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KOSZUL AND LOCAL COHOMOLOGY, AND A QUESTION OF DUTTA

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ABSTRACT. For $(A,\mathfrak{m})$ a local ring, we study the natural map from the Koszul cohomology module $H^{\dim A}(\mathfrak{m};A)$ to the local cohomology module $H^{\dim A}_{\mathfrak{m}}(A)$. We prove that the injectivity of this map characterizes the Cohen-Macaulay property of the ring A. We also answer a question of Dutta by constructing normal rings A for which this map is zero.

1. INTRODUCTION

For a commutative Noetherian local ring $(A,\mathfrak{m})$, we study the natural map from the Koszul cohomology module $H^{\dim A}(\mathfrak{m};A)$ to the local cohomology module $H^{\dim A}_{\mathfrak{m}}(A)$, and use this to answer a question raised by Dutta [Du], Question 1.1 below. The motivation for Dutta's question stems from Hochster's *monomial conjecture* [Ho1, page 33] that occupies a central place in local algebra; this is the conjecture that if $\boldsymbol{z} := z_1,\dots,z_n$ form a system of parameters for a local ring A, then for each $t\in\mathbb{N}$ one has

$$(z_1\cdots z_n)^t \;\notin\; (z_1^{t+1},\dots,z_n^{t+1})A.$$

An equivalent formulation of the conjecture is that the natural map

$$\varphi^n_{\boldsymbol{z}}\colon H^n(\boldsymbol{z};A)\longrightarrow H^n_{\boldsymbol{z}A}(A),$$

as discussed in §2, is nonzero. The monomial conjecture was proved for rings containing a field by Hochster, in the same paper where it was first formulated. The case of rings of dimension at most two is straightforward; for decades, the conjecture remained unresolved for mixed characteristic rings of dimension greater than or equal to three, as did its equivalent formulations, *the direct summand conjecture*, *the canonical element conjecture*, and the *improved new intersection conjecture*. In [He] Heitmann proved these equivalent conjectures for mixed characteristic rings of dimension three; more recently, André [An] settled the mixed characteristic case in full generality, with Bhatt [Bh] establishing a derived variant. Related homological conjectures including *Auslander's zerodivisor conjecture* and *Bass's conjecture* had been settled earlier by Roberts [Ro].

For the setup of Dutta's question, let A be a complete local ring. Using the Cohen structure theorem, A can be written as the homomorphic image of a complete regular local ring; this surjection may be factored so as to obtain a complete Gorenstein local ring R, such that A is the homomorphic image of R, and $\dim R = \dim A$. Dutta [Du, page 50] asked:

Question 1.1. Let A be a complete normal local ring. Let $(R,\mathfrak{m})$ be a Gorenstein ring with a surjective homomorphism $R\twoheadrightarrow A$, such that $n:=\dim R=\dim A$. Does the natural map

$$\operatorname{Ext}^n_R(R/\mathfrak{m},A)\longrightarrow H^n_{\mathfrak{m}}(A) \tag{1.1.1}$$

have a nonzero image?

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We prove that the answer to the above is negative in the following strong sense: we construct a complete normal local ring A such that for *each* Gorenstein ring R with $R \twoheadrightarrow A$ and $\dim R = \dim A$, the map (1.1.1) is zero. Our approach is via studying a related question on maps from Koszul cohomology to local cohomology: Let $\boldsymbol{z} := z_1, \dots, z_t$ be elements of R that generate an $\mathfrak{m}$-primary ideal. Let $P_\bullet$ be a projective resolution of $R/\mathfrak{m}$ as an R-module. The canonical surjection $R/\boldsymbol{z}R \twoheadrightarrow R/\mathfrak{m}$ lifts to a map of complexes

$$K_\bullet(\boldsymbol{z}; R) \longrightarrow P_\bullet,$$

where $K_\bullet(\boldsymbol{z}; R)$ denotes the homological Koszul complex. Applying $\operatorname{Hom}_R(-, A)$ to the above and taking cohomology, one obtains the map

$$\operatorname{Ext}^n_R(R/\mathfrak{m}, A) \longrightarrow H^n(\boldsymbol{z}; A), \tag{1.1.2}$$

where $H^n(\boldsymbol{z}; A)$ denotes Koszul cohomology. The map (1.1.1) factors as a composition of (1.1.2) and the natural map from Koszul cohomology to local cohomology

$$H^n(\boldsymbol{z}; A) \longrightarrow H^n_{\mathfrak{m}}(A); \tag{1.1.3}$$

the map above is described explicitly in §2. Note that in (1.1.3), the ring R no longer plays a role: the elements $\boldsymbol{z}$ may be replaced by their images in A; likewise, the maximal ideal of R may be replaced by that of A. In Theorem 3.2 we construct normal graded rings A for which the map (1.1.3) is zero; localizing at the homogeneous maximal ideal and taking the completion, one obtains examples where the answer to Question 1.1 is negative.

Quite generally, for $(A, \mathfrak{m})$ a local ring and $n := \dim A$, we prove that the injectivity of the natural map $H^n(\mathfrak{a}; A) \longrightarrow H^n_{\mathfrak{a}}(A)$ for some (or each) $\mathfrak{m}$-primary $\mathfrak{a}$ is equivalent to the ring A being Cohen-Macaulay, Theorem 3.1; here, and in the sequel, we use $K^\bullet(\mathfrak{a}; A)$ to denote the cohomological Koszul complex on a *minimal* set of generators for an ideal $\mathfrak{a}$, and $H^\bullet(\mathfrak{a}; A)$ for its cohomology. In §2 we record definitions and preliminary material. While §3 is largely devoted to the injectivity of the map $H^n(\mathfrak{m}; A) \longrightarrow H^n_{\mathfrak{m}}(A)$, §4 investigates the nonvanishing and the kernel. Theorem 4.2 records a case where we obtain precise information on the kernel of the map $H^n(\mathfrak{m}; A) \longrightarrow H^n_{\mathfrak{m}}(A)$, that we then illustrate with several examples, including some involving Stanley-Reisner rings, §5.

All rings under consideration in this paper are Noetherian; by a *local* ring $(A, \mathfrak{m})$, we mean a Noetherian ring A with a unique maximal ideal $\mathfrak{m}$.

2. Graded Koszul and local cohomology, and limit closure

We record some preliminaries on Koszul and local cohomology; the discussion below is in the graded context, in the form used in the proof of Theorems 3.2. Ignoring the grading and degree shifts, one has similar statements outside of the graded setting.

