A CONNECTEDNESS RESULT IN POSITIVE CHARACTERISTIC

ANURAG K. SINGH AND ULI WALTHER

1. Introduction

All rings considered in this note are commutative and Noetherian. We give a simple proof of the following result due to Lyubeznik:

Theorem 1.1. [Ly, Corollary 4.6] *Let $(R, \mathfrak{m})$ be a complete local ring of positive dimension with a separably closed coefficient field of positive characteristic. Then the e-th iteration of the Frobenius map*

$$F : H^1_{\mathfrak{m}}(R) \longrightarrow H^1_{\mathfrak{m}}(R)$$

is zero for $e \gg 0$ if and only if $\dim R \geq 2$ and $\operatorname{Spec} R \setminus \{\mathfrak{m}\}$ is connected in the Zariski topology.

We also obtain, by similar methods, the following theorem:

Theorem 1.2. *Let $(R, \mathfrak{m})$ be a complete local ring of positive dimension with an algebraically closed coefficient field of positive characteristic. Then the number of connected components of $\operatorname{Spec} R \setminus \{\mathfrak{m}\}$ is*

$$1 + \dim_K \bigcap_{e \in \mathbb{N}} F^e(H^1_{\mathfrak{m}}(R)).$$

Theorem 1.1 is obtained in [Ly] as a corollary of the following theorems of Lyubeznik and Peskine-Szpiro:

Theorem 1.3. [Ly, Theorem 1.1] *Let $(A, \mathfrak{M})$ be a regular local ring containing a field of positive characteristic, and let $\mathfrak{A}$ be an ideal of A. Then $H^i_{\mathfrak{A}}(A) = 0$ if and only if there exists an integer $e \geq 1$ such that the e-th Frobenius iteration*

$$F^e : H^{\dim A - i}_{\mathfrak{M}}(A/\mathfrak{A}) \longrightarrow H^{\dim A - i}_{\mathfrak{M}}(A/\mathfrak{A})$$

is the zero map.

Theorem 1.4. [PS, Chapter III, Theorem 5.5] *Let $(A, \mathfrak{M})$ be a complete regular local ring with a separably closed coefficient field of positive characteristic, and let $\mathfrak{A}$ be an ideal of A. Then $H^i_{\mathfrak{A}}(A) = 0$ for $i \geq \dim A - 1$ if and only if $\dim(A/\mathfrak{A}) \geq 2$ and $\operatorname{Spec}(A/\mathfrak{A}) \setminus \{\mathfrak{M}\}$ is connected.*

Our proof of Theorem 1.1 is "simple" in the sense that it does not rely on vanishing theorems such as those of [PS]—indeed the only ingredient, aside from elementary considerations, is the local duality theorem.

We would like to mention that results analogous to Theorem 1.4 were discovered by Hartshorne in the projective case [HaR, Theorem 7.5], and by Ogus in equicharacteristic zero via de Rham cohomology [Og, Corollary 2.11].

Date: December 25, 2007.
The first author is supported in part by the NSF under Grant DMS 0300600.
The second author is supported in part by the NSF under Grant DMS 0100509.

Theorem 1.5. *Let $(A, \mathfrak{M})$ be a regular local ring containing a field, and let $\mathfrak{A}$ be an ideal of A. Then $H^i_{\mathfrak{A}}(A) = 0$ for $i \geq \dim A - 1$ if and only if*

(1) $\dim(A/\mathfrak{A}) \geq 2$, *and*
(2) $\operatorname{Spec}(A/\mathfrak{A}) \setminus \{\mathfrak{M}\}$ *is formally geometrically connected (see Definition 2.1).*

Huneke and Lyubeznik gave a characteristic free proof of this in [HL, Theorem 2.9] using a generalization of a result of Faltings, [Fa, Satz 1]. We do not know a criterion to detect connectedness of the punctured spectrum of $A/\mathfrak{A}$ in terms of $H^i_{\mathfrak{M}}(A/\mathfrak{A})$, except when A has positive characteristic.

2. Preliminary remarks

Notation: When R is the homomorphic image of a ring A, we use upper-case letters $\mathfrak{P}, \mathfrak{Q}, \mathfrak{M}, \mathfrak{A}, \mathfrak{B}$ for ideals of A, and corresponding lower-case letters $\mathfrak{p}, \mathfrak{q}, \mathfrak{m}, \mathfrak{a}, \mathfrak{b}$ for their images in R.

Definition 2.1. Let $(R, \mathfrak{m})$ be a local ring. A field $K \subseteq R$ is a *coefficient field* for R if the composition $K \hookrightarrow R \twoheadrightarrow R/\mathfrak{m}$ is an isomorphism. Every complete local ring containing a field contains a coefficient field.

