π_* and onto $\tau(\overline{c})$ under π'_*. Thus, $\tau(\overline{c}) = \overline{\pi'(\sum(-1)^j a^{m-j,i-m+j})} \in H^i(\mathrm{Tot}^\bullet(B^\bullet \otimes C^\bullet))$. This construction is additive and hence a group homomorphism.

Now we shall investigate the inverse map. The idea is similar but complicated by the fact that $A^\bullet$ may be non-exact in more than one place.

This time we have to start the diagram chase at the right "boundary" of the double complex. Let $\overline{c'} = \overline{c^{m,i-m} + c^{m-1,i-m+1} + \ldots + c^{i-l,l}}$ be a cohomology class in $H^i(\mathrm{Tot}^\bullet(B^\bullet \otimes C^\bullet))$ with $c^{j,i-j} \in B^j \otimes C^{i-j}$. Clearly $\delta^{i-l,l}(c^{i-l,l}) = 0$, and since $B^{i-l} \otimes A^\bullet \xrightarrow{\cong} B^{i-l} \otimes C^\bullet$ there is $a^{i-l,l} \in \ker(\alpha^{i-l,l})$, $\pi'(a^{i-l,l}) = c^{i-l,l}$. Since $A^{l+1} = 0$, $\alpha^{i-l+1,l}(\beta^{i-l,l}(a^{i-l,l})) = 0$. So $\beta^{i-l,l}(a^{i-l,l})$ represents a cohomology class in $B^{i-l+1} \otimes A^\bullet$. Since by assumption on c' we have $\pi'(\beta^{i-l,l}(a^{i-l,l})) = -\delta^{i-l+1,l-1}(c^{i-l+1,l-1})$, $\beta^{i-l,l}(a^{i-l,l})$ represents the zero cohomology class and therefore there is $a_1^{i-l+1,l-1} \in B^{i-l+1} \otimes A^{l-1}$ with $\alpha^{i-l+1,l-1}(a_1^{i-l+1,l-1}) = -\beta^{i-l,l}(a^{i-l,l})$.

There is unfortunately little reason to hope that $\pi'(a_1^{i-l+1,l-1}) = c^{i-l+1,l-1}$, which is what we really want. But, $c^{i-l+1,l-1} - \pi'(a_1^{i-l+1,l-1}) \in \ker(\delta^{i-l+1,l-1})$ by construction. Therefore there is $a_2^{i-l+1,l-1} \in \ker(\alpha^{i-l+1,l-1})$ such that $\pi'(a_2^{i-l+1,l-1}) = c^{i-l+1,l-1} - \pi'(a_1^{i-l+1,l-1})$. We set

$$a^{i-l+1,l-1} = a_1^{i-l+1,l-1} + a_2^{i-l+1,l-1} \in B^{i-l+1} \otimes A^{l-1}$$

which maps to $c^{i-l+1,l-1}$ under π' and satisfies

$$\alpha^{i-l+1,l-1}(a^{i-l+1,l-1}) = -\beta^{i-l,l}(a^{i-l,l}).$$

Now we repeat this procedure with $a^{i-l+1,l-1}$ as follows. By construction $\alpha^{i-l+2,l-1}(\beta^{i-l+1,l-1}(a^{i-l+1,l-1})) = 0$, and $\pi'(\beta^{i-l+1,l-1}(a^{i-l+1,l-1})) = -\delta^{i-l+2,l-2}(c^{i-l+2,l-2})$. Hence $\beta^{i-l+1,l-1}(a^{i-l+1,l-1})$ represents the zero class in $B^{i-l+2} \otimes A^\bullet$. So we can pick $a_1^{i-l+2,l-2} \in B^{i-l+2} \otimes A^{l-2}$ such that $\alpha^{i-l+2,l-2}(a_1^{i-l+2,l-2}) = -\beta^{i-l+1,l-1}(a^{i-l+1,l-1})$. Since $\delta^{i-l+2,l-2}(c^{i-l+2,l-2} - \pi'(a_1^{i-l+2,l-2})) = 0$ we can then pick $a_2^{i-l+2,l-2} \in B^{i-l+2} \otimes A^{l-2}$ satisfying $\pi'(a_2^{i-l+2,l-2}) = c^{i-l+2,l-2} - \pi'(a_1^{i-l+2,l-2})$ and $\alpha^{i-l+2,l-2}(a_2^{i-l+2,l-2}) = 0$. Set

$$a^{i-l+2,l-2} = a_1^{i-l+2,l-2} + a_2^{i-l+2,l-2} \in B^{i-l+2} \otimes A^{l-2}.$$

So $\pi'(a^{i-l+2,l-2}) = c^{i-l+2,l-2}$ and

$$\alpha^{i-l+2,l-2}(a^{i-l+2,l-2}) = -\beta^{i-l+1,l-1}(a^{i-l+1,l-1}).$$

Proceeding in this way we obtain elements $a^{i-l+j,l-j} \in B^{i-l+j} \otimes A^{l-j}$ from which we construct $a = \sum a^{i-l+j,l-j}$. a is by construction in the kernel of

$\mathrm{Tot}^\bullet(B^\bullet \otimes A^\bullet)$ and its cohomology class $\overline{a}$ in $H^\bullet(B^\bullet \otimes A^\bullet)$ satisfies

$$\pi'_*(\overline{a}) = \overline{c'}.$$

This construction yields the desired inverse to the map τ:

$$\overline{\pi(a)} = \overline{\pi(a^{m,i-m})} = \tau^{-1}(\overline{c'}).$$

In these computations we used that we can compute preimages under maps between free modules, and images under such maps. These constructions are all algorithmic because the construction of preimages of elements under maps between free D_n-modules is an application of Gröbner bases. This finishes the proof of the transfer theorem. $\square$

2.3. A particular and important application for the theorem is given if we set $(B^\bullet, \beta^\bullet) = (\Omega^\bullet, \varepsilon^\bullet)$, and let $(C^\bullet, \delta^\bullet)$ be the Mayer-Vietoris complex associated to a variety. This is possible because $\Omega^\bullet$ is a resolution for $\Omega(n)$. In this case the theorem enables us to compute first generators for the cohomology of $\Omega(n) \otimes A^\bullet$ via Algorithm 6.1. in [16], and then from the generators for this homology to compute actual differential forms representing the de Rham cohomology of U (represented by elements of $\mathrm{Tot}^\bullet(\Omega^\bullet \otimes \check{C}^\bullet)$).

The philosophy for evaluating the cup product is as follows. After computing a basis for the cohomology of $\Omega(n) \otimes A^\bullet$ one lifts these basis elements into cohomology generators of $\mathrm{Tot}^\bullet(\Omega^\bullet \otimes A^\bullet)$ and computes their transfers in $\mathrm{Tot}^\bullet(\Omega^\bullet \otimes \check{C}^\bullet)$. We can multiply those classes which are now actual differential forms. Then one transfers the products back to $\Omega(n) \otimes A^\bullet$. (The point of the second transfer above is that one wishes to express the products in terms of the chosen basis, but this is only possible in a truncation of $\Omega \otimes A^\bullet$ because $\Omega \otimes A^\bullet$ as well as $\mathrm{Tot}^\bullet(\Omega^\bullet \otimes \check{C}^\bullet)$ are too large.)