Let A be an $\mathbb{N}$-graded ring, and z a homogeneous ring element. Then one has a degree-preserving map from the Koszul complex $K^\bullet(z; A)$ to the Čech complex $C^\bullet(z; A)$ as below:

$$\begin{array}{ccccccc} 0 & \longrightarrow & A & \xrightarrow{z} & A(\deg z) & \longrightarrow & 0 \\ & & {\scriptstyle 1}\big\downarrow & & {\scriptstyle \frac{1}{z}}\big\downarrow & & \\ 0 & \longrightarrow & A & \longrightarrow & A_z & \longrightarrow & 0. \end{array}$$

For a sequence of homogeneous elements $\boldsymbol{z} := z_1, \dots, z_t$, one similarly has

$$K^\bullet(\boldsymbol{z}; A) := \bigotimes_i K^\bullet(z_i; A) \longrightarrow \bigotimes_i C^\bullet(z_i; A) =: C^\bullet(\boldsymbol{z}; A).$$

For each $m \geqslant 0$, the induced map from Koszul cohomology to local cohomology modules

$$\varphi^m_{\boldsymbol{z}} \colon H^m(\boldsymbol{z}; A) \longrightarrow H^m_{\boldsymbol{z}A}(A)$$

is degree-preserving, and what we refer to as the *natural* map. For homogeneous elements $\boldsymbol{z}$ and w in A, one has a commutative diagram with degree-preserving maps and exacts rows:
(2.0.1)

$$\begin{array}{ccccccccc}
\longrightarrow & H^{m-1}(\boldsymbol{z};A) & \xrightarrow{\pm w} & H^{m-1}(\boldsymbol{z};A)(\deg w) & \longrightarrow & H^m(\boldsymbol{z},w;A) & \longrightarrow & H^m(\boldsymbol{z};A) & \longrightarrow \\
& \Big\downarrow{\scriptstyle \varphi_{\boldsymbol{z}}^{m-1}} & & \Big\downarrow{\scriptstyle \pm\frac{1}{w}\varphi_{\boldsymbol{z}}^{m-1}} & & \Big\downarrow{\scriptstyle \varphi_{\boldsymbol{z},w}^{m}} & & \Big\downarrow{\scriptstyle \varphi_{\boldsymbol{z}}^{m}} & \\
\longrightarrow & H^{m-1}_{(\boldsymbol{z})}(A) & \longrightarrow & H^{m-1}_{(\boldsymbol{z})}(A_w) & \longrightarrow & H^m_{(\boldsymbol{z},w)}(A) & \longrightarrow & H^m_{(\boldsymbol{z})}(A) & \longrightarrow
\end{array}$$

Set $n := \dim A$, and fix a homogeneous system of parameters $\boldsymbol{z} := z_1,\dots,z_n$ for A. The map $\varphi_{\boldsymbol{z}}^n \colon H^n(\boldsymbol{z};A) \longrightarrow H^n_{\boldsymbol{z}A}(A)$ then takes form

$$\frac{A}{\boldsymbol{z}A}\Big(\sum \deg z_i\Big) \longrightarrow \frac{A_{z_1\cdots z_n}}{\sum A_{z_1\cdots\hat{z_i}\cdots z_n}}, \qquad 1 \longmapsto \left[\frac{1}{z_1\cdots z_n}\right]. \tag{2.0.2}$$

Following [HS, §2.5], the *limit closure* of the parameter ideal $\boldsymbol{z}A$ is the ideal

$$(\boldsymbol{z}A)^{\lim} := \{x \in A \mid xz_1^j\cdots z_n^j \in (z_1^{j+1},\dots,z_n^{j+1})A \ \text{ for } j \gg 0\}.$$

Using (2.0.2), it is readily seen that $(\boldsymbol{z}A)^{\lim}/\boldsymbol{z}A$ is the kernel of $\varphi_{\boldsymbol{z}}^n \colon H^n(\boldsymbol{z};A) \longrightarrow H^n_{\boldsymbol{z}A}(A)$. It follows that $(\boldsymbol{z}A)^{\lim}$ is unchanged if one takes a different choice of *minimal* generators for the ideal $\boldsymbol{z}A$. When $\boldsymbol{z}$ is a regular sequence, it is easily checked that $(\boldsymbol{z}A)^{\lim} = \boldsymbol{z}A$.

Example 2.1. Consider $A := \mathbb{R}[x,y,ix,iy]$, which is a subring of the polynomial ring $\mathbb{C}[x,y]$. Then A is a standard graded ring, with homogeneous system of parameters x,y. Since

$$ix(xy) = iy(x^2) \ \in \ (x^2,y^2),$$

one has $ix \in (x,y)^{\lim}$. Similarly, $iy \in (x,y)^{\lim}$, so $(x,y)^{\lim}$ equals the homogeneous maximal ideal $\mathfrak{m}$ of A. The map $H^2(\mathfrak{m};A) \longrightarrow H^2_{\mathfrak{m}}(A)$ is zero by an argument similar to the one used in the proof of Theorem 3.2; alternatively, see Example 5.2.

The ring in the example above is not normal; this leads to:

Question 2.2. Does there exist a non-regular complete normal local ring $(A,\mathfrak{m})$, with a system of parameters $\boldsymbol{z}$, such that $(\boldsymbol{z}A)^{\lim} = \mathfrak{m}$?

For a ring A that meets the conditions above, the answer to Question 1.1 is negative: let $\boldsymbol{z} := z_1,\dots,z_n$ be a system of parameters with $(\boldsymbol{z}A)^{\lim} = \mathfrak{m}$, and take R as in Question 1.1. The map $\operatorname{Ext}^n_R(R/\mathfrak{m}_R,A) \longrightarrow H^n_{\mathfrak{m}}(A)$ factors through $H^n(\boldsymbol{z};A) = A/\boldsymbol{z}A$. As $\operatorname{Ext}^n_R(R/\mathfrak{m}_R,A)$ is annihilated by $\mathfrak{m}$, so is its image in $A/\boldsymbol{z}A$. Hence this image is a submodule of

$$0 :_{A/\boldsymbol{z}A} \mathfrak{m} \ \subseteq \ \mathfrak{m}/\boldsymbol{z}A,$$

where the containment above holds since A is not regular. Since $H^n(\boldsymbol{z};A) \longrightarrow H^n_{\mathfrak{m}}(A)$ has kernel $(\boldsymbol{z}A)^{\lim}/\boldsymbol{z}A = \mathfrak{m}/\boldsymbol{z}A$, it follows that $\operatorname{Ext}^n_R(R/\mathfrak{m}_R,A) \longrightarrow H^n_{\mathfrak{m}}(A)$ must be zero.

3. Injectivity of the map from Koszul to local cohomology

We begin with a characterization of the Cohen-Macaulay property:

Theorem 3.1. *Let $(A,\mathfrak{m})$ be a local ring; set $n := \dim A$. Then the following are equivalent:*

(1) *The ring A is Cohen-Macaulay.*
(2) *The natural map $H^n(\mathfrak{a};A) \longrightarrow H^n_{\mathfrak{a}}(A)$ is injective for each $\mathfrak{m}$-primary ideal $\mathfrak{a}$ of A.*
(3) *The natural map $H^n(\mathfrak{a};A) \longrightarrow H^n_{\mathfrak{a}}(A)$ is injective for some $\mathfrak{m}$-primary ideal $\mathfrak{a}$ of A.*

Proof. Suppose $\mathfrak{a}$ is an $\mathfrak{m}$-primary ideal. We claim that there exists a minimal generating set $z_1,\dots,z_t$ for $\mathfrak{a}$ such that $z_1,\dots,z_n$ is a system of parameters. For $i<n$, it suffices to choose an element z_{i+1}, not in any minimal prime of $(z_1,\dots,z_i)A$, such that

$$z_{i+1}\in\mathfrak{a}\setminus\mathfrak{m}\mathfrak{a}.$$

This may be accomplished using the version of prime avoidance where up to two of the ideals need not be prime, see for example [Ka, Theorem 81].