We recall some notions from [Ra, Chapitre VIII]. Let $(R, \mathfrak{m}, K)$ be a local ring and let $\overline{f(T)} \in K[T]$ denote the image of a polynomial $f(T) \in R[T]$. Then R is *Henselian* if for every monic polynomial $f(T) \in R[T]$, every factorization of $\overline{f(T)}$ as a product of relatively prime monic polynomials in $K[T]$ lifts to a factorization of $f(T)$ as a product of monic polynomials in $R[T]$. Hensel's Lemma is precisely the statement that every complete local ring is Henselian. The *Henselization* of a local ring R is a local ring R^{h}, with the property that every local homomorphism from R to a Henselian local ring factors uniquely through R^{h}. The ring R^{h} is obtained by taking the direct limit of all local étale extensions S of R for which $(R, \mathfrak{m}) \longrightarrow (S, \mathfrak{n})$ induces an isomorphism of residue fields $R/\mathfrak{m} \xrightarrow{\cong} S/\mathfrak{n}$.

A local ring $(R, \mathfrak{m}, K)$ is said to be *strictly Henselian* if it is Henselian and its residue field K is separably closed. It is easily seen that R is strictly Henselian if and only if every monic polynomial $f(T) \in R[T]$ for which $\overline{f(T)} \in K[T]$ is separable splits into linear factors in $R[T]$. Every local ring has a *strict Henselization* R^{sh}, such that every local homomorphism from R to a strictly Henselian ring factors through R^{sh}. The strict Henselization of a field K is its separable closure K^{sep}. In general, the strict Henselization of a local ring $(R, \mathfrak{m}, K)$ is obtained by fixing an embedding $\iota : K \longrightarrow K^{\mathrm{sep}}$, and taking the direct limit of local étale extensions $(S, \mathfrak{n}, L)$ of $(R, \mathfrak{m}, K)$ such that there is an induced map $K \longrightarrow L \longrightarrow K^{\mathrm{sep}}$ which agrees with $\iota : K \longrightarrow K^{\mathrm{sep}}$.

The *punctured spectrum* of a local ring $(R, \mathfrak{m})$ is the set $\operatorname{Spec} R \setminus \{\mathfrak{m}\}$ with the Zariski topology. We say that *the punctured spectrum of R is formally geometrically connected* if the punctured spectrum of $\widehat{\widehat{R}^{\mathrm{sh}}}$, the completion of the strict Henselization of the completion of R, is connected.

Definition 2.2. Let $\mathfrak{a}$ be an ideal of a ring R. A ring homomorphism $\varphi : R \longrightarrow S$ induces a map of local cohomology modules $H^i_{\mathfrak{a}}(R) \xrightarrow{\varphi} H^i_{\mathfrak{a}S}(S)$. In particular, if R contains a field of characteristic $p > 0$, then the Frobenius homomorphism $F : R \longrightarrow R$ induces an additive map

$$H^i_{\mathfrak{a}}(R) \xrightarrow{F} H^i_{\mathfrak{a}^{[p]}}(R) = H^i_{\mathfrak{a}}(R),$$

called the *Frobenius action* on $H^i_{\mathfrak a}(R)$. An element $\eta \in H^i_{\mathfrak a}(R)$ is *F-torsion* if there exists $e \in \mathbb{N}$ such that $F^e(\eta) = 0$. The module $H^i_{\mathfrak a}(R)$ is *F-torsion* if every element of $H^i_{\mathfrak a}(R)$ is F-torsion. The image of F^e is not, in general, an R-module. However, it is a K-vector space when K is perfect, and in this case the *F-stable* part

$$H^i_{\mathfrak a}(R)_{\mathrm{st}} = \bigcap_{e\in\mathbb{N}} F^e(H^i_{\mathfrak a}(R))$$

of $H^i_{\mathfrak a}(R)$ is a K-vector space as well. Results about F-torsion modules and F-stable subspaces are summarized in §5.

Remark 2.3. Consider a local ring $(R, \mathfrak m)$ of positive dimension. The punctured spectrum of R is disconnected if and only if the minimal primes of R can be partitioned into two sets $\mathfrak p_1, \dots, \mathfrak p_m$ and $\mathfrak q_1, \dots, \mathfrak q_n$ such that $\operatorname{rad}(\mathfrak p_i + \mathfrak q_j) = \mathfrak m$ for all pairs $\mathfrak p_i, \mathfrak q_j$. Consider the graph Γ whose vertices are the minimal primes of R, and there is an edge between minimal primes $\mathfrak p$ and $\mathfrak p'$ if and only if $\operatorname{rad}(\mathfrak p + \mathfrak p') \neq \mathfrak m$. It follows that the punctured spectrum of R is connected if and only if the graph Γ is connected. If the graph Γ is connected, take a spanning tree. The spanning tree must contain a vertex $\mathfrak p_i$ with only one edge, so $\Gamma \setminus \{\mathfrak p_i\}$ is connected as well.

Let $\mathfrak P_1, \dots, \mathfrak P_n$ be incomparable prime ideals of a local domain A. Their images $\mathfrak p_1, \dots, \mathfrak p_n$ are precisely the minimal primes of the ring $R = A/(\mathfrak P_1 \cap \cdots \cap \mathfrak P_n)$. From the above discussion, we conclude that if the punctured spectrum of R is connected, then there exists i such that the punctured spectrum of the ring

$$A/(\mathfrak P_1 \cap \cdots \cap \widehat{\mathfrak P_i} \cap \cdots \cap \mathfrak P_n)$$

is connected as well.