2.4. Let us consider how the Transfer Theorem works in our Example 2.1. (This is a very special case of the Transfer Theorem since the complex $\check{C}^\bullet$ has only one nonzero entry.) To this end we consider the double complex

$$(2.4)\qquad \begin{tikzcd}
\Omega(2)\otimes D_2 \arrow[r] & \Omega(2)\otimes(D_2\oplus D_2) \arrow[r] & \Omega(2)\otimes D_2 & \\
D_2\otimes D_2 \arrow[r, "\alpha^{2,-2}"] \arrow[u] & D_2\otimes(D_2\oplus D_2) \arrow[r, "\alpha^{2,-1}"] \arrow[u] & D_2\otimes D_2 \arrow[r] \arrow[u] & D_2\otimes M \\
\begin{pmatrix} D_2[-1] \\ \oplus \\ D_2[-1] \end{pmatrix}\otimes D_2 \arrow[r, "\alpha^{1,-2}"] \arrow[u, "\varepsilon^{1,-2}"'] & \begin{pmatrix} D_2[-1] \\ \oplus \\ D_2[-1] \end{pmatrix}\otimes(D_2\oplus D_2) \arrow[r, "\alpha^{1,-1}"] \arrow[u, "-\varepsilon^{1,-1}"'] & \begin{pmatrix} D_2[-1] \\ \oplus \\ D_2[-1] \end{pmatrix}\otimes D_2 \arrow[r] \arrow[u, "\varepsilon^{1,0}"'] & \begin{pmatrix} D_2[-1] \\ \oplus \\ D_2[-1] \end{pmatrix}\otimes M \arrow[u] \\
D_2[-2]\otimes D_2 \arrow[r, "\alpha^{0,-2}"] \arrow[u, "\varepsilon^{0,-2}"'] & D_2[-2]\otimes(D_2\oplus D_2) \arrow[r, "\alpha^{0,-1}"] \arrow[u, "-\varepsilon^{0,-1}"'] & D_2[-2]\otimes D_2 \arrow[r] \arrow[u, "\varepsilon^{0,0}"'] & D_2[-2]\otimes M \arrow[u]
\end{tikzcd}$$

obtained by tensoring the resolution (2.2) for M with the complex $\Omega^\bullet$. Note that we have the following maps in the complex:

$$\begin{aligned}
\varepsilon^{1,-2}((a,b)\otimes c) &= (-\partial_y a + \partial_x b)\otimes c,\\
\varepsilon^{0,-2}(a\otimes c) &= (\partial_x a, \partial_y a)\otimes c,\\
\varepsilon^{1,-1}((a,b)\otimes (c,d)) &= (\partial_y a - \partial_x b)\otimes (c,d),\\
\varepsilon^{0,-1}(a\otimes (c,d)) &= (\partial_x a, \partial_y a)\otimes (c,d),\\
\varepsilon^{1,0}((a,b)\otimes c) &= (-\partial_y a + \partial_x b)\otimes c,\\
\varepsilon^{0,0}(a\otimes c) &= (\partial_x a, \partial_y a)\otimes c.
\end{aligned}$$

Similarly,

$$\begin{aligned}
\alpha^{2,-2}(a\otimes c) &= a\otimes (c\partial_y y, -c\partial_x x),\\
\alpha^{2,-1}(a\otimes (c,d)) &= a\otimes (c\partial_x x + d\partial_y y),\\
\alpha^{1,-2}((a,b)\otimes c) &= (a,b)\otimes (c\partial_y y, -c\partial_x x),\\
\alpha^{1,-1}((a,b)\otimes (c,d)) &= (a,b)\otimes (c\partial_x x + d\partial_y y),\\
\alpha^{0,-2}(a\otimes c) &= a\otimes (c\partial_y y, -c\partial_x x),\\
\alpha^{0,-1}(a\otimes (c,d)) &= a\otimes (c\partial_x x + d\partial_y y).
\end{aligned}$$

Let us now multiply our two cohomology classes $\overline{(1,0)}$ and $\overline{(0,1)}$ in $H^1(\Omega\otimes A^\bullet)$ from the top row in (2.4).

First we lift the two classes into $D_2\otimes(D_2\oplus D_2)$, for example to $1\otimes(1,0)$ and $1\otimes(0,1)$. Now we find their transfers. Apply $\alpha^{2,-1}$ to both to obtain $1\otimes\partial_x x = \partial_x\otimes x$ and $1\otimes\partial_y y = \partial_y\otimes y$ respectively. These are now in the kernel of the projection $D_2\otimes D_2 \to \Omega\otimes D_2$. Thus we can lift them into $\begin{pmatrix} D_2[-1] \\ \oplus \\ D_2[-1]\end{pmatrix}\otimes D_2$, resulting in $(0,1)\otimes x$ and $(-1,0)\otimes y$. Multiplying by $(-1)^{2-1}$, we find $(0,-1)\otimes x$ and $(1,0)\otimes y$. These we can push into $\begin{pmatrix} D_2[1] \\ \oplus \\ D_2[1]\end{pmatrix}\otimes M$ where they become the cosets of $(0,-1)\otimes(x)$ and $(1,0)\otimes(y)$. Since M is generated by $\frac{1}{xy}$, these elements correspond to the forms $\frac{-x\,dy}{xy} = \frac{-dy}{y}$ and $\frac{y\,dx}{xy} = \frac{dx}{x}$. The product is $\frac{1}{xy}\cdot dx\wedge dy$ in $D_2\otimes M$, which corresponds to the lifted operator $1\otimes 1$ in $D_2\otimes D_2$. We obtain the element $\overline{(1\otimes 1)}$ in $\Omega\otimes D_2$, establishing the multiplication rule $\overline{(1,0)}\cup\overline{(0,1)} = \overline{1}$.

In our example we have tacitly been assuming the following things which we need to explain a bit.

2.5. The output of Algorithm 2.4 is set of a cohomology classes in the complex $\frac{\tilde{F}^{k_1}(\Omega(n)\otimes A^\bullet[\mathfrak{m}_\bullet])}{\tilde{F}^{k_0-1}(\Omega(n)\otimes A^\bullet[\mathfrak{m}_\bullet])}$. We however pretended that they actually live in $\Omega(n) \otimes A^\bullet[\mathfrak{m}_\bullet]$. This was crucial for applying the Transfer Theorem to it. The following lemma explains the relationship between the cohomologies of these two complexes.

Lemma 2.6. *Let $(A^\bullet[\mathfrak{m}_\bullet], \alpha^\bullet)$ be a $\tilde{V}_n$-strict free complex of left D_n modules with holonomic cohomology. Denote by $b(s)$ the b-function of $(A^\bullet[\mathfrak{m}_\bullet], \alpha^\bullet)$ for integration along $\partial_1, \ldots, \partial_n$ and let k_1 be the largest integral root of $b(s)$. Then*

- *$\tilde{F}^{k_1}(\Omega \otimes A^i[\mathfrak{m}_i])$ is a finite dimensional vector space for all i;*
- *$\overline{1} \otimes P \in \tilde{F}^{k_1}(\Omega \otimes A^i[\mathfrak{m}_i]) \subseteq \Omega \otimes A^i[\mathfrak{m}_i]$ is in $\ker(\tilde{F}^{k_1}(\Omega \otimes \alpha^i))$ if and only if it is in $\ker(\Omega \otimes \alpha^i)$;*
- *$\overline{1} \otimes P \in \tilde{F}^{k_1}(\Omega \otimes A^i[\mathfrak{m}_i])$ is in $\alpha^{i-1}(\tilde{F}^{k_1}(\Omega \otimes A^{i-1}[\mathfrak{m}_{i-1}]))$ if and only if it is in $\alpha^{i-1}(\Omega \otimes A^{i-1}[\mathfrak{m}_{i-1}])$;*
- *if $k_1' > k_1$ and $\overline{1} \otimes P \in \ker(\tilde{F}^{k_1'}(\Omega \otimes \alpha^i))$ then there exists an operator $Q \in \tilde{F}^{k_1'}(A^{i-1}[\mathfrak{m}_{i-1}])$ such that*
$$\alpha^{i-1}(Q) + P = P' + P''$$
where P' is of $\tilde{V}_n$-degree at most k_1 and $P'' \in (\partial_1, \ldots, \partial_n) \cdot A^i$ (we will see in 2.7 how construct such a Q and P' from a given P).