Assume (1). Suppose $\boldsymbol{z}$ generates an $\mathfrak{m}$-primary ideal, and $w\in\mathfrak{m}$ is an additional element. Using (2.0.1), the vanishing of $H^{n-1}(\boldsymbol{z};A)$ implies that the map

$$H^n(\boldsymbol{z},w;A)\longrightarrow H^n(\boldsymbol{z};A)$$

is injective. Thus, if the map $\varphi^n_{\boldsymbol{z}}\colon H^n(\boldsymbol{z};A)\longrightarrow H^n_{\boldsymbol{z}A}(A)=H^n_{\mathfrak{m}}(A)$ is injective, then so is the map $\varphi^n_{\boldsymbol{z},w}\colon H^n(\boldsymbol{z},w;A)\longrightarrow H^n_{(\boldsymbol{z},w)A}(A)=H^n_{\mathfrak{m}}(A)$. Hence the proof of (2) reduces to the case where $\mathfrak{a}$ is generated by a system of parameters $\boldsymbol{z}$. But then $\boldsymbol{z}$ is a regular sequence, and the injectivity follows.

It is immediate that (2) implies (3). Next, assume (3), i.e., that $\mathfrak{a}$ is an $\mathfrak{m}$-primary ideal and that $H^n(\mathfrak{a};A)\longrightarrow H^n_{\mathfrak{a}}(A)$ is injective. We first consider the case where $\mathfrak{a}$ is generated by a system of parameters $\boldsymbol{z}$. In this case, the fact that $\boldsymbol{z}$ is a regular sequence on A follows from [CHL, Corollary 2.4], though one may also argue as follows: The injectivity translates as $(\boldsymbol{z}A)^{\lim}=\boldsymbol{z}A$. Using $e(\boldsymbol{z}A)$ to denote the multiplicity of the ideal $\boldsymbol{z}A$, one has

$$e(\boldsymbol{z}A)\leqslant\ell(A/\boldsymbol{z}A),$$

with equality holding precisely if A is Cohen-Macaulay, see [BH, Corollary 4.7.11]. But

$$\ell(A/(\boldsymbol{z}A)^{\lim})\leqslant e(\boldsymbol{z}A)$$

by [MQS, Theorem 9], so A is Cohen-Macaulay.

For the general case, take a minimal generating set $z_1,\dots,z_t$ for $\mathfrak{a}$ such that $z_1,\dots,z_n$ is a system of parameters. Suppose $n<t$. Using (2.0.1), one has a commutative diagram

$$\begin{tikzcd}
H^n(z_1,\dots,z_t;A) \arrow[r] \arrow[d, "\varphi^n_{\boldsymbol{z}}"'] & H^n(z_1,\dots,z_{t-1};A) \arrow[d] \\
H^n_{\mathfrak{a}}(A) \arrow[r, equal] & H^n_{\mathfrak{a}}(A)
\end{tikzcd}$$

Since $\varphi^n_{\boldsymbol{z}}$ is injective by assumption, it follows that

$$H^n(z_1,\dots,z_t;A)\longrightarrow H^n(z_1,\dots,z_{t-1};A)$$

is injective, and hence that multiplication by z_t on $H^{n-1}(z_1,\dots,z_{t-1};A)$ is surjective. But then, by Nakayama's lemma, $H^{n-1}(z_1,\dots,z_{t-1};A)=0$, so A is Cohen-Macaulay. $\square$

The next theorem provides a large class of rings for which the answer to Question 1.1 is negative; while the rings below are graded, the relevant issues are unchanged under localization and completion.

Theorem 3.2. *Let k be a field; take polynomial rings $k[x_1,\dots,x_b]$ and $C:=k[y_1,\dots,y_c]$, where $b\geqslant 3$ and $c\geqslant 2$. Let $f(\boldsymbol{x})$ be a homogeneous polynomial such that the hypersurface*

$$B:=k[x_1,\dots,x_b]/(f(\boldsymbol{x}))$$

is normal. Then the Segre product $A:=B\#C$ is a normal ring of dimension $b+c-2$.

Let $\mathfrak{m}$ denote the homogeneous maximal ideal of A. Then the following are equivalent:

(1) *The ring A is Cohen-Macaulay.*
(2) *The polynomial $f(\boldsymbol{x})$ has degree less than b.*

(3) *The natural map* $H^{b+c-2}(\mathfrak{m};A) \longrightarrow H^{b+c-2}_{\mathfrak{m}}(A)$ *is injective.*
(4) *The natural map* $H^{b+c-2}(\mathfrak{m};A) \longrightarrow H^{b+c-2}_{\mathfrak{m}}(A)$ *is nonzero.*

Proof. Regarding the normality of A, note that $B \otimes_k C$ is a polynomial ring over B, hence normal; it follows that its pure subring A is normal as well. The dimension of A and the equivalence of (1) and (2) may be obtained from the Künneth formula for local cohomology [GW, Theorem 4.1.5]. Note that (1) and (3) are equivalent by Theorem 3.1, and that (3) trivially implies (4).

Set $d := \deg f(\boldsymbol{x})$ and assume that $d \geqslant b$. To complete the proof, we show that the map

$$H^{b+c-2}(\mathfrak{m};A) \longrightarrow H^{b+c-2}_{\mathfrak{m}}(A)$$

is zero. Fix the polynomial ring

$$S := k[z_{ij} \mid 1 \leqslant i \leqslant b,\ 1 \leqslant j \leqslant c]$$

with the k-algebra surjection $\pi\colon S \twoheadrightarrow A$, where $z_{ij} \longmapsto x_i y_j$. We work with the standard $\mathbb{N}$-gradings on S and A, i.e., $\deg z_{ij} = 1 = \deg x_i y_j$. Note that the minimal generators for $\ker \pi$ have degree 2 and degree d, with the degree 2 generators being

$$z_{ij}z_{rs} - z_{is}z_{rj}.$$

Specifically,

$$A_t = [S/I_2(Z)]_t \qquad \text{for } t \leqslant d-1, \tag{3.2.1}$$

where $I_2(Z)$ is the ideal generated by the size 2 minors of the matrix $Z := (z_{ij})$.