Theorems 1.1 and 1.2 state that connectedness issues for $\operatorname{Spec} R \setminus \{\mathfrak m\}$ are determined by the Frobenius action on $H^1_{\mathfrak m}(R)$. We next record an observation about the length of $H^1_{\mathfrak m}(R)$.

Proposition 2.4. *Let $(R, \mathfrak m)$ be a local ring which is a homomorphic image of a Gorenstein domain. Then $H^1_{\mathfrak m}(R)$ has finite length if and only if $\operatorname{ann} \mathfrak p = 0$ for every prime ideal $\mathfrak p$ of R with $\dim R/\mathfrak p = 1$.*

Proof. If $\dim R = 0$ then $H^1_{\mathfrak m}(R) = 0$ and R has no primes with $\dim R/\mathfrak p = 1$. If $\dim R = 1$ then $H^1_{\mathfrak m}(R)$ has infinite length and $\dim R/\mathfrak p = 1$ for some minimal, hence associated, prime $\mathfrak p$ of R. For the rest of the proof we hence assume that $\dim R \geq 2$.

Let $R = A/\mathfrak Q$ where A is a Gorenstein domain. Localizing A at the inverse image of $\mathfrak m$, we may assume that $(A, \mathfrak M)$ is a local ring. Using local duality over A, the module $H^1_{\mathfrak m}(R) = H^1_{\mathfrak M}(A/\mathfrak Q)$ has finite length if and only if $\operatorname{Ext}_A^{\dim A - 1}(A/\mathfrak Q, A)$ has finite length as an A-module. Since $\operatorname{Ext}_A^{\dim A-1}(A/\mathfrak Q, A)$ is finitely generated, this is equivalent to the vanishing of

$$\operatorname{Ext}_A^{\dim A-1}(A/\mathfrak Q, A)_{\mathfrak P} = \operatorname{Ext}_{A_{\mathfrak P}}^{\dim A-1}(A_{\mathfrak P}/\mathfrak Q A_{\mathfrak P}, A_{\mathfrak P})$$

for all $\mathfrak P \in \operatorname{Spec} A \setminus \{\mathfrak M\}$. Using local duality over the Gorenstein local ring $(A_{\mathfrak P}, \mathfrak P A_{\mathfrak P})$, this is equivalent to the vanishing of

$$H^{\dim A_{\mathfrak P} - \dim A + 1}_{\mathfrak P A_{\mathfrak P}}(A_{\mathfrak P}/\mathfrak Q A_{\mathfrak P}) = H^{\dim A_{\mathfrak P} - \dim A + 1}_{\mathfrak p R_{\mathfrak p}}(R_{\mathfrak p})$$

for all $\mathfrak P \in \operatorname{Spec} A \setminus \{\mathfrak M\}$. This local cohomology module vanishes for $\mathfrak P \notin V(\mathfrak Q)$. Since $\dim A_{\mathfrak P} - \dim A + 1 \leq 0$ for $\mathfrak P \in \operatorname{Spec} A \setminus \{\mathfrak M\}$, we need only consider primes

$\mathfrak{P} \in V(\mathfrak{Q})$ with $\dim A_{\mathfrak{P}} = \dim A - 1$. Since A is a catenary local domain, $\dim A_{\mathfrak{P}}$ equals $\dim A - 1$ precisely when $\dim A/\mathfrak{P} = 1$, which is equivalent to $\dim R/\mathfrak{p} = 1$. Hence $H^1_{\mathfrak{m}}(R)$ has finite length if and only if $H^0_{\mathfrak{p}R_{\mathfrak{p}}}(R_{\mathfrak{p}}) = H^0_{\mathfrak{p}}(R)$ vanishes for all $\mathfrak{p} \in \operatorname{Spec} R$ with $\dim R/\mathfrak{p} = 1$, i.e., if and only if $\operatorname{ann} \mathfrak{p} = 0$ for all $\mathfrak{p} \in \operatorname{Spec} R$ with $\dim R/\mathfrak{p} = 1$. $\square$

3. Main results

Theorem 3.1. *Let $(R, \mathfrak{m})$ be a strictly Henselian local domain containing a field of positive characteristic. If $\dim R \geq 2$ and R is a homomorphic image of a Gorenstein domain, then $H^1_{\mathfrak{m}}(R)$ is F-torsion.*