Proof. The main points of this lemma have been proven in [11, 16]. There it is shown that if $B^\bullet[\mathfrak{n}_\bullet]$ is V_n-strict and l_1 is the largest integral root of the nonzero b-function of $B^\bullet[\mathfrak{n}_\bullet]$ for restriction to the origin, then the inclusion of complexes

$$F^{l_1}(\tilde{\Omega} \otimes B^\bullet[\mathfrak{n}_\bullet]) \hookrightarrow F^{l_1'}(\tilde{\Omega} \otimes B^\bullet[\mathfrak{n}_\bullet])$$

is a quasi-isomorphism for $l_1' > l_1$ in our situation. By the symmetry of integration and restriction,

$$\tilde{F}^{k_1}(\Omega \otimes A^\bullet[\mathfrak{m}_\bullet]) \quad \hookrightarrow \quad \tilde{F}^{k_1'}(\Omega \otimes A^\bullet[\mathfrak{m}_\bullet]) \tag{2.5}$$

is a quasi-isomorphism for $k_1' > k_1$. This implies the final three claims. The first claim is a consequence of the facts that $\tilde{\mathrm{gr}}^k(\Omega \otimes A^\bullet[\mathfrak{m}_\bullet])$ is finite dimensional in each degree, and $\tilde{F}^k(\Omega \otimes A^\bullet[\mathfrak{m}_\bullet])$ is the zero complex for sufficiently small k (because $\tilde{V}_n \deg(P[j]) \le j - 1$ implies $P \in (\partial_1, \ldots, \partial_n) \cdot D_n$ and hence $\overline{1} \otimes P = 0$ in $\Omega \otimes D_n$). $\square$

By the lemma, taking the k_0 of Algorithm 2.4 sufficiently small results in $\tilde{F}^{k_0-1}(\Omega \otimes A^\bullet[\mathfrak{m}_\bullet]) = 0$ and thus $\frac{\tilde{F}^{k_1}(\Omega \otimes A^\bullet[\mathfrak{m}_\bullet])}{\tilde{F}^{k_0-1}(\Omega \otimes A^\bullet[\mathfrak{m}_\bullet])} = \tilde{F}^{k_1}(\Omega \otimes A^\bullet[\mathfrak{m}_\bullet])$. Furthermore, any cohomology class $\overline{1} \otimes P$ in $\tilde{F}^{k_1}(\Omega \otimes A^\bullet[\mathfrak{m}_\bullet])$ lifts immediately to the class of $\overline{1} \otimes P$ in $\Omega \otimes A^\bullet[\mathfrak{m}_\bullet]$. It is hence safe to assume that the output of Algorithm 2.4 consists of classes of the complex $\tilde{F}^{k_1}(\Omega \otimes A^\bullet[\mathfrak{m}_\bullet])$.

2.6. Another problem we evaded in the example is the following. Choose two cohomology classes $\overline{c}, \overline{c'}$ in $\tilde{F}^{k_1}(\Omega \otimes A^\bullet[\mathfrak{m}_\bullet])$, find their transfer and take the cup product. The representative c'' which we compute for the transfer of this cup product may be of $\tilde{V}_n$-degree strictly bigger than k_1

(see Example 2.8). This new representative therefore would not fit into the complex $\tilde{F}^{k_1}(\Omega\otimes A^\bullet[\mathfrak{m}_\bullet])$ within which we are carrying out all computations. However, if $k_1' = \tilde{V}_n\deg(c'')$ then the complexes $\tilde{F}^{k_1}(\Omega\otimes A^\bullet[\mathfrak{m}_\bullet])$ and $\tilde{F}^{k_1'}(\Omega\otimes A^\bullet[\mathfrak{m}_\bullet])$ are quasi-isomorphic according to Lemma 2.5. Thus, c'' is simply a bad representative for $\overline{c''}$.

The following algorithm shows how to find for such a $\overline{c''}$ a good representative, namely one in $\tilde{F}^{k_1}(\Omega\otimes A^\bullet[\mathfrak{m}_\bullet])$, and in effect finds implicitly the Q and explicitly the P' in the fourth part of Lemma 2.6 for a given P. (We continue to assume that the hypotheses of Lemma 2.6 hold, in particular $\tilde{F}^{k_1}(\Omega\otimes A^\bullet[\mathfrak{m}_\bullet]) \cong \Omega\otimes A^\bullet[\mathfrak{m}_\bullet]$ is part of this assumption.)

Algorithm 2.7 (Minimal $\tilde{V}$-degree representatives in $\Omega\otimes A^\bullet[\mathfrak{m}_\bullet]$).
INPUT: $P\in A^i[\mathfrak{m}_i]$ such that $\tilde{V}_n\deg(P[\mathfrak{m}_i]) = k_1' > k_1$ and $\overline{1}\otimes P$ generates a cohomology class in $\tilde{F}^{k_1'}(\Omega\otimes A^\bullet[\mathfrak{m}_\bullet])$.
OUTPUT: $P'\in A^i[\mathfrak{m}_i]$ such that $\tilde{V}_n\deg(P'[\mathfrak{m}_i]) \le k_1$ and $\overline{1}\otimes P'$ generates the cohomology class in $\tilde{F}^{k_1}(\Omega\otimes A^\bullet[\mathfrak{m}_\bullet])$ that corresponds to the class of $\overline{1}\otimes P$ under the quasi-isomorphism (2.5).
Begin

1. We use the presentation (1.5) for $\tilde{F}^{k_1}(\Omega\otimes A^\bullet[\mathfrak{m}_\bullet])$. In particular, we assume that $\overline{1}\otimes P$ contains no ∂_j.
2. Let M^{i-1} be the $\mathbb{C}$-matrix that represents (by right multiplication by M^{i-1}) the $(i-1)$-st differential in $\tilde{F}^{k_1'}(\Omega\otimes A^{i-1}[\mathfrak{m}_{i-1}]) \to \tilde{F}^{k_1'}(\Omega\otimes A^i[\mathfrak{m}_i])$. The $\mathbb{C}$-module of rows of M^{i-1} represents the i-boundaries in $\tilde{F}^{k_1'}(\Omega\otimes A^\bullet[\mathfrak{m}_\bullet])$.
3. Apply the Buchberger algorithm to the $\mathbb{C}$-module of rows of M^{i-1}. For this we assume that the order on the columns we use refines $\tilde{V}_n$-degree of the elements of A^i the columns correspond to (see the example below).
4. Apply Buchberger reduction to $\overline{1}\otimes P$ relative to a Gröbner basis for the rows of M^{i-1}.
5. Return the remainder, which now lies in $\tilde{F}^{k_1}(\Omega\otimes A^\bullet[\mathfrak{m}_\bullet])$ and in the same cohomology class as the input.

End.

Example 2.8. Suppose in our running example we had chosen the operator $\overline{1}\otimes(1+x^2\partial_x+x)\in\Omega^2\otimes D_2$ as a lift for $\frac{1}{xy}dx\wedge dy\in\Omega^2\otimes R_2[f^{-1}]$. This is conceivable since $(x^2\partial_x+x)\bullet\frac{1}{xy}=0$. Then $\overline{\overline{1}\otimes(1+x^2\partial_x+x)} = \overline{\overline{1}\otimes(1+{\partial_x}^2-x)} = \overline{\overline{1}\otimes(1-x)}\in\Omega\otimes A^0$.

Consider the matrix M^{-1} representing the map $\tilde{F}^1(\Omega\otimes A^{-1}[\mathfrak{m}_{-1}]) \to \tilde{F}^1(\Omega\otimes A^0[\mathfrak{m}_0])$. The monomials $P_{j,\beta}$ from (1.5) in $\tilde{F}^1(\Omega\otimes A^{-1}[\mathfrak{m}_{-1}]) = \tilde{F}^1(\Omega\otimes {D_2}^2[0,0])$ are $(x,0)$, $(y,0)$, $(0,x)$, $(0,y)$, $(1,0)$, $(0,1)$. The monomials in $\tilde{F}^1(\Omega\otimes A^0[\mathfrak{m}_0]) = \tilde{F}^1(\Omega\otimes D_2[0])$ are x, y and 1. Then one can check that with these bases $M^{-1} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. We have already ordered the columns of this matrix by the $\tilde{V}_2$-degree of x, y and 1 (so that the last column, corresponding to the operator 1, is the least important one). The Buchberger algorithm (in this case equivalent to Gauss row reduction) applied to this matrix yields a standard basis $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$. This allows the reduction of $P = 1 - x$ corresponding to $(-1, 0, 1)$ to the element $(0, 0, 1)$ representing 1. Thus $1 \otimes (1 - x)$ and $1 \otimes 1$ represent the same cohomology class but we prefer the latter as representative because it has smallest degree in its class.