The Künneth formula gives

$$H^{b+c-2}_{\mathfrak{m}}(A) = H^{b-1}_{\mathfrak{m}_B}(B) \# H^{c}_{\mathfrak{m}_C}(C).$$

Since $\left[H^c_{\mathfrak{m}_C}(C)\right]_j = 0$ for $j > -c$, it follows that

$$\left[H^{b+c-2}_{\mathfrak{m}}(A)\right]_j = 0 \qquad \text{for } j > -c.$$

The images of $\boldsymbol{z} := z_{11}, \dots, z_{bc}$ are minimal generators for $\mathfrak{m}$, so it suffices to show that

$$\left[H^{b+c-2}(\boldsymbol{z};A)\right]_j = 0 \qquad \text{for } j \leqslant -c.$$

Since $\deg z_{ij} = 1$ for each i, j, the Koszul complex $K^\bullet(\boldsymbol{z};A)$ has the form

$$0 \longrightarrow A \longrightarrow \bigoplus A(1) \longrightarrow \bigoplus A(2) \longrightarrow \bigoplus A(3) \longrightarrow \cdots.$$

Fix j with $j \leqslant -c$. The graded strand of the Koszul complex computing $\left[H^{b+c-2}(\boldsymbol{z};A)\right]_j$ is

$$\bigoplus A_{b+c-3+j} \xrightarrow{\ \alpha\ } \bigoplus A_{b+c-2+j} \xrightarrow{\ \beta\ } \bigoplus A_{b+c-1+j}, \tag{3.2.2}$$

where the nonzero entries of the matrices for α and β are linear forms in the z_{ij}. The condition $j \leqslant -c$ implies that $b+c-1+j \leqslant b-1 \leqslant d-1$. In light of (3.2.1), it follows that the cohomology of (3.2.2) coincides with that of

$$\bigoplus [S/I_2(Z)]_{b+c-3+j} \xrightarrow{\ \alpha\ } \bigoplus [S/I_2(Z)]_{b+c-2+j} \xrightarrow{\ \beta\ } \bigoplus [S/I_2(Z)]_{b+c-1+j}.$$

But the Koszul cohomology module $H^{b+c-2}(\boldsymbol{z};S/I_2(Z))$ is zero since $S/I_2(Z)$ is a Cohen-Macaulay ring of dimension $b+c-1$. $\square$

4. Nonvanishing of the map from Koszul to local cohomology

Let $(A,\mathfrak{m})$ be a local ring; set $n := \dim A$. Theorem 3.1 characterizes the injectivity of the map $\varphi^n_{\mathfrak{m}}\colon H^n(\mathfrak{m};A) \longrightarrow H^n_{\mathfrak{m}}(A)$. We next discuss when this map is nonzero.

A *canonical module* for a local ring $(A,\mathfrak{m})$ is a finitely generated A-module ω_A with

$$\operatorname{Hom}_A(\omega_A, E) \;\cong\; H^{\dim A}_{\mathfrak{m}}(A),$$

where E is the injective hull of the residue field $A/\mathfrak{m}$ in the category of A-modules. The canonical module of A—when it exists—is unique up to isomorphism. Suppose A is the homomorphic image of a Gorenstein local ring R. Then

$$\operatorname{Ext}^{\dim R-\dim A}_R(A, R)$$

is an A-module satisfying the Serre condition S_2, and is a canonical module for A.

A local ring A is said to be *quasi-Gorenstein* if it is the homomorphic image of a Gorenstein local ring, and ω_A is isomorphic to A. Using André's Theorem [An], one obtains:

Theorem 4.1. *Let $(A,\mathfrak{m})$ be a local ring that is a homomorphic image of a Gorenstein local ring. Set $n := \dim A$. Then the natural map $H^n(\mathfrak{m};\omega_A) \longrightarrow H^n_{\mathfrak{m}}(\omega_A)$ is nonzero. In particular, if A is quasi-Gorenstein, then the natural map $H^n(\mathfrak{m};A) \longrightarrow H^n_{\mathfrak{m}}(A)$ is nonzero.*

Proof. Since the direct summand conjecture is true by [An], the local ring A satisfies the canonical element property, see [Ho3, Theorem 2.8]. By [Ho3, Theorem 4.3], this is equivalent to the map $\operatorname{Ext}^n_A(A/\mathfrak{m}, \omega_A) \longrightarrow H^n_{\mathfrak{m}}(\omega_A)$ being nonzero. This map factors as

$$\operatorname{Ext}^n_A(A/\mathfrak{m}, \omega_A) \longrightarrow H^n(\mathfrak{m};\omega_A) \longrightarrow H^n_{\mathfrak{m}}(\omega_A),$$

implying that the map $H^n(\mathfrak{m};\omega_A) \longrightarrow H^n_{\mathfrak{m}}(\omega_A)$ is nonzero. □

The next theorem records an interesting case where we have substantial information on the kernel of the map $H^n(\mathfrak{m};A) \longrightarrow H^n_{\mathfrak{m}}(A)$.

Theorem 4.2. *Let A be a standard graded ring that is finitely generated over a field $k := A_0$. Set $n := \dim A$ and $e := \operatorname{edim} A$, and let $\mathfrak{m}$ denote the homogeneous maximal ideal of A. Suppose there exists an integer d such that for each integer j with $j < n$, one has*

$$H^j_{\mathfrak{m}}(A) \;=\; [H^j_{\mathfrak{m}}(A)]_d.$$

Set $s_j := \operatorname{rank}[H^j_{\mathfrak{m}}(A)]_d$. Then:

(1) *The kernel of the natural map $\varphi^n_{\mathfrak{m}}\colon H^n(\mathfrak{m};A) \longrightarrow H^n_{\mathfrak{m}}(A)$ has Hilbert series*

$$\sum_i \operatorname{rank}\big[\ker\varphi^n_{\mathfrak{m}}\big]_i T^i \;=\; s_{n-1}\binom{e}{1}T^{d-1} + s_{n-2}\binom{e}{2}T^{d-2} + \cdots + s_0\binom{e}{n}T^{d-n}.$$

(2) *If $\mathfrak{a}$ is an $\mathfrak{m}$-primary ideal that is minimally generated by r homogeneous elements, each of degree t, then the kernel of $H^n(\mathfrak{a};A) \longrightarrow H^n_{\mathfrak{m}}(A)$ has Hilbert series*

$$s_{n-1}\binom{r}{1}T^{d-t} + s_{n-2}\binom{r}{2}T^{d-2t} + \cdots + s_0\binom{r}{n}T^{d-nt}.$$

Proof. It suffices to prove the second assertion. Fix minimal generators $\boldsymbol{z}$ of $\mathfrak{a}$. The Koszul complex $K^\bullet := K^\bullet(\boldsymbol{z};A)$ takes the form

$$0 \longrightarrow A \longrightarrow A(t)^{\binom{r}{1}} \longrightarrow A(2t)^{\binom{r}{2}} \longrightarrow \cdots \longrightarrow A(rt)^{\binom{r}{r}} \longrightarrow 0,$$

and lives in cohomology degree $0,1,\dots,r$.

Let $\omega_A^\bullet$ be the graded normalized dualizing complex of A, in which case $\omega_A = H^{-n}(\omega_A^\bullet)$ is the graded canonical module of A. Set $(-)^\vee = \operatorname{Hom}_A(-, {}^*E)$, where ${}^*E$ is the injective hull of $A/\mathfrak{m}$ in the category of graded A-modules. Applying $R\operatorname{Hom}_A(K^\bullet, -)$ to the triangle

$$\omega_A[n] \longrightarrow \omega_A^\bullet \longrightarrow \tau_{>-n}\omega_A^\bullet \xrightarrow{+1}$$

gives

$$R\operatorname{Hom}_A(K^\bullet, \omega_A[n]) \longrightarrow R\operatorname{Hom}_A(K^\bullet, \omega_A^\bullet) \longrightarrow R\operatorname{Hom}_A(K^\bullet, \tau_{>-n}\omega_A^\bullet) \xrightarrow{+1} .$$