Proof. Suppose there exists $\eta \in H^1_{\mathfrak{m}}(R)$ which is not F-torsion. Since R is a domain, Proposition 2.4 implies that $H^1_{\mathfrak{m}}(A)$ has finite length. Hence for all integers $e \gg 0$, the element $F^e(\eta)$ belongs to the R-module spanned by $\eta, F(\eta), F^2(\eta), \dots, F^{e-1}(\eta)$. Amongst all equations of the form

$$F^{e+k}(\eta) + r_1 F^{e+k-1}(\eta) + \cdots + r_e F^k(\eta) = 0 \tag{1}$$

with $r_i \in R$ for all i, choose one where the number of nonzero coefficients r_i that occur is minimal. We claim that r_e must be a unit. Note that $H^1_{\mathfrak{m}}(A)$ is killed by $\mathfrak{m}^{q'}$ for some $q' = p^{e'}$. If $r_e \in \mathfrak{m}$, then applying $F^{e'}$ to equation (1), we get

$$F^{e'+e+k}(\eta) + r_1^{q'} F^{e'+e+k-1}(\eta) + \cdots + r_e^{q'} F^{e'+k}(\eta) = 0.$$

But $r_e^{q'} F^{e'+k}(\eta) \in \mathfrak{m}^{q'} H^1_{\mathfrak{m}}(R) = 0$, so this is an equation with fewer nonzero coefficients, contradicting the minimality assumption. This shows that $r_e \in R$ is a unit. Since η is not F-torsion neither is $F^k(\eta)$, so after a change of notation we have an equation

$$F^e(\eta) + r_1 F^{e-1}(\eta) + \cdots + r_e \eta = 0 \tag{2}$$

where r_e is a unit and $\eta \in H^1_{\mathfrak{m}}(R)$ is not F-torsion. Let $\eta = [(y_1/x_1, \dots, y_d/x_d)]$ where $H^1_{\mathfrak{m}}(R)$ is regarded as the cohomology of a Čech complex on a system of parameters $x_1, \dots, x_d$ for R. Then (2) implies that there exists $r_{e+1} \in R$ such that each $y_i/x_i \in R_{x_i}$ is a root of the polynomial

$$f(T) = T^{p^e} + r_1 T^{p^{e-1}} + \cdots + r_e T + r_{e+1} \in R[T].$$

Now $f'(T) = r_e$ is a unit, so $\overline{f(T)} \in R/\mathfrak{m}[T]$ is a separable polynomial. Since R is strictly Henselian, the polynomial $f(T)$ splits in $R[T]$, and hence any root of $f(T)$ in the fraction field of R must be an element of R. In particular, $y_1/x_1 = \cdots = y_d/x_d \in R$, and so $\eta = 0$. $\square$

We next prove the connectedness criterion. Proposition 5.1 states that $H^1_{\mathfrak{m}}(R)$ is F-torsion if and only if there exists e such that $F^e(H^1_{\mathfrak{m}}(R)) = 0$. Hence the following theorem is equivalent to Theorem 1.1.

Theorem 3.2. *Let $(R, \mathfrak{m})$ be a local ring with $\dim R > 0$, which contains a field of positive characteristic. Then $H^1_{\mathfrak{m}}(R)$ is F-torsion if and only if $\dim R \geq 2$ and the punctured spectrum of R is formally geometrically connected.*

Proof. Quite generally, for a local ring $(R, \mathfrak{m})$ we have $H^i_{\mathfrak{m}}(\widehat{R}) = H^i_{\mathfrak{m}}(R)$. Moreover, $S = \widehat{\widehat{R}^{\text{sh}}}$ is a faithfully flat extension of R and so $H^i_{\mathfrak{m}}(R) \otimes_R S \cong H^i_{\mathfrak{m}S}(S)$ is F-torsion if and only if $H^1_{\mathfrak{m}}(R)$ is F-torsion. Hence we may assume that R is a complete local ring with a separably closed coefficient field.

Suppose that $H^1_{\mathfrak{m}}(R)$ is F-torsion. The local cohomology module $H^{\dim R}_{\mathfrak{m}}(R)$ is not F-torsion by Lemma 5.2, so $\dim R \geq 2$. Let $\mathfrak{a}$ and $\mathfrak{b}$ be ideals of R such that $\mathfrak{a}+\mathfrak{b}$ is $\mathfrak{m}$-primary and $\mathfrak{a} \cap \mathfrak{b} = 0$. Let

$$x_1 = y_1 + z_1, \quad \ldots, \quad x_d = y_d + z_d$$

be a system of parameters for R where $y_i \in \mathfrak{a}$ and $z_i \in \mathfrak{b}$. Since $\mathfrak{a}\mathfrak{b} \subseteq \mathfrak{a} \cap \mathfrak{b} = 0$, we have $y_i z_j = 0$ for all i, j, and hence

$$y_i(y_j + z_j) = y_j(y_i + z_i).$$

These relations give an element of $H^1_{\mathfrak{m}}(R)$ regarded as the cohomology of a Čech complex on $x_1, \ldots, x_d$, namely

$$\eta = \left[\left(\frac{y_1}{x_1}, \ldots, \frac{y_d}{x_d}\right)\right] \in H^1_{\mathfrak{m}}(R).$$

The hypothesis implies that $F^e(\eta) = 0$ for some e, so there exists $q = p^e$ and $r \in R$ such that $(y_i/x_i)^q = r$ in R_{x_i} for all $1 \leq i \leq d$. Hence there exists $t \in \mathbb{N}$ such that $x_i^t y_i^q = r x_i^{q+t}$, i.e.,

$$(y_i + z_i)^t y_i^q = r(y_i + z_i)^{q+t}.$$

But $y_i z_i = 0$, so these equations simplify to give $(1-r)y_i^{q+t} = r z_i^{q+t}$. Since R is a local ring, either r or $1-r$ must be a unit. If r is a unit, then $z_i^{q+t} \in \mathfrak{a}$ for all i, and so $\mathfrak{a}$ is $\mathfrak{m}$-primary. Similarly if $1-r$ is a unit, then $\mathfrak{b}$ is $\mathfrak{m}$-primary. This proves that the punctured spectrum of R is connected.