That Algorithm 2.7 works not only in our example but in general is easily seen from the fact that

$$\tilde{F}^{k_1}(\Omega \otimes A^\bullet[\mathfrak{m}_\bullet]) \hookrightarrow \tilde{F}^{k'_1}(\Omega \otimes A^\bullet[\mathfrak{m}_\bullet])$$

is a quasi-isomorphism.

2.7. At this point we have explained each step of Example 2.1 with the exception of how to obtain elements of $\Omega^i \otimes A^0$ that map onto given elements of $\Omega^i \otimes R_n[f^{-1}]$, a crucial step of the Transfer mechanism.

The question reduces to the following: given is $R_n[f^{-1}]$ as a left D_n-module via the presentation $R_n[f^{-1}] = D_n \bullet (1/f)$; for $g/f^k \in R_n[f^{-1}]$, find $P \in D_n$ such that $P \bullet (1/f) = g/f^k$. This also addresses the problem of finding elements in $\Omega^i \otimes A^\bullet$ that project onto a given sum of elements of the form $\frac{g}{f^k} dx_{s_1} \wedge \ldots \wedge dx_{s_i}$ since we can produce lifts componentwise.

The answer lies in the Bernstein operator associated to f introduced in Section 1. Let $P_f(s) \bullet f^{s+1} = b_f(s) f^s$. The construction of such $P_f(s)$ is performed as follows. As in [15] find $J^\Delta(f^s) = \{P \in D_n[s] | P(s) \bullet f^s = 0\}$. Then compute $(J^\Delta(f^s) + D_n[s]f) \cap \mathbb{C}[s] = \mathbb{C}[s] \cdot b_f(s)$. The computation of the intersection involves finding a relation $Q(s) + P(s)f = b_f(s) \in D_n[s]$ where $Q(s) \in J^\Delta(f^s)$. $P(s)$ is the desired operator.

Since $1/f$ generates $R_n[f^{-1}]$, $b_f(s)$ is nonzero for negative integers. Hence we have

$$g \frac{P_f(1-k) \cdot \ldots \cdot P_f(-1)}{b_f(1-k) \cdot \ldots \cdot b_f(-1)} \bullet \left(\frac{1}{f}\right) = \frac{g}{f^k}.$$

We obtain thus

Algorithm 2.9 (The ring structure of a hypersurface complement).
INPUT: $f \in R_n$.
OUTPUT: A multiplication table for $H^\bullet_{dR}(X \setminus \mathrm{Var}(f); \mathbb{C})$.
Begin

1. Replacing, if necessary, f by a power of itself find a presentation $R_n[f^{-1}] = D_n \bullet (\frac{1}{f}) \cong D_n/I_0$.
2. Compute a $\tilde{V}_n$-strict resolution $(A^\bullet[\mathfrak{m}_\bullet], \alpha^\bullet)$ of the D_n-module D_n/I_0 by finitely generated free D_n-modules where D_n/I_0 is positioned in cohomological degree 0.
3. Write down the double complex $\Omega^\bullet \otimes A^\bullet[\mathfrak{m}_\bullet]$.
4. Compute the b-function $b(s)$ of D_n/I_0 for integration along $\partial_1, \dots, \partial_n$.
5. Let $k_1 \in \mathbb{Z}$ be the largest integral root of $b(s)$. Then
$$\tilde{F}^{k_1}(\Omega \otimes A^\bullet[\mathfrak{m}_\bullet]) \xrightarrow{\cong} \Omega \otimes A^\bullet[\mathfrak{m}_\bullet]$$
(and $\tilde{F}^{k_1}(\Omega \otimes A^\bullet[\mathfrak{m}_\bullet])$ is a complex of finite dimensional vector spaces over $\mathbb{C}$, which is true for all k_1).
6. Compute generators $\overline{c_1}, \dots, \overline{c_t}$ for the cohomology of $\tilde{F}^{k_1}(\Omega(n) \otimes A^\bullet[\mathfrak{m}_\bullet])$ with $\overline{c_i} \in H^{d_i}(\tilde{F}^{k_1}(\Omega(n) \otimes A^\bullet[\mathfrak{m}_\bullet])) \cong H^{d_i}_{dR}(X \setminus Y; \mathbb{C})$.
7. For each i compute $\tau(\overline{c_i}) \in H^{d_i}(\Omega^\bullet \otimes D_n/I_0) \cong H^{d_i}(\Omega^\bullet \otimes R_n[f^{-1}])$. For all pairs i, j compute $\tau(\overline{c_i}) \cup \tau(\overline{c_j})$ and list the product $\overline{c'_{i,j}} \in H^{d_i+d_j}(\Omega^\bullet \otimes D_n/I_0)$.
8. Compute $\overline{c_{i,j}} = \tau^{-1}(\overline{c'_{i,j}})$ and if necessary use linear algebra (Algorithm 2.7) to convert the result into the coset of a representative of degree at most k_1. Express $\overline{c_{i,j}}$ in terms of the $\overline{c_i}$.
9. Establish the product relation $\overline{c_i} \cup \overline{c_j} = \overline{c_{i,j}}$.

End.

In Step 7, one picks a representative for $\tau(\overline{c_i}) \in H^i(\Omega^\bullet \otimes D_n/I_0)$ by an element in $\Omega^{d_i} \otimes D_n$ and applies it to $\frac{1}{f}$ as a generator of $R_n[f^{-1}]$ which gives the differential forms to be multiplied.

Example 2.10. Here we give an example based on *Macaulay 2*. Let $n = 3$, $R_3 = \mathbb{C}[x, y, z]$, $f = x^3 + y^3 + z^3$. Then one computes with *Macaulay 2* that $R_3[f^{-1}]$ is generated over $D_3 = \mathbb{C}\langle x, y, z, \partial_x, \partial_y, \partial_z\rangle$ by f^{-2}. In fact the entire set of commands is the following

```
load "Dloadfile.m2"
R = QQ[x,y,z]
f=x^3+y^3+z^3
deRhamAll(f)
```

Macaulay 2 computes then that

$$\begin{aligned} R_3[f^{-1}] \quad \cong \quad & D_3/D_3 \cdot (x\partial_x + y\partial_y + z\partial_z + 6, \\ & z^2\partial_y - y^2\partial_z, \\ & z^2\partial_x - x^2\partial_z, \\ & y^2\partial_x - x^2\partial_y, \\ & x^3\partial_z + y^3\partial_z + z^3\partial_z + 6z^2, \\ & x^3\partial_y + y^3\partial_y + y^2z\partial_z + 6y^2), \end{aligned}$$

and constructs a $\tilde{V}_3$-strict free resolution of $R_3[f^{-1}]$ of length 4 of the form

$$A^\bullet : {D_3}^8 \to {D_3}^{23} \to {D_3}^{27} \to {D_3}^{12} \to D_3 \to 0$$

and a b-function for integration of $R_3[f^{-1}]$ of $b(s) = s - 3$.

Truncating the complex $\Omega \otimes A^\bullet$ to forms of degree at most 3 one obtains a complex of vector spaces $\mathbb{C}^{138} \to \mathbb{C}^{495} \to \mathbb{C}^{641} \to \mathbb{C}^{294} \to \mathbb{C}^{20} \to 0$. From this complex *Macaulay 2* computes the following dimensions and generators of the de Rham cohomology of U:

Group	Dimension	Generators
$H^0_{dR}(U;\mathbb{C})$	1	$e := \frac{f^2}{f^2}$
$H^1_{dR}(U;\mathbb{C})$	1	$o := \frac{(x^2dx - y^2dy + z^2dz)f}{f^2}$
$H^2_{dR}(U;\mathbb{C})$	2	$t_1 := \frac{xyz(zdx\wedge dy + ydz\wedge dx + xdy\wedge dz)}{f^2}$ $t_2 := \frac{z^3(zdx\wedge dy + ydz\wedge dx + xdy\wedge dz)}{f^2}$
$H^3_{dR}(U;\mathbb{C})$	2	$d_1 := \frac{xyzdx\wedge dy\wedge dz}{f^2}$ $d_2 := \frac{z^3dx\wedge dy\wedge dz}{f^2}$

From this output one can determine the products of the classes. By multiplying by hand one sees that $o \cup t_1 = d_1$, $o \cup t_2 = d_2$, $o \cup o = 0$ and e operates as the identity. Hence the cohomology ring is the quotient of the free $\mathbb{C}$-algebra on 3 generators o, t_1, t_2 subject to $o^2 = {t_1}^2 = {t_2}^2 = t_1t_2 = t_2t_1 = ot_1 - t_1o = ot_2 - t_2o = 0$.