Since the complex $K^\bullet$ has Artinian cohomology, applying the functor $(-)^\vee$ gives

$$R\operatorname{Hom}_A(K^\bullet, \tau_{>-n}\omega_A^\bullet)^\vee \longrightarrow K^\bullet \longrightarrow R\operatorname{Hom}_A(K^\bullet, \omega_A[n])^\vee \xrightarrow{+1} .$$

This induces the exact sequence

$$0 \longrightarrow H^n(R\operatorname{Hom}_A(K^\bullet, \tau_{>-n}\omega_A^\bullet)^\vee) \longrightarrow H^n(K^\bullet) \longrightarrow H^n(R\operatorname{Hom}_A(K^\bullet, \omega_A[n])^\vee),$$

where the zero on the left is because $R\operatorname{Hom}_A(K^\bullet, \omega_A[n])^\vee$ lives in cohomological degrees $n, n+1, \dots, n+r$. The module $H^n(R\operatorname{Hom}_A(K^\bullet, \omega_A[n])^\vee)$ is the kernel of the map

$$H^n_\mathfrak{m}(A) \longrightarrow H^n_\mathfrak{m}(A)(t)^{\binom{r}{1}}$$

given by multiplication by z_i in the i-th coordinate. It follows that

$$H^n(R\operatorname{Hom}_A(K^\bullet, \omega_A[n])^\vee) = 0 :_{H^n_\mathfrak{m}(A)} \mathfrak{a}.$$

The map $H^n(K^\bullet) \longrightarrow H^n(R\operatorname{Hom}_A(K^\bullet, \omega_A[n])^\vee)$ may be identified naturally with

$$H^n(\mathfrak{a}; A) \longrightarrow 0 :_{H^n_\mathfrak{m}(A)} \mathfrak{a},$$

which has the same kernel as $H^n(\mathfrak{a}; A) \longrightarrow H^n_\mathfrak{m}(A)$ since $H^n(\mathfrak{a}; A)$ is annihilated by $\mathfrak{a}$. Summarizing, one has

$$H^n(R\operatorname{Hom}_A(K^\bullet, \tau_{>-n}\omega_A^\bullet)^\vee) = \ker\big(H^n(\mathfrak{a}; A) \longrightarrow H^n_\mathfrak{m}(A)\big).$$

The hypothesis that $H^j_\mathfrak{m}(A) = [H^j_\mathfrak{m}(A)]_d$ for each integer j with $j < n$ implies that R is Buchsbaum, see [Sch, Theorem 3.1]. Acknowledging the abuse of notation, we reuse the symbol k below for the residue field $A/\mathfrak{m}$. By [Sch, Theorem 2.3], $\tau_{>-n}\omega_A^\bullet$ is quasi-isomorphic to the complex

$$0 \longrightarrow k^{s_{n-1}}(d) \longrightarrow \cdots \longrightarrow k^{s_1}(d) \longrightarrow k^{s_0}(d) \longrightarrow 0$$

of graded k-vector spaces, each in degree $-d$, with zero differentials; note that k^{s_j} in the complex above has cohomology degree $-j$. Using this representative of $\tau_{>-n}\omega_A^\bullet$ to compute $R\operatorname{Hom}_A(K^\bullet, \tau_{>-n}\omega_A^\bullet)^\vee$, the corresponding double complex takes the form:

$$\begin{array}{ccccccccc}
 & & \uparrow & & \uparrow & & & & \uparrow & & \\
0 & \longrightarrow & k(2t-d)^{\binom{r}{2}s_0} & \longrightarrow & k(2t-d)^{\binom{r}{2}s_1} & \longrightarrow & \cdots & \longrightarrow & k(2t-d)^{\binom{r}{2}s_{n-1}} & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & & & \uparrow & & \\
0 & \longrightarrow & k(t-d)^{\binom{r}{1}s_0} & \longrightarrow & k(t-d)^{\binom{r}{1}s_1} & \longrightarrow & \cdots & \longrightarrow & k(t-d)^{\binom{r}{1}s_{n-1}} & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & & & \uparrow & & \\
0 & \longrightarrow & k(-d)^{s_0} & \longrightarrow & k(-d)^{s_1} & \longrightarrow & \cdots & \longrightarrow & k(-d)^{s_{n-1}} & \longrightarrow & 0
\end{array}$$

Note that all differentials are zero: the horizontal ones in light of the chosen representative for $\tau_{>-n}\omega_A^\bullet$, and the vertical ones since each column is a Koszul complex of the form $K^\bullet(\mathfrak{a}; k^{s_j})(-d)$. The bottom row sits in cohomology degree $0,1,\dots,n-1$ as it is the graded dual of $\tau_{>-n}\omega_A^\bullet$. Taking the total complex, one obtains

$$H^n(R\operatorname{Hom}_A(K^\bullet,\, \tau_{>-n}\omega_A^\bullet)^\vee) \;=\; k(t-d)^{\binom{r}{1}s_{n-1}} \oplus k(2t-d)^{\binom{r}{2}s_{n-2}} \oplus \cdots \oplus k(nt-d)^{\binom{r}{n}s_0}. \;\square$$

We illustrate the preceding results with a number of examples, beginning with an elementary example taken from [Du, Remark 3.9]:

Example 4.3. Let k be a field. Set $A := k[x,y]/(x^2,xy)$. Then $H^0_\mathfrak{m}(A) = xA$, which is a rank 1 vector space, concentrated in degree 1. By Theorem 4.2, $\ker\big(H^1(\mathfrak{m};A) \longrightarrow H^1_\mathfrak{m}(A)\big)$ has rank 2, and is concentrated in degree 0. This is confirmed by examining $K^\bullet(\mathfrak{m};A)$, i.e.,

$$0 \longrightarrow A \xrightarrow{\begin{pmatrix} x \\ y \end{pmatrix}} A^2(1) \xrightarrow{\begin{pmatrix} y & -x \end{pmatrix}} A(2) \longrightarrow 0,$$

to observe that $H^1(\mathfrak{m};A) \cong k^2$, with the generators corresponding to

$$\begin{pmatrix} 0 \\ x \end{pmatrix}, \quad \begin{pmatrix} 0 \\ y \end{pmatrix},$$

and that each of these maps to zero under $H^1(\mathfrak{m};A) \longrightarrow H^1_\mathfrak{m}(A)$.

For integers $t \geqslant 2$, Theorem 4.2 says that the kernel of $H^1(y^t;A) \longrightarrow H^1_\mathfrak{m}(A)$ has Hilbert series T^{1-t}. Indeed, this map is

$$A(t)/Ay^t \longrightarrow A_y/A,$$

with the kernel being the rank 1 vector space generated by $x \in [A(t)/Ay^t]_{1-t}$.

Lastly, note that the image of $H^1(y^t;A) \longrightarrow H^1_\mathfrak{m}(A)$ has Hilbert series

$$T^{-1} + T^{-2} + T^{-3} + \cdots + T^{-t}$$

which agrees with the Hilbert series of $H^1_\mathfrak{m}(A)$ as $t \longrightarrow \infty$.