For the converse, assume that $\dim R \geq 2$ and that the punctured spectrum of R is connected. Let $\mathfrak{n}$ denote the nilradical of R. Note that $\operatorname{Spec} R$ is homeomorphic to $\operatorname{Spec} R/\mathfrak{n}$. Moreover, $\mathfrak{n}$ supports a Frobenius action and is F-torsion. The long exact sequence of local cohomology relating $H^1_{\mathfrak{m}}(R)$ and $H^1_{\mathfrak{m}}(R/\mathfrak{n})$ implies that if $H^1_{\mathfrak{m}}(R/\mathfrak{n})$ is F-torsion then so is $H^1_{\mathfrak{m}}(R)$, and hence there is no loss of generality in assuming that R is reduced. Let $R = A/(\mathfrak{P}_1 \cap \cdots \cap \mathfrak{P}_n)$ where $\mathfrak{P}_1, \ldots, \mathfrak{P}_n$ are incomparable prime ideals of a power series ring $A = K[[x_1, \ldots, x_m]]$ over a separably closed field K. We use induction on n to prove that $H^1_{\mathfrak{m}}(R)$ is F-torsion; the case $n = 1$ follows from Theorem 3.1, so we assume $n > 1$ below.

If $\dim R/\mathfrak{p}_i = 1$ for some i, then $\operatorname{Spec} R \setminus \{\mathfrak{m}\}$ is the disjoint union of $V(\mathfrak{p}_i) \setminus \{\mathfrak{m}\}$ and $V(\mathfrak{p}_1 \cap \cdots \cap \widehat{\mathfrak{p}_i} \cap \cdots \cap \mathfrak{p}_n) \setminus \{\mathfrak{m}\}$, contradicting the connectedness assumption. Hence $\dim R/\mathfrak{p}_i \geq 2$ for all i. By Remark 2.3, after relabeling the minimal primes if necessary, we may assume that the punctured spectrum of $A/\mathfrak{Q}$ is connected where $\mathfrak{Q} = \mathfrak{P}_2 \cap \cdots \cap \mathfrak{P}_n$. The short exact sequence

$$0 \longrightarrow A/(\mathfrak{P}_1 \cap \mathfrak{Q}) \longrightarrow A/\mathfrak{P}_1 \oplus A/\mathfrak{Q} \longrightarrow A/(\mathfrak{P}_1 + \mathfrak{Q}) \longrightarrow 0$$

induces a long exact sequence of local cohomology modules containing the piece

$$(3) \quad H^0_{\mathfrak{M}}(A/(\mathfrak{P}_1 + \mathfrak{Q})) \longrightarrow H^1_{\mathfrak{M}}(A/(\mathfrak{P}_1 \cap \mathfrak{Q})) \longrightarrow H^1_{\mathfrak{M}}(A/\mathfrak{P}_1) \oplus H^1_{\mathfrak{M}}(A/\mathfrak{Q}).$$

Since $\operatorname{rad}(\mathfrak{P}_1 + \mathfrak{P}_i) \neq \mathfrak{M}$ for some $i > 1$, it follows that $\dim A/(\mathfrak{P}_1 + \mathfrak{Q}) \geq 1$. Proposition 5.2 (1) now implies that $H^0_{\mathfrak{M}}(A/(\mathfrak{P}_1+\mathfrak{Q}))$ is F-torsion. By the inductive hypothesis, $H^1_{\mathfrak{M}}(A/\mathfrak{P}_1)$ and $H^1_{\mathfrak{M}}(A/\mathfrak{Q})$ are F-torsion as well. The exact sequence (3) implies that $H^1_{\mathfrak{M}}(A/(\mathfrak{P}_1 \cap \mathfrak{Q})) = H^1_{\mathfrak{m}}(R)$ is F-torsion. □

The following lemma will be used in the proof of Theorem 1.2.

Lemma 3.3. *Let $(R, \mathfrak{m})$ be a complete local domain with an algebraically closed coefficient field of positive characteristic. Then $H^1_{\mathfrak{m}}(R)_{\mathrm{st}}$, the F-stable part of $H^1_{\mathfrak{m}}(R)$, is zero.*

Proof. If $\dim R = 0$ then $H^1_{\mathfrak{m}}(R) = 0$; if $\dim R \geq 2$ then the assertion follows from Theorem 3.1. The remaining case is $\dim R = 1$. Theorem 5.3 implies that $H^1_{\mathfrak{m}}(R)_{\mathrm{st}}$ has a vector space basis $\eta_1, \dots, \eta_r$ such that $F(\eta_i) = \eta_i$.