Remark 2.11. In the previous example we did not need to go through Algorithm 2.7. In general this is will be necessary, but this step is not implemented yet in *Macaulay 2*.

3. V-strict resolutions for complexes

In [16] we gave an algorithm to construct for a given right bounded complex $(C^\bullet, \delta^\bullet)$ of finitely generated D_n-modules a V_n-strict complex of free D_n-modules that is quasi-isomorphic to $(C^\bullet, \delta^\bullet)$. This was accomplished by means of a Cartan-Eilenberg resolution constructed by splitting $C^\bullet$ into short exact sequences.

The purpose of this section is to outline an improvement on this algorithm. We show by example that the new method can be distinctly more efficient.

Definition 3.1. Let R be an associative ring with identity and $\mu_1, \ldots, \mu_r$ elements of a left R-module M. If $\bigoplus_{i=1}^r R \cdot e_{\mu_i}$ denotes the free left R-module on generators (*symbols*) $e_{\mu_1}, \ldots, e_{\mu_r}$ then the map $\bigoplus_{i=1}^r R\cdot e_{\mu_i} \to M$, $e_{\mu_i} \to \mu_i$, is called the *symbol map*.

Proposition 3.2. *Suppose $(C^\bullet, \delta^\bullet)$ is a right bounded complex of finitely generated left modules over the left Noetherian ring R which is associative* *with identity element 1. The following recipe produces a right bounded complex* $(A^\bullet, \alpha^\bullet)$ *of finitely generated free left* R*-modules that is quasi-isomorphic to* $(C^\bullet, \delta^\bullet)$ *together with a quasi-isomorphism* $\pi^\bullet : A^\bullet \xrightarrow{\cong} C^\bullet$.

1. *Let* b *be the largest index for which* $C^b \neq 0$. *Set* $A^i = 0$, $\alpha^i = 0$ *and* $\pi^i = 0$ *for* $i > b$.
2. *For* $i \leq b$ *set* $K^i =$ *a free* R*-module generated by symbols for a set of generators for* $\ker(\delta^i : C^i \to C^{i+1})$.
3. *For* $i \leq b$ *set* $J^i =$ *a free* R*-module generated by symbols for generators of the submodule* N^{i+1} *of* A^{i+1} *given by*

$$\ker\left(\alpha^{i+1} : A^{i+1} \to A^{i+2}\right) \cap \left((\pi^{i+1})^{-1}(\mathrm{im}(\delta^i : C^i \to C^{i+1}))\right). \tag{3.1}$$

4. *Set* $A^i = J^i \oplus K^i$.
5. *Define* $\alpha^i : A^i \to A^{i+1}$ *to be the zero map on* K^i *and the symbol map on* J^i.
6. *Define* π^i *to be the symbol map on* K^i.
7. *Let* π^i *on* J^i *be a lift into* C^i *for* $\pi^{i+1} \circ \alpha^i$, *that is to say, an* R*-linear map that makes the diagram*

$$\begin{tikzcd} C^i \arrow[r, "\delta^i"] & C^{i+1} \\ J^i \arrow[u, dotted, "\pi^i|_{J^i}"] \arrow[r, "\alpha^i|_{J^i}"] & K^{i+1} \oplus J^{i+1} \arrow[u, "\pi^{i+1}"'] \end{tikzcd}$$

   commutative.

Proof. The proof is straightforward. Noetherianness implies that all K^i, J^i are finitely generated. By definition, $\alpha^{i+1} \circ \alpha^i = 0$ since α^i maps into $\ker(\alpha^{i+1})$. By construction, π is a chain map. By definition, $\pi^i : K^i \twoheadrightarrow \ker(\delta^i)$ and $K^i \subseteq \ker(\alpha^i)$. Hence π is surjective on cohomology level. J^i surjects onto N^{i+1}, hence all $P \in \ker(\alpha^{i+1})$ with $\pi^{i+1}(P) \in \mathrm{im}(\delta^i)$ are in $\mathrm{im}(\alpha^i)$ so that π is injective on cohomology. □

The proposition, which is perhaps well known, allows the following construction relevant for us.

Corollary 3.3. *In the situation of the proposition, let* $R = D_n$. *We modify the construction as follows by inserting 3 steps:*

2.5. *For the generators of* K^i, $i \leq b$, *choose an arbitrary shift vector* $\mathfrak{m}_{K,i}$.
3.5. *Starting with* $i = b$ *and decreasing* i *arrange* J^i *to map its generators onto a* V_n*-strict reduced Gröbner basis for* N^{i+1} *inside* $A^{i+1} = J^{i+1} \oplus K^{i+1}[\mathfrak{m}_{J,i+1}, \mathfrak{m}_{K,i+1}]$.
4.5. *For each generator* $e_{\mu^i_j}$ *of* J^i *define the corresponding shift to be*

$$V_n \deg(\alpha^i(e_{\mu^i_j})[\mathfrak{m}_{J,i+1}, \mathfrak{m}_{K,i+1}]).$$

Then $(A^\bullet[\mathfrak{m}_\bullet], \alpha^\bullet)$ *is* V_n*-strict.*

Proof. It is clear that the resulting complex is V_n-adapted. In order to check V_n-strictness we then have to make sure that each element of V_n-degree say d in the image of α^i is realized as the image of an element of V_n-degree d. But this is guaranteed by the fact that J^i maps its generators onto a V_n-strict Gröbner basis for the image N^{i+1}. (Compare Section 2 in [11].) $\square$

We note that since kernels and inverse images are computable with Gröbner basis techniques, this gives an alternative method to compute V-strict resolutions for complexes.

Remark 3.4. Of course, changing Steps 3.5 and 4.5 to

3.5.' Starting with $i = b$ and decreasing i arrange J^i to map its generators onto a $\tilde{V}_n$-strict reduced Gröbner basis for N^{i+1} inside $A^{i+1} = J^{i+1} \oplus K^{i+1}[\mathfrak{m}_{J,i+1}, \mathfrak{m}_{K,i+1}]$.

4.5.' For each generator $e_{\mu_j^i}$ of J^i define the corresponding shift to be

$$\tilde{V}_n \deg(\alpha^i(e_{\mu_j^i})[\mathfrak{m}_{J,i+1}, \mathfrak{m}_{K,i+1}]).$$

leads to $\tilde{V}_n$-strict resolutions.

Remark 3.5. In [16], Example 3.21 we showed that the method for computing V_n-strict quasi-isomorphic free complexes given in [16] may produce unnaturally large complexes if the input is already a V_n-strict free resolution. It is clear that our new method (from Corollary 3.3) does not suffer from this flaw because the new algorithm will simply compute the same resolution as the given one (or perhaps a smaller one).

But also otherwise the new method appears to be better:

Example 3.6. Consider the complex

$$\check{C}^\bullet = \left(R_3[(xy)^{-1}] \oplus R_3[(xz)^{-1}] \to R_3[(xyz)^{-1}]\right)$$

where $R_3 = \mathbb{C}[x, y, z]$. This is the Mayer-Vietoris complex associated to the complement of the variety $\mathrm{Var}(xy, xz)$. Its Fourier image $C^\bullet$ is given by

$$(D_3/D_3 \cdot (x\partial_x, y\partial_y, z) \oplus D_3/D_3 \cdot (x\partial_x, y, z\partial_z) \to D_3/D_3 \cdot (x\partial_x, y\partial_y, z\partial_z)).$$

We compute a V_3-strict resolution for this complex using the method of the corollary.

Observation 1. $H^1(C^\bullet) = D_3/D_3 \cdot (x\partial_x, \partial_y, \partial_z)$, $H^0(C^\bullet) = D_3/(x\partial_x, y, z)$. $H^1(C^\bullet)$ is generated by the Fourier image of $\frac{1}{xyz}$ in C^1, $H^0(C^\bullet)$ by that of $(\frac{1}{x}, \frac{1}{x})$ in C^0. We conclude that $K^1 = D_3$, $K^0 = D_3$, $J^1 = 0$, $A^1 = D_3[0]$, $\pi^1(P) = P \mod D_3 \cdot (x\partial_x, \partial_y, \partial_z)$.