Example 4.4. Let A be as in Theorem 3.2 where $f(\boldsymbol{x})$ has degree b. Then the ring A is not Cohen-Macaulay, and the Künneth formula gives

$$H^j_\mathfrak{m}(A) \;=\begin{cases} k & \text{if } j = b-1, \\ 0 & \text{if } j \neq b-1,\;\; b+c-2. \end{cases}$$

The map $H^{b+c-2}(\mathfrak{m};A) \longrightarrow H^{b+c-2}_\mathfrak{m}(A)$ is zero by Theorem 3.2, while Theorem 4.2 says that its kernel, i.e., $H^{b+c-2}(\mathfrak{m};A)$, has Hilbert series

$$\binom{bc}{c-1} T^{-(c-1)}.$$

Example 4.5. Let k be a field; take polynomial rings $k[x_1,\dots,x_b]$ and $C := k[y_1,\dots,y_c]$, where $b \geqslant 3$ and $c \geqslant 3$. Let $f(\boldsymbol{x})$ and $g(\boldsymbol{y})$ be homogeneous polynomials of degrees b and c respectively, such that the hypersurfaces

$$B := k[x_1,\dots,x_b]/(f(\boldsymbol{x})) \qquad \text{and} \qquad C := k[y_1,\dots,y_c]/(g(\boldsymbol{y}))$$

are normal. The ring $A := B\#C$ is normal, and of dimension $b+c-3$; let $\mathfrak{m}$ denote the homogeneous maximal ideal of A. Then

$$H^j_\mathfrak{m}(A) \;=\begin{cases} 0 & \text{if } j \neq b-1,\;\; c-1,\;\; \text{or } b+c-3, \\ k & \text{if } b \neq c, \text{ and } j \text{ equals either } b-1 \text{ or } c-1, \\ k^2 & \text{if } b = c \text{ and } j = b-1. \end{cases}$$

Since B and C are each Gorenstein with a-invariant 0, the ring A is quasi-Gorenstein by [GW, Theorem 4.3.1]. Hence Theorem 4.1 says that $H^{b+c-3}(\mathfrak{m};A) \longrightarrow H^{b+c-3}_{\mathfrak{m}}(A)$ is nonzero, while Theorem 4.2 implies that the kernel has Hilbert series

$$\binom{bc}{c-2}T^{-(c-2)} + \binom{bc}{b-2}T^{-(b-2)}.$$

We conclude this section with an example where A is not Cohen-Macaulay or quasi-Gorenstein, but the map $H^{\dim A}(\mathfrak{m};A) \longrightarrow H^{\dim A}_{\mathfrak{m}}(A)$ is nonzero:

Example 4.6. Let k be a field, and set $A := k[x,y,z]/(xz,y^2,yz,z^2)$. Then $\dim A = 1$, and x is a homogeneous parameter. The Koszul complex $K^\bullet(\mathfrak{m};A)$ is

$$0 \longrightarrow A \xrightarrow{\begin{pmatrix} x \\ y \\ z \end{pmatrix}} A^3(1) \xrightarrow{\begin{pmatrix} 0 & -z & y \\ -z & 0 & x \\ -y & x & 0 \end{pmatrix}} A^3(2) \xrightarrow{\begin{pmatrix} x & -y & z \end{pmatrix}} A(3) \longrightarrow 0,$$

from which it follows that $H^1(\mathfrak{m};A) \cong k^4$, with the four generators corresponding to

$$\begin{pmatrix} z \\ 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ z \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \\ z \end{pmatrix}, \quad \begin{pmatrix} y \\ 0 \\ 0 \end{pmatrix}.$$

The first three generators map to zero under $H^1(\mathfrak{m};A) \longrightarrow H^1_{\mathfrak{m}}(A)$, whereas the fourth maps to the nonzero element of $H^1_{\mathfrak{m}}(A)$ represented by the Čech cocycle

$$\left(\frac{y}{x}, 0, 0\right) \in A_x \oplus A_y \oplus A_z.$$

The ring A is not S_2, and hence not quasi-Gorenstein.

Note that $H^0_{\mathfrak{m}}(A) = zA$ is a rank 1 vector space concentrated in degree 1, so Theorem 4.2 also confirms that $\ker\left(H^1(\mathfrak{m};A) \longrightarrow H^1_{\mathfrak{m}}(A)\right)$ has rank 3, and is concentrated in degree 0.

For integers $t \geqslant 2$, Theorem 4.2 says that the kernel of $H^1(x^t;A) \longrightarrow H^1_{\mathfrak{m}}(A)$ has Hilbert series T^{1-t}. Indeed, $H^1(x^t;A)$ has Hilbert series

$$1 + 2T^{-1} + 2T^{-2} + \cdots + 2T^{2-t} + 3T^{1-t} + T^{-t},$$

while $H^1_{\mathfrak{m}}(A)$ has Hilbert series $1 + 2T^{-1} + 2T^{-2} + 2T^{-3} + \cdots$.

5. Stanley-Reisner rings

Theorem 4.2 is perhaps best viewed in the broader context of filtering a local cohomology module $H^n_{\mathfrak{m}}(A)$—that is typically not finitely generated—using natural finitely generated submodules such as the images of Koszul cohomology or Ext modules, themes that are pursued at length in [BBLSZ] and [SW]. This turns out to be fascinating even in the context of Stanley-Reisner rings; in this section, we examine the implications of Theorem 4.2 in some extensively studied examples such as triangulations of the torus and of the real projective plane. First, some generalities:

Let Δ be a simplicial complex with vertices $1,\ldots,e$. For k a field, consider the polynomial ring $k[x_1,\ldots,x_e]$ and the ideal $\mathfrak{a}$ generated by the square-free monomials

$$x_{i_1}\cdots x_{i_r}$$

such that $\{i_1,\ldots,i_r\}$ is not a face of Δ. The *Stanley-Reisner ring* of Δ over k is the ring

$$A := k[x_1,\ldots,x_e]/\mathfrak{a}.$$

The ring A has a $\mathbb{Z}^e$-grading with $\deg x_i$ being the i-th unit vector; this induces a grading on the Čech complex $C^\bullet(\boldsymbol{x};A)$. The module

$$A_{x_{i_1}\cdots x_{i_r}}$$

is nonzero precisely if $x_{i_1}\cdots x_{i_r}\notin\mathfrak{a}$, equivalently $\{i_1,\dots,i_r\}\in\Delta$. Hence the graded strand of $C^\bullet(\boldsymbol{x};A)$ in degree $\mathbf{0}:=(0,\dots,0)$ is the complex that computes the reduced simplicial cohomology $\widetilde{H}^\bullet(\Delta;k)$, with the indices shifted by one, so

$$[H^j_\mathfrak{m}(A)]_\mathbf{0}\ \cong\ \widetilde{H}^{j-1}(\Delta;k)\qquad\text{for } j\geqslant 0.$$

Theorem 4.2 yields the following corollary for Stanley-Reisner rings:

Corollary 5.1. *Let $A:=k[x_1,\dots,x_e]/\mathfrak{a}$ be an equidimensional Stanley-Reisner ring over a field k, such that $A_\mathfrak{p}$ is Cohen-Macaulay for each $\mathfrak{p}\in\operatorname{Spec}A\smallsetminus\{\mathfrak{m}\}$. Set $n:=\dim A$. For t a positive integer, set $\mathfrak{m}^{[t]}$ to be the ideal of A generated by the images of $x_1^t,\dots,x_e^t$.*

Then the kernel of the natural map $H^n(\mathfrak{m}^{[t]};A)\longrightarrow H^n_\mathfrak{m}(A)$ has Hilbert series

$$s_{n-1}\binom{e}{1}T^{-t}+s_{n-2}\binom{e}{2}T^{-2t}+\cdots+s_0\binom{e}{n}T^{-nt},$$

where $s_j:=\operatorname{rank}\widetilde{H}^{j-1}(\Delta;k)$, with Δ denoting the underlying simplicial complex.