Let $\eta \in H^1_{\mathfrak{m}}(R)_{\mathrm{st}}$ be an element with $F(\eta) = \eta$. Considering $H^1_{\mathfrak{m}}(R)$ as the cohomology of a suitable Čech complex, let η be the class of y/x in $R_x/R = H^1_{\mathfrak{m}}(R)$ where $y \in R$ and $x \in \mathfrak{m}$. Since $F(\eta) = \eta$, there exists $r \in R$ such that

$$\left(\frac{y}{x}\right)^p - \frac{y}{x} - r = 0,$$

and so $y/x \in R_x$ is a root of the polynomial $f(T) = T^p - T - r \in R[T]$. The polynomial $\overline{f(T)} \in K[T]$ is separable and R is strictly Henselian, so $f(T)$ splits in $R[T]$. Since y/x is a root of $f(T)$ in the fraction field of R, it must then be an element of R, and hence $\eta = 0$. $\square$

Proof of Theorem 1.2. We may assume R to be reduced by Lemma 5.5. First consider the case where the punctured spectrum of R is connected. If $\dim R \geq 2$ then $H^1_{\mathfrak{m}}(R)$ is F-torsion by Theorem 3.2, so $H^1_{\mathfrak{m}}(R)_{\mathrm{st}} = 0$. If $\dim R = 1$ then R is a domain, and Lemma 3.3 implies that $H^1_{\mathfrak{m}}(R)_{\mathrm{st}} = 0$.

We continue by induction on the number of connected components of the punctured spectrum of R. If the punctured spectrum of R is disconnected, then $R = A/(\mathfrak{A} \cap \mathfrak{B})$ where $(A, \mathfrak{M})$ is a power series ring over the field K, and $\mathfrak{A}$ and $\mathfrak{B}$ are radical ideals of A which are not $\mathfrak{M}$-primary, but $\mathfrak{A} + \mathfrak{B}$ is $\mathfrak{M}$-primary. There is a short exact sequence

$$0 \longrightarrow A/(\mathfrak{A} \cap \mathfrak{B}) \longrightarrow A/\mathfrak{A} \oplus A/\mathfrak{B} \longrightarrow A/(\mathfrak{A} + \mathfrak{B}) \longrightarrow 0.$$

Since $H^0_{\mathfrak{M}}(A/\mathfrak{A}) = H^0_{\mathfrak{M}}(A/\mathfrak{B}) = H^1_{\mathfrak{M}}(A/(\mathfrak{A} + \mathfrak{B})) = 0$, the resulting long exact sequence of local cohomology modules gives a short exact sequence

$$0 \longrightarrow H^0_{\mathfrak{M}}(A/(\mathfrak{A} + \mathfrak{B})) \longrightarrow H^1_{\mathfrak{M}}(A/(\mathfrak{A} \cap \mathfrak{B})) \longrightarrow H^1_{\mathfrak{M}}(A/\mathfrak{A}) \oplus H^1_{\mathfrak{M}}(A/\mathfrak{B}) \longrightarrow 0.$$

By Theorem 5.4, we have a K-vector space isomorphism

$$H^1_{\mathfrak{m}}(R)_{\mathrm{st}} = H^1_{\mathfrak{M}}(A/(\mathfrak{A} \cap \mathfrak{B}))_{\mathrm{st}} \cong H^0_{\mathfrak{M}}(A/(\mathfrak{A} + \mathfrak{B}))_{\mathrm{st}} \oplus H^1_{\mathfrak{M}}(A/\mathfrak{A})_{\mathrm{st}} \oplus H^1_{\mathfrak{M}}(A/\mathfrak{B})_{\mathrm{st}}.$$

Since $H^0_{\mathfrak{M}}(A/(\mathfrak{A} + \mathfrak{B}))_{\mathrm{st}} = K$ by Proposition 5.2 (3), the inductive hypothesis completes the proof. $\square$

We record the graded analogues of the result proved in this section:

Theorem 3.4. *Let R be an $\mathbb{N}$-graded ring of positive dimension which is finitely generated over a field $R_0 = K$ of characteristic $p > 0$.*

1. *If K is separably closed and R is a domain with $\dim R \geq 2$, then $H^1_{\mathfrak{m}}(R)$ is F-torsion.*
2. *Let K^{sep} denote the separable closure of K. Then $H^1_{\mathfrak{m}}(R)$ is F-torsion if and only if $\dim R \geq 2$ and $\operatorname{Proj}(R \otimes_K K^{\mathrm{sep}})$ is connected.*
3. *If K is algebraically closed, then the number of connected components of $\operatorname{Proj} R$ is*

$$1 + \dim_K \bigcap_{e \in \mathbb{N}} F^e(H^1_{\mathfrak{m}}(R)) = 1 + \dim_K \bigcap_{e \in \mathbb{N}} F^e([H^1_{\mathfrak{m}}(R)]_0).$$