Observation 2. J^0 should map its generators onto a V_3-strict basis for $D_3 \cdot (-\partial_z, \partial_y, x\partial_x)$. These three elements already form a V_3-strict basis. So $J^0 = {D_3}^3$, α^0 is right multiplication by $(-\partial_z, \partial_y, x\partial_x, 0)^T$, $A^0 = {D_3}^3 \oplus D_3[1, 1, 0, 2]$, $\pi^0(P_1, P_2, P_3, P_4) = \overline{(-P_4 \cdot \partial_y, -P_4 \cdot \partial_z)} \in C^0$. (We will in Remark 4.4 explain the somewhat mysterious 2 in the shift for A^0.)

Observation 3. J^{-1} should map its generators onto a V_3-strict basis for $\ker({D_3}^3 \oplus D_3 \to D_3) \cap {\pi_0}^{-1}(0)$. The kernel is generated by the rows of

$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ \partial_y & \partial_z & 0 & 0 \\ 0 & x\partial_x & -\partial_y & 0 \\ x\partial_x & 0 & \partial_z & 0 \end{pmatrix}$$

and these rows form a V_3-strict basis. The preimage of $0 \subset C^0$ inside A^0 is generated by the rows of

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & x\partial_x \\ 0 & 0 & 0 & y \\ 0 & 0 & 0 & z \end{pmatrix}.$$

The intersection of this preimage with $\ker(\alpha^0)$ is generated by the rows of the matrix

$$\begin{pmatrix} \partial_y & \partial_z & 0 & 0 \\ 0 & x\partial_x & -\partial_y & 0 \\ x\partial_x & 0 & \partial_z & 0 \\ 0 & 0 & 0 & y \\ 0 & 0 & 0 & z \\ 0 & 0 & 0 & x\partial_x \end{pmatrix}.$$

The rows form a V_3-strict Gröbner basis. Hence $J^{-1} = {D_3}^6$, α^{-1} is given by right multiplication by the above 6×4-matrix and π^{-1} is the zero map of course. For the shift we find $A^{-1} = {D_3}^3 \oplus {D_3}^3[2,1,1,1,1,2]$.

Observation 5. J^{-2} must map onto $\ker(\alpha^{-1})$, J^{-3} onto $\ker(\alpha^{-2})$. This is because $K^i = C^i = 0$ for negative i.

Observation 6. A V_3-strict basis for N^{-1} is given by the rows of

$$\begin{pmatrix} x\partial_x & -\partial_z & -\partial_y & 0 & 0 & 0 \\ 0 & 0 & 0 & z & -y & 0 \\ 0 & 0 & 0 & x\partial_x & 0 & -y \\ 0 & 0 & 0 & 0 & x\partial_x & -z \end{pmatrix}.$$

Hence $J^{-2} = {D_3}^4$ and α^{-2} is given by the 4×6 matrix above. Then N^{-2} has a V_3-strict basis given by $(0, x\partial_x, -z, y)$ representing α^{-3} and $J^{-3} = D_3$ while $J^i = 0$ for $i < -3$. So $A^{-2} = D_3 \oplus {D_3}^3[2,0,1,1]$ and $A^{-3} = D_3[0]$.

So in this resolution we get "Betti" numbers $1,4,6,4,1$ and note that this is considerably better than $3,12,23,17,3$ as computed in Example 3.19 in [16].

Example 3.7. In a similar way one computes a $\tilde{V}_3$-strict resolution of $\check{C}^\bullet$ with matrices $M^0 = (-z, y, -\partial_x x, 0)^T$,

$$M^{-1} = \begin{pmatrix} y & z & 0 & 0 \\ 0 & -\partial_x x & -y & 0 \\ -\partial_x x & 0 & z & 0 \\ 0 & 0 & 0 & -\partial_y \\ 0 & 0 & 0 & -\partial_z \\ 0 & 0 & 0 & -\partial_x x \end{pmatrix},$$

$$M^{-2} = \begin{pmatrix} -\partial_x x & -z & -y & 0 & 0 & 0 \\ 0 & 0 & 0 & -\partial_z & \partial_y & 0 \\ 0 & 0 & 0 & -\partial_x x & 0 & \partial_y \\ 0 & 0 & 0 & 0 & -\partial_x x & \partial_z \end{pmatrix},$$

and $M^{-3} = (0, -\partial_x x, \partial_z, -\partial_y)$ with shift vectors $\mathfrak{m}_1 = [0]$, $\mathfrak{m}_0 = [1, 1, 0, 2]$, $\mathfrak{m}_{-1} = [2, 1, 1, 1, 1, 2]$, $\mathfrak{m}_{-2} = [2, 0, 1, 1]$ and $\mathfrak{m}_{-3} = [0]$.

4. The general case

In this section we generalize Algorithm 2.9 to complements U of general affine varieties Y in X, defined by $f_0 = \ldots = f_r = 0$. Let $(\check{C}^\bullet, \delta^\bullet)$ be the reduced Čech complex (1.1) to $f_0, \ldots, f_r$ and for a list $I = (\iota_0, \ldots, \iota_i)$ of integers (with repetition allowed) between 0 and r set $U_I = \bigcap_{j=0}^i U_{\iota_i}$. Recall that the de Rham cohomology of U is in a natural way the cohomology of the complex $\mathrm{Tot}^\bullet(\Omega^\bullet \otimes_{D_n} \check{C}^\bullet)$ (cf. [8, 16]).

A general element $\omega \in \Omega^i(\check{C}^j)$ in the algebraic Čech-de Rham complex on U relative to the cover $\bigcup_{\iota=0}^r U_\iota$ is characterized by the set of forms $\omega(U_I)$ on U_I where I runs over $(j+1)$-tuples of distinct integers between 0 and r. For bookkeeping purposes we shall extend the index set to *ordered* tuples. Then we write $\omega(U_I) = 0$ if I has repeated entries, and we set $\omega(U_I) = -\omega(U_{I'})$ if the strings I and I' differ by an elementary permutation.

The following theorem can be found for example in [2] (but also see our Remark 2.3).

Theorem 4.1 ([2], p.174, compare also Remark 2.3.). *If $\omega' \in \Omega^{j'}(\check{C}^{k'})$ and $\omega'' \in \Omega^{j''}(\check{C}^{k''})$ then one can define a bilinear product $\omega' \cup \omega'' \in \Omega^{j'+j''}(\check{C}^{k'+k''})$ as the form η for which*

$$\eta(U_{(\iota_0, \ldots, \iota_{k'+k''})}) = (-1)^{k' \cdot j''} \omega'(U_{(\iota_0, \ldots, \iota_{k'})}) \wedge \omega''(U_{(\iota_{k'}, \ldots, \iota_{k'+k''})}).$$

This product descends to a product on cohomology in $\Omega^\bullet(\check{C}^\bullet)$ and then agrees with the usual cup product in singular cohomology under the de Rham isomorphism. □

We will now establish an algorithm similar to Algorithm 2.9. We assume that we already computed $\check{C}^\bullet$ explicitly, which may be computed as follows:

1. find generators for $\check{C}^l$,
2. write $\check{C}^l = (D_n)^{a_l} / I_l$,

3. compute matrices over D_n that represent the differentials in $\check{C}^\bullet$ in this presentation of $\check{C}^\bullet$.

This is explained in detail in [15].

Let $(A^\bullet, \alpha^\bullet)$ be a free $\tilde{V}_n$-strict complex surjecting onto $(\check{C}^\bullet, \delta^\bullet)$ via the quasi-isomorphism π' constructed either as in our Section 3 or by the techniques of [16], Section 3. The induced maps in the double complexes $\Omega^\bullet \otimes A^\bullet$ and $\Omega^\bullet \otimes \check{C}^\bullet$ will be denoted by $\alpha^{i,j} = \Omega^i \otimes \alpha^j$, $\delta^{i,j} = \Omega^i \otimes \delta^j$ and $\varepsilon^{i,j} = (-1)^j \varepsilon^i \otimes A^j$ (resp. $(-1)^j \varepsilon^i \otimes \check{C}^j$).