Proof. The hypotheses that A is equidimensional and that $A_\mathfrak{p}$ is Cohen-Macaulay for $\mathfrak{p}\neq\mathfrak{m}$ imply that each local cohomology module $H^j_\mathfrak{m}(A)$, for $j<n$, has finite length. Let F denote the k-algebra endomorphism of A with $x_i\longmapsto x_i^2$ for each i. Then F is a pure endomorphism by [SW, Example 2.2], so the induced map

$$\widetilde{F}\colon H^j_\mathfrak{m}(A)\longrightarrow H^j_\mathfrak{m}(A)$$

is injective for each j. Using the $\mathbb{Z}^e$-grading from the preceding discussion, $\widetilde{F}$ restricts to an injective map

$$[H^j_\mathfrak{m}(A)]_{\boldsymbol{i}}\longrightarrow[H^j_\mathfrak{m}(A)]_{2\boldsymbol{i}}$$

for each $\boldsymbol{i}\in\mathbb{Z}^e$. But $H^j_\mathfrak{m}(A)$ has finite length for $j<n$, so

$$H^j_\mathfrak{m}(A)\ =\ [H^j_\mathfrak{m}(A)]_\mathbf{0}$$

for each $j<n$. The result now follows by Theorem 4.2. □

We begin by using Corollary 5.1 to shed light on Example 2.1:

Example 5.2. Consider the simplicial complex corresponding to two disjoint line segments; the corresponding Stanley-Reisner ring A, over a field k, is

$$k[x_1,x_2,x_3,x_4]/(x_1x_3,\ x_1x_4,\ x_2x_3,\ x_2x_4)$$

The ring A has dimension 2, and is not Cohen-Macaulay since

$$[H^1_\mathfrak{m}(A)]_\mathbf{0}\ \cong\ \widetilde{H}^0(\Delta;k)\ \cong\ k.$$

Since A is equidimensional and Cohen-Macaulay on the punctured spectrum, Corollary 5.1 says that the kernel of $H^2(\mathfrak{m}^{[t]};A)\longrightarrow H^2_\mathfrak{m}(A)$ has Hilbert series $4T^{-t}$ for each $t\geqslant 1$. This is consistent with the following table, where we record the rank of the vector spaces

$$[H^2(\mathfrak{m}^{[t]};A)]_j$$

as computed by *Macaulay2* [GS], in the case k is the field of rational numbers. Entries that are 0 are omitted. The last row records the rank of

$$\Big[\lim_{t\to\infty}H^2(\mathfrak{m}^{[t]};A)\Big]_j\ =\ \big[H^2_\mathfrak{m}(A)\big]_j,$$

which may be obtained using the exact sequence

$$0 \longrightarrow A \longrightarrow A/(x_1,x_2) \oplus A/(x_3,x_3) \longrightarrow A/\mathfrak{m} \longrightarrow 0$$

and the induced isomorphism $H^2_{\mathfrak{m}}(A) \cong H^2_{\mathfrak{m}}(A/(x_1,x_2)) \oplus H^2_{\mathfrak{m}}(A/(x_3,x_4))$.

t \ j	-1	-2	-3	-4	-5	-6	-7	-8	-9	-10
1	4									
2		6	4							
3		2	8	6	4					
4		2	4	10	8	6	4			
5		2	4	6	12	10	8	6	4	
6		2	4	6	8	14	12	10	8	6
7		2	4	6	8	10	16	14	12	10
8		2	4	6	8	10	12	18	16	14
9		2	4	6	8	10	12	14	20	18
10		2	4	6	8	10	12	14	16	22
$\lim_{t\to\infty}$		2	4	6	8	10	12	14	16	18

Note that for the ring $A := \mathbb{R}[x,y,ix,iy]$ in Example 2.1, the tensor product $A \otimes_{\mathbb{R}} \mathbb{C}$ is isomorphic to $\mathbb{C}[x_1,x_2,x_3,x_4]/(x_1x_3,\ x_1x_4,\ x_2x_3,\ x_2x_4)$ since

$$A \cong \mathbb{R}[x,y,u,v]/(x^2+u^2,\ y^2+v^2,\ vx-uy,\ xy+uv),$$

and hence $A \otimes_{\mathbb{R}} \mathbb{C}$ is isomorphic to $\mathbb{C}[x,y,u,v]/\big((u-ix,\ v-iy) \cap (u+ix,\ v+iy)\big)$.

Example 5.3. Consider the triangulation of the torus below.

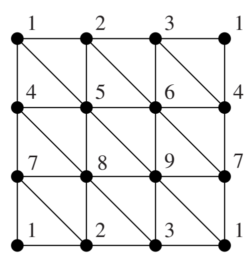

Figure 1: A triangulation of the torus

The corresponding Stanley-Reisner ring A is the homomorphic image of $k[x_1,\dots,x_9]$ modulo the ideal generated by the monomials x_1x_6, x_1x_8, x_2x_4, x_2x_9, x_3x_5, x_3x_7, x_4x_9, x_5x_7, x_6x_8, $x_1x_2x_3$, $x_1x_4x_7$, $x_1x_5x_9$, $x_2x_5x_8$, $x_2x_6x_7$, $x_3x_4x_8$, $x_3x_6x_9$, $x_4x_5x_6$, and $x_7x_8x_9$. The ring A has dimension 3, and is not Cohen-Macaulay since

$$[H^2_{\mathfrak{m}}(A)]_{\mathbf{0}} \cong \widetilde{H}^1(\Delta; k) \cong k^2.$$

It is readily verified that A is Cohen-Macaulay on the punctured spectrum. The canonical module of A may be computed as

$$\omega_A = \operatorname{Ext}^6_{k[\boldsymbol{x}]}(A,\, \omega_{k[\boldsymbol{x}]}) \cong A,$$

so the ring A is quasi-Gorenstein. Hence the natural map $H^3(\mathfrak{m};A) \longrightarrow H^3_{\mathfrak{m}}(A)$ is nonzero, though not injective. Indeed, the module $H^3(\mathfrak{m};A)$ has Hilbert series $1+18T^{-1}$, while Corollary 5.1 implies that the kernel of $H^3(\mathfrak{m}^{[t]};A) \longrightarrow H^3_{\mathfrak{m}}(A)$ has Hilbert series $18T^{-t}$ for each integer $t \geqslant 1$. The following table records the rank of the vector spaces

$$\left[H^3(\mathfrak{m}^{[t]};A)\right]_j,$$

as computed by *Macaulay2* in the case k is the field of rational numbers. The last row records the rank of

$$\left[\lim_{t\to\infty} H^3(\mathfrak{m}^{[t]};A)\right]_j \;=\; \left[H^3_{\mathfrak{m}}(A)\right]_j,$$

which may be obtained using the Hilbert series of A: since the ring A is quasi-Gorenstein with a-invariant 0, one has $\operatorname{rank}\left[H^3_{\mathfrak{m}}(A)\right]_j = \operatorname{rank}\,[A]_{-j}$.