Proof. The proofs are similar to those in the complete case, so we only sketch a proof of (1). Note that $H^1_{\mathfrak{m}}(R)$ is a $\mathbb{Z}$-graded R-module, and that

$$F : [H^1_{\mathfrak{m}}(R)]_n \longrightarrow [H^1_{\mathfrak{m}}(R)]_{np} \quad \text{for all } n \in \mathbb{Z}.$$

The module $H^1_{\mathfrak{m}}(R)$ has finite length, so all elements of $H^1_{\mathfrak{m}}(R)$ of positive or negative degree are F-torsion; it remains to show that elements $\eta \in [H^1_{\mathfrak{m}}(R)]_0$ are F-torsion as well. Let η be a element of $[H^1_{\mathfrak{m}}(R)]_0$ which is not F-torsion. As in the proof of Theorem 3.1, after a change of notation we may assume that

$$F^e(\eta) + r_1 F^{e-1}(\eta) + \cdots + r_e \eta = 0$$

where all r_i are in $[R]_0 = K$, and r_e is nonzero. Let $\eta = [(y_1/x_1, \ldots, y_d/x_d)]$ where $H^1_{\mathfrak{m}}(R)$ is regarded as the cohomology of a homogeneous Čech complex. Then there exists $r_{e+1} \in K$ such that $y_i/x_i \in R_{x_i}$ is a root of the polynomial

$$f(T) = T^{p^e} + r_1 T^{p^{e-1}} + \cdots + r_e T + r_{e+1} \in K[T].$$

But $f(T)$ is a separable polynomial, so it splits in $K[T]$. The element $y_i/x_i = y_j/x_j$ is a root of $f(T)$ in the fraction field of R, so it must be one of the roots of $f(T)$ in K. It follows that $\eta = 0$. $\square$

4. F-PURITY

A ring homomorphism $\varphi : R \longrightarrow S$ is *pure* if $\varphi \otimes 1 : R \otimes_R M \longrightarrow S \otimes_R M$ is injective for every R-module M. If R is a ring containing a field of characteristic $p > 0$, then R is *F-pure* if the Frobenius homomorphism $F : R \longrightarrow R$ is pure. The notion was introduced by Hochster and Roberts in the course of their study of rings of invariants in [HR1, HR2].

Examples of F-pure rings include regular rings of positive characteristic and their pure subrings. If $\mathfrak{a}$ is generated by square-free monomials in the variables $x_1, \ldots, x_n$ and K is a field of positive characteristic, then $K[x_1, \ldots, x_n]/\mathfrak{a}$ is F-pure.

Goto and Watanabe classified one-dimensional F-pure rings in [GW]: let $(R, \mathfrak{m})$ be a local ring containing a field of positive characteristic such that $R/\mathfrak{m} = K$ is algebraically closed, $F : R \longrightarrow R$ is finite, and $\dim R = 1$. Then R is F-pure if and only if

$$\widehat{R} \cong K[[x_1, \cdots, x_n]]/(x_i x_j : i < j).$$

Two-dimensional F-pure rings have attracted a lot of attention: in [Wa1] Watanabe proved that F-pure normal Gorenstein local rings of dimension two are either rational double points, simple elliptic singularities, or cusp singularities. Watanabe also obtained a classification of two-dimensional normal $\mathbb{N}$-graded rings R over an algebraically closed field R_0, in terms of the associated $\mathbb{Q}$-divisor on the curve $\operatorname{Proj} R$, [Wa2]. In [MS] Mehta and Srinivas obtained a classification of two-dimensional F-pure normal singularities in terms of the resolution of the singularity. Hara completed the classification of two-dimensional normal F-pure singularities in terms of the dual graph of the minimal resolution of the singularity, [HaN].

The results of §3 imply that over separably closed fields, F-pure domains of dimension two are Cohen-Macaulay. The point is that if R is an F-pure ring, then the Frobenius action $F : H^i_{\mathfrak{m}}(R) \longrightarrow H^i_{\mathfrak{m}}(R)$ is an injective map.

Corollary 4.1. *Let R be a local ring with $\dim R \geq 2$ which contains a field of positive characteristic. If R is F-pure and the punctured spectrum of R is formally geometrically connected, then* $\operatorname{depth} R \geq 2$.

In particular, if R is a complete local F-pure domain of dimension two, with a separably closed coefficient field, then R is Cohen-Macaulay.

Proof. An F-pure ring is reduced, so $H^0_{\mathfrak{m}}(R) = 0$. By Theorem 3.1, $H^1_{\mathfrak{m}}(R)$ is F-torsion. Since R is F-pure, it follows that $H^1_{\mathfrak{m}}(R) = 0$. □

In the graded case, we similarly have:

Corollary 4.2. *Let R be an $\mathbb{N}$-graded ring with $\dim R \geq 2$ which is finitely generated over a field $R_0 = K$ of positive characteristic. If R is F-pure and $\operatorname{Proj}(R \otimes_K K^{\mathrm{sep}})$ is connected, then $\operatorname{depth} R \geq 2$.*

In the following example, R is a graded F-pure domain of dimension 2, but $\operatorname{depth} R = 1$. The issue is that $\operatorname{Proj} R$ is connected though $\operatorname{Proj}(R \otimes_K K^{\mathrm{sep}})$ is not.