In the remainder of this section we will explain the following algorithm.

Algorithm 4.2 (Cup products on Zariski-open sets in $\mathbb{C}^n$).
INPUT: $f_0, \dots, f_r \subseteq R_n$, $Y = \mathrm{Var}(f_0, \dots, f_r)$.
OUTPUT: The ring structure of $H^\bullet_{dR}(X \setminus Y; \mathbb{C})$.

Begin

1. Compute the reduced Čech complex $\check{C}^\bullet$ to $f_0, \dots, f_r$ as a complex of finitely generated D_n-modules, where $\bigoplus_{\iota=0}^r R_n[f_\iota{}^{-1}]$ is placed in cohomological degree 0 ([15]).
2. Compute a $\tilde{V}_n$-strict complex $(A^\bullet[\mathfrak{m}_\bullet], \alpha^\bullet)$ that is quasi-isomorphic to $\check{C}^\bullet$ using Corollary 3.3 or the recipe from [16], Section 3. In particular, $A^j = 0$ for $j \geq n + r$.
3. Write down the double complex $\Omega^\bullet \otimes A^\bullet[\mathfrak{m}_\bullet]$ and find the b-function $b(s)$ for integration of $A^\bullet[\mathfrak{m}_\bullet]$.
4. Let $k_1 = \max_{a \in \mathbb{Z}}\{b(a) = 0\}$. Then $\tilde{F}^{k_1}(\Omega \otimes A^\bullet[\mathfrak{m}_\bullet]) \cong \Omega \otimes A^\bullet[\mathfrak{m}_\bullet]$.
5. Compute generators $\overline{c_1}, \dots, \overline{c_t}$ for the cohomology of $\tilde{F}^{k_1}(\Omega(n) \otimes A^\bullet[\mathfrak{m}_\bullet])$ with $\overline{c_i} \in H^{d_i}(\tilde{F}^{k_1}(\Omega(n) \otimes A^\bullet[\mathfrak{m}_\bullet])) \cong H^{d_i}_{dR}(X \setminus Y; \mathbb{C})$.
6. For each i compute $\tau(\overline{c_i}) \in H^{d_i}(\mathrm{Tot}^\bullet(\Omega^\bullet \otimes \check{C}^\bullet))$. For all pairs i, j compute $\tau(\overline{c_i}) \cup \tau(\overline{c_j})$ and list the product $\overline{c'_{i,j}} \in H^{d_i + d_j}(\mathrm{Tot}^\bullet(\Omega^\bullet \otimes \check{C}^\bullet))$.
7. Compute $\overline{c_{i,j}} = \tau^{-1}(\overline{c'_{i,j}}) \in H^{d_i + d_j}(\Omega(n) \otimes A^\bullet[\mathfrak{m}_\bullet])$.
8. If necessary use linear algebra (Algorithm 2.7) to convert the result into a representative of $\tilde{V}_n$-degree at most k_1.
9. Establish the product rule $\overline{c_i} \cup \overline{c_j} = \overline{c_{i,j}}$.

End.

Steps 1-5 are nothing but Theorem 6.1. of [16]. The complex $\tilde{F}^{k_1}(\Omega(n) \otimes A^\bullet[\mathfrak{m}_\bullet])$ is always a complex of finite dimensional vector spaces over $\mathbb{C}$, no matter what our choice for k_1 is.

Most parts of Algorithm 4.2 are executable precisely the way they can be done in the hypersurface case. The new challenges that occur come from the process of lifting cohomology classes from $\mathrm{Tot}^\bullet(\Omega^\bullet \otimes \check{C}^\bullet)$ into $\mathrm{Tot}^\bullet(\Omega^\bullet \otimes A^\bullet)$.

4.1. Recall that $\check{C}^l \cong (D_n)^{a_l}/I_l$. Replacing the f_ι by powers of themselves if necessary we assume that $\check{C}^l = \bigoplus_{|I|=l+1} R_n[F_I{}^{-1}]$ is generated by elements of the form $c_I = F_I{}^{-1} \cdot \kappa_I$ where $\kappa_I = (\dots, 0, 1, 0, \dots)$. Let $c \in \check{C}^l$ be given, so c is a sum $c = \sum g_I \cdot \kappa_I$ with $g_I \in R_n[F_I{}^{-1}]$. Then using Bernstein operators (i.e., $P_I(s)$ with $P_I(s) \bullet F_I{}^{s+1} = b_I(s) F_I{}^s$ and $b_I(s) \in \mathbb{C}[s]$, compare

1.2) it is not difficult to see that one can find an operator $P \in (D_n)^{a_i}$ such that P projects onto c via application to the ${F_I}^{-1}$, similarly to the hypersurface case in Subsection 2.7. This shows how to find preimages under the surjection $\bigoplus_{|I|=l+1} D_n \twoheadrightarrow \check{C}^l$.

The situation is however slightly more complicated because if $1 \otimes c \in \ker(\Omega^j \otimes (\check{C}^l \to \check{C}^{l+1}))$ then we would like the corresponding $1 \otimes P$ to be in $\ker(\Omega^j \otimes (A^l \to A^{l+1}))$. This means of course that $P \in A^l$, $\alpha^l(P) = 0$ and $\pi'(P) = c$ since Ω^i is D_n-free. Here is how this can be done:

4.2. For $c \in \ker(\check{C}^l \to \check{C}^{l+1})$ let $(P \mod I_l) = c$ under the isomorphism $\check{C}^l = (D_n)^{a_l}/I_l$. By construction of $A^l[\mathfrak{m}_l]$ ([16], Section 3 or Proposition 3.2 and its corollary), one can then lift $(P \mod I_l)$ to $Q \in A^l$ such that $Q \in \ker(\alpha^l)$, and $\pi'(Q) = (P \mod I_l)$.

4.3. Let c, P be as in 4.1 but assume this time in addition that $c = (P \mod I_l) \in \operatorname{im}(\check{C}^{l-1} \to \check{C}^l)$. From the construction of $A^l[\mathfrak{m}_l]$ ([16], Section 3 or Proposition 3.2 and its corollary) one can read off $Q \in A^l$ with $\pi'(Q) = (P \mod I_l)$ and $Q \in \operatorname{im}(\alpha^{l-1})$.

The operator Q in both 4.2 and 4.3 can be found as follows.

- Case a, we used [16]:
  By construction, A^l is then the free module on
  1. symbols for a generating set for $\operatorname{im}(\check{C}^{l-1} \to \check{C}^l)$,
  2. symbols for a generating set for $H^l(\check{C}^\bullet)$,
  3. symbols for a generating set for $\operatorname{im}(\check{C}^l \to \check{C}^{l+1})$,

  plus another free module related to $\check{C}^j$ with $j > l$. Moreover, the free module spanned by the elements from 1. and 2. projects by construction onto the kernel of $\check{C}^l \to \check{C}^{l+1}$. Since $(P \mod I_l) \in \check{C}^l$ maps to zero in $\check{C}^{l+1} = (D_n)^{a_{l+1}}/I_{l+1}$, P is in the submodule of $(D_n)^{a_l}$ generated by the images under π' of the elements from 1., 2., and I_l. Reduce P modulo a Gröbner basis for this submodule to find a syzygy which can be used to find an element in the kernel of $A^l \to A^{l+1}$ projecting to the same coset as P modulo I_l.

  If we want to lift an image $(P \mod I_l) \in \operatorname{im}(\check{C}^{l-1} \to \check{C}^l)$ into $\operatorname{im}(A^{l-1} \to A^l)$ we reduce P modulo a Gröbner basis for I_l and the elements from 1. only.
- Case b, we used Section 3:
  K^l maps onto $\ker((D_n)^{a_l}/I_l \to (D_n)^{a_{l+1}}/I_{l+1})$. If $(P \mod I_l)$ is in this kernel, then reduce P modulo a Gröbner basis for $K^l + I_l \subseteq {D_n}^{a_l}$ to obtain the desired syzygy. If on the other hand $(P \mod I_l)$ is in $\operatorname{im}((D_n)^{a_{l-1}}/I_{l-1} \to (D_n)^{a_l}/I_l)$, reduce P modulo $I_l + \pi'(\alpha^{l-1}(J^{l-1}))$.