$t \backslash j$	0	-1	-2	-3	-4	-5	-6	-7	-8	-9	-10
1	1	18									
2	1	9	45	18							
3	1	9	36	90	81	54	18				
4	1	9	36	81	153	162	153	108	54	18	
5	1	9	36	81	144	234	261	270	243	180	108
6	1	9	36	81	144	225	333	378	405	396	351
7	1	9	36	81	144	225	324	450	513	558	567
8	1	9	36	81	144	225	324	441	585	666	729
9	1	9	36	81	144	225	324	441	576	738	837
10	1	9	36	81	144	225	324	441	576	729	909
$\lim_{t\to\infty}$	1	9	36	81	144	225	324	441	576	729	900

We conclude with an example where the cohomology is characteristic-dependent:

Example 5.4. Consider the simplicial complex corresponding to the triangulation of the real projective plane $\mathbb{RP}^2$ in Figure 2. The corresponding Stanley-Reisner ring A is the

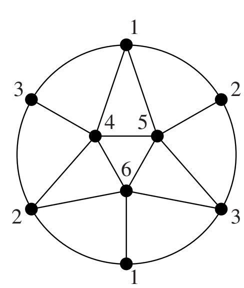

Figure 2: A triangulation of the real projective plane

homomorphic image of $k[x_1,\dots,x_6]$ modulo the ideal generated $x_1x_2x_3$, $x_1x_2x_4$, $x_1x_3x_5$, $x_1x_4x_6$, $x_1x_5x_6$, $x_2x_3x_6$, $x_2x_4x_5$, $x_2x_5x_6$, $x_3x_4x_5$, and $x_3x_4x_6$. Note that $\dim A = 3$.

Suppose the field k has characteristic other than 2. Then A is Cohen-Macaulay, see for example [Ho2, page 180], so the natural map $H^3(\mathfrak{m};A) \longrightarrow H^3_{\mathfrak{m}}(A)$ is injective by

Theorem 3.1. However, A is not Gorenstein: the socle of the homomorphic image of A modulo a system of parameters has rank 6.

Next, suppose k has characteristic 2; then A is not Cohen-Macaulay since

$$[H^2_{\mathfrak{m}}(A)]_{\mathbf{0}} \;\cong\; \widetilde{H}^1(\Delta;k) \;\cong\; k.$$

Indeed, in this case, A has depth 2, and the canonical module of A may be computed as

$$\omega_A \;=\; \operatorname{Ext}^3_{k[\boldsymbol{x}]}(A,\,\omega_{k[\boldsymbol{x}]}) \;\cong\; A,$$

so A is quasi-Gorenstein. Hence the natural map $H^3(\mathfrak{m};A) \longrightarrow H^3_{\mathfrak{m}}(A)$ is nonzero, though not injective. The module $H^3(\mathfrak{m};A)$ has Hilbert series $1+6T^{-1}$, and Corollary 5.1 implies that the kernel of $H^3(\mathfrak{m}^{[t]};A) \longrightarrow H^3_{\mathfrak{m}}(A)$ has Hilbert series $6T^{-t}$ for each $t \geqslant 1$. The ranks of the vector spaces

$$[H^3(\mathfrak{m}^{[t]};A)]_j$$

are recorded in the next table, with the last row computed as in the preceding example, using that A is quasi-Gorenstein, with a-invariant 0.

t \ j	0	-1	-2	-3	-4	-5	-6	-7	-8	-9	-10
1	1	6									
2	1	6	21	10							
3	1	6	21	46	45	30	10				
4	1	6	21	46	81	90	85	60	30	10	
5	1	6	21	46	81	126	145	150	135	100	60
6	1	6	21	46	81	126	181	210	225	220	195
7	1	6	21	46	81	126	181	246	285	310	315
8	1	6	21	46	81	126	181	246	321	370	405
9	1	6	21	46	81	126	181	246	321	406	465
10	1	6	21	46	81	126	181	246	321	406	501
$\lim_{t\to\infty}$	1	6	21	46	81	126	181	246	321	406	501

References

[An] Y. André, *La conjecture du facteur direct*, Publ. Math. Inst. Hautes Études Sci. **127** (2018), 71–93. 1, 6

[Bh] B. Bhatt, *On the direct summand conjecture and its derived variant*, Invent. Math. **212** (2018), 297–317. 1

[BBLSZ] B. Bhatt, M. Blickle, G. Lyubeznik, A. K. Singh, and W. Zhang, *Stabilization of the cohomology of thickenings*, Amer. J. Math. **141** (2019), 531–561. 9

[BH] W. Bruns and J. Herzog, *Cohen-Macaulay rings*, revised edition, Cambridge Stud. Adv. Math. **39**, Cambridge University Press, Cambridge, 1998. 4

[CHL] N. T. Cuong, N. T. Hoa, and N. T. H. Loan, *On certain length functions associated to a system of parameters in local rings*, Vietnam J. Math. **27** (1999), 259–272. 4

[Du] S. P. Dutta, *Dualizing complex and the canonical element conjecture. II*, J. London Math. Soc. (2) **56** (1997), 49–63. 1, 8

[GW] S. Goto and K.-i. Watanabe, *On graded rings I*, J. Math. Soc. Japan **30** (1978), 179–213. 5, 9

[GS] D. Grayson and M. E. Stillman, *Macaulay2, a software system for research in algebraic geometry*, available at `http://www.math.uiuc.edu/Macaulay2/`. 10

[He] R. C. Heitmann, *The direct summand conjecture in dimension three*, Ann. of Math. (2) **156** (2002), 695–712. 1

[Ho1] M. Hochster, *Contracted ideals from integral extensions of regular rings*, Nagoya Math. J. **51** (1973), 25–43. 1

[Ho2] M. Hochster, *Cohen-Macaulay rings, combinatorics, and simplicial complexes*, in: Ring theory II (Oklahoma, 1975), 171–223, Lecture Notes in Pure and Appl. Math. **26**, Dekker, New York, 1977. 12

[Ho3] M. Hochster, *Canonical elements in local cohomology modules and the direct summand conjecture*, J. Algebra **84** (1983), 503–553. 6

[HS] C. Huneke and K. E. Smith, *Tight closure and the Kodaira vanishing theorem*, J. Reine Angew. Math. **484** (1997), 127–152. 3

[Ka] I. Kaplansky, *Commutative rings*, revised edition, The University of Chicago Press, Chicago-London, 1974. 4

[MQS] L. Ma, P. H. Quy, and I. Smirnov, *Colength, multiplicity, and ideal closure operations*, Comm. Algebra **48** (2020), 1601–1607. 4

[Ro] P. Roberts, *Le théorème d'intersection*, C. R. Acad. Sci. Paris Sér. I Math. **304** (1987), 177–180. 1

[Sch] P. Schenzel, *Applications of dualizing complexes to Buchsbaum rings*, Adv. in Math. **44** (1982), no. 1, 61–77. 7

[SW] A. K. Singh and U. Walther, *Local cohomology and pure morphisms*, Illinois J. Math. **51** (2007), 287–298. 9, 10

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