Example 4.3. Let K be a field of characteristic $p > 2$, and $a \in K$ an element such that $\sqrt{a} \notin K$. Let $R = K[x, y, x\sqrt{a}, y\sqrt{a}]$. The domain R has a presentation

$$R = K[x, y, u, v]/(u^2 - ax^2, v^2 - ay^2, uv - axy, vx - uy),$$

and if K^{sep} denotes the separable closure of K, then

$$R \otimes_K K^{\mathrm{sep}} \cong K^{\mathrm{sep}}[x, y, u, v]/(u - x\sqrt{a}, v - y\sqrt{a})(u + x\sqrt{a}, v + y\sqrt{a}).$$

Using a change of variables, $R \otimes_K K^{\mathrm{sep}} \cong K^{\mathrm{sep}}[x', y', u', v']/(x', y')(u', v')$. Since $(x', y')(u', v')$ is a square-free monomial ideal, $R \otimes_K K^{\mathrm{sep}}$ is F-pure and it follows that R is F-pure. However, R is not Cohen-Macaulay since x, y is a homogeneous system of parameters with a non-trivial relation

$$(x\sqrt{a})y = (y\sqrt{a})x.$$

Using the Čech complex on x, y to compute $H^1_{\mathfrak{m}}(R)$, we see that it is a 1-dimensional K-vector space generated by the element

$$\eta = \left[\left(\frac{x\sqrt{a}}{x}, \frac{y\sqrt{a}}{y}\right)\right] \in H^1_{\mathfrak{m}}(R)$$

corresponding to the relation above. Given $e \in \mathbb{N}$, let $p^e = 2k + 1$. Then

$$F^e(\eta) = a^k \eta,$$

which is a nonzero element of $H^1_{\mathfrak{m}}(R)$. Consequently $H^1_{\mathfrak{m}}(R)$ is not F-torsion, corresponding to that fact that $\operatorname{Spec} R_{\mathfrak{m}}$ is not formally geometrically connected.

The corollaries obtained in this section imply that over a separably closed field, a graded or complete local F-pure domain of dimension two is Cohen-Macaulay. We include an example to show that this is not true for rings of higher dimension.

Example 4.4. Let K be a field of characteristic $p > 0$, and take

$$A = K[x_1, \dots, x_d]/(x_1^d + \cdots + x_d^d)$$

where $d \geq 3$. Let R be the Segre product of A and the polynomial ring $B = K[s, t]$. Then $\dim R = d$ and the Künneth formula for local cohomology implies that

$$H^{d-1}_{\mathfrak{m}_R}(R) \;\cong\; [H^{d-1}_{\mathfrak{m}_A}(A)]_0 \otimes_K [B]_0 \;\cong\; K,$$

so R is not Cohen-Macaulay. If $p \equiv 1 \mod d$ then A is F-pure by [HR2, Proposition 5.21], hence $A \otimes_K B$ and its direct summand R are F-pure as well.

5. Appendix: F-torsion modules and F-stable vector spaces

Let R be a commutative ring containing a field K of characteristic $p > 0$. A *Frobenius action* on an R-module M is an additive map $F : M \longrightarrow M$ such that $F(rm) = r^p F(m)$ for all $r \in R$ and $m \in M$. In this case $\ker F$ is a submodule of M, and we have an ascending sequence of submodules of M,

$$\ker F \subseteq \ker F^2 \subseteq \ker F^3 \subseteq \cdots .$$

The union of these is the *F-nilpotent* submodule of M, denoted $M_{\mathrm{nil}} = \bigcup_{e\in\mathbb{N}} \ker F^e$. We say M is *F-torsion* if $M_{\mathrm{nil}} = M$.

The following proposition is proved in [HS] under the hypothesis that $(R, \mathfrak{m})$ has a perfect coefficient field, but the general case follows from this as we record below.

Proposition 5.1. *Let $(R, \mathfrak{m})$ be a local ring containing a field of positive characteristic, and let M be an Artinian R-module with a Frobenius action. Then there exists $e \in \mathbb{N}$ such that $F^e(M_{\mathrm{nil}}) = 0$.*

In particular, an Artinian module M is F-torsion if and only if $F^e(M) = 0$ for some $e \in \mathbb{N}$.

Proof. Since M is Artinian, it is also a module over $\widehat{R}$, so we may assume that $(R, \mathfrak{m})$ is a complete local ring. Now M is also a module over a power series ring mapping onto R, so there is no loss of generality in taking R to be the power series ring $R = K[[x_1, \ldots, x_d]]$. If K is perfect, the desired result is [HS, Proposition 1.11], and we shall derive the general case from this. Note that we may replace M by M_{nil} and assume that M is F-torsion.

Let $L = \varinjlim K_i$ be the perfect closure of K, where each K_i is a finite field extension of