Since the Transfer Theorem allows us to turn cohomology classes of $\Omega(n) \otimes A^\bullet$ into de Rham cohomology classes and vice versa as in the case $r = 0$, the justification of Algorithm 4.2 is then as follows:

- Step 1 is explained in [15], Step 2 in [16] and Section 3.
- Step 3 is automatic.

- Step 4 is explained in [11, 16].
- Step 5 is linear algebra over the base field.
- Step 6 is the Transfer Theorem; the multiplications take place inside $\Omega^\bullet(\check{C}^\bullet)$ according to Theorem 4.1.
- Step 7 is the Transfer Theorem and Subsections 4.1, 4.2, 4.3 (for the computation of good lifts from $\check{C}^\bullet$ to $A^\bullet$).
- Step 8 is Algorithm 2.7.

Example 4.3. We now move on to compute the de Rham cohomology ring of the variety of Example 3.6. For that we need to tensor the resolution of Example 3.7 with $\Omega(3)$. Recall that appropriate shifts are: $\mathfrak{m}_1 = [0]$, $\mathfrak{m}_0 = [1,1,0,2]$, $\mathfrak{m}_{-1} = [2,1,1,1,1,2]$, $\mathfrak{m}_{-2} = [2,0,1,1]$ and $\mathfrak{m}_{-3} = [0]$. The fact that there was some choice involved warrants the following

Remark 4.4. The shift of K^i in Corollary 3.3 can be chosen arbitrarily for all i since $\alpha^i(K^i) = 0$. However, in the case where $\check{C}^\bullet$ is a Čech complex that comes from homogeneous polynomials, there is a canonical degree associated to all elements of $\check{C}^i$ for all i as we explain now. $R_n[(f_0 \cdot \ldots \cdot f_r)^{-1}]$ is generated by $(f_0 \cdot \ldots \cdot f_r)^{-a}$ for some integer a. As soon as we fix the $\tilde{V}_n$-degree of this generator all elements in the Čech complex inherit a natural $\tilde{V}_n$-degree via the inclusion $R_n[{F_I}^{-1}] \hookrightarrow R_n[(f_0 \cdot \ldots \cdot f_r)^{-1}]$. Choosing this natural degree in a sense minimizes the b-function of the entire complex, and hence the truncated integration complex. This is because the b-functions for restriction or integration of submodules of the module M (equipped with the filtration inherited from M) divide the corresponding b-function of M (compare Remark 3.5 of [16]).

If not all f_ι are homogeneous, we can still pull back the V_n-degree from $\check{C}^r$ to $R_n[{F_I}^{-1}]$, but the effects may not be so drastic as in the homogeneous case. Choosing this natural shift has good effects on the complexity.

In our example this natural degree is 2 for the generator of K^0 as can easily be seen form the fact that $\ker(\check{C}^0 \to \check{C}^1)$ is generated by $(\frac{1}{x}, \frac{1}{x})$ and $\check{C}^1$ is generated by $\frac{1}{xyz}$.

To compute the b-function of this complex we first need to identify generators for the cohomology of the complex. These are by construction given by the generators of K^1 and K^0. K^1 generates a module isomorphic to $D_3/D_3 \cdot (\partial_x x, y, z)[0]$ which has a b-function equal to $b_1(s) = s$. K^0 generates a module isomorphic to $D_3/D_3 \cdot (\partial_x x, \partial_y, \partial_z)[2]$ with a b-function $b_0(s) = s$ as well, since the b-function of $D_3/D_3 \cdot (\partial_x x, \partial_y, \partial_z)[0]$ is $s+2$. Hence the b-function for the complex $(A^\bullet[\mathfrak{m}_\bullet], \alpha^\bullet)$ is $b(s) = s$ and each cohomology class of its integration will live in degree 0.

We find $H^0_{dR}(U;\mathbb{C}) = \mathbb{C}$ generated by $\overline{1} \otimes 1 \in \Omega(3) \otimes A^{-3}$, $H^1_{dR}(U;\mathbb{C}) = \mathbb{C}$ generated by $\overline{1} \otimes (0,1,0,0) \in \Omega(3) \otimes A^{-2}$, $H^3_{dR}(U;\mathbb{C}) = \mathbb{C}$ generated by $\overline{1} \otimes (0,0,1,0) \in \Omega(3) \otimes A^0$ and $H^4_{dR}(U;\mathbb{C}) = \mathbb{C}$ generated by $\overline{1} \otimes 1 \in \Omega(3) \otimes A^1$. We shall now compute the product of the two generators in degrees 1 and 3. We compute first the transfer of the class of $\overline{1} \otimes (0,1,0,0)$

in $\Omega^3 \otimes A^{-2}$ representing $H^1_{dR}(U;\mathbb{C})$. $\alpha^{3,-2}(1 \otimes (0,1,0,0)dx\,dy\,dz) = 1 \otimes (0,0,0,-\partial_z,\partial_y,0)dx\,dy\,dz \in \Omega^3 \otimes A^{-1}$ which lifts to $1\otimes(0,0,0,0,0,0)dy\,dz + 1 \otimes (0,0,0,0,-1,0)dx\,dz + 1 \otimes (0,0,0,-1,0,0)dy\,dz$ (where as before we use $dx\,dz$ etc. to distinguish the summands in Ω^2. This maps under $\alpha^{2,-1}$ to $(0,0,0,0)dy\,dz + (0,0,0,\partial_z)dx\,dz + (0,0,0,\partial_y)dx\,dy$ in $\Omega^2 \otimes A^0$. Pulling back to $\Omega^1 \otimes A^0$ yields $(0,0,0,1)dx + (0,0,0,0)dy + (0,0,0,0)dz$. Application of $\alpha^{1,0}$ results in zero. Thus starting with $\overline{a}$ = the class of $\overline{1} \otimes (0,1,0,0)$ we obtain $a^{3,-2} = 1 \otimes (0,1,0,0)dx\,dy\,dz$, $a^{2,-1} = 1 \otimes (0,0,0,0,-1,0)dxdz + 1 \otimes (0,0,0,-1,0,0)dydz$, $a^{1,0} = (0,0,0,1)dx$ and $a^{0,1} = 0$. Applying $(a^{3,-2} - a^{2,-1} + a^{1,0} - a^{0,1})$ to the generators 0, 0, $(\frac{1}{x},\frac{1}{x})$ and $\frac{1}{xyz}$ of $\pi(K^\bullet)$ we obtain $(\frac{dx}{x},\frac{dx}{x}) \in H^1_{dR}(U;\mathbb{C})$.

Similarly, we see that $\overline{1} \otimes (0,0,1,0)dx\,dy\,dz \in \Omega^3 \otimes A^0$ becomes the form $\frac{-x}{xyz}dy\,dz \in \Omega^2(\check{C}^1)$. Since $(\frac{dx}{x},\frac{dx}{x}) \cup \frac{dydz}{yz} = 2\frac{dxdydz}{xyz}$, $H^1_{dR}(U;\mathbb{C}) \cup H^3_{dR}(U;\mathbb{C}) = H^4_{dR}(U;\mathbb{C})$.

Remark 4.5. If in the above example we choose the shift for K^0 to be $d \in \mathbb{Z}$, then the b-function of $A^\bullet$ is $s(s+2-d)$, much to our disadvantage for $d \neq 2$. In general, if $\check{C}^\bullet$ comes from homogeneous polynomials, its b-function for integration is $s + n - \sum_{\iota=0}^r \deg(f_\iota)\cdot a_i$ if $\prod_{\iota=0}^r {f_\iota}^{-a_\iota}$ is the generator for $R_n[(f_0 \cdot \ldots \cdot f_r)^{-1}]$ provided we choose the natural shifts. (This is because this is the only nontrivial divisor of the b-function for integration of $\check{C}^r$.)

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Mathematical Sciences Research Institute, Berkeley, CA, 94720
E-mail address: `walther@msri.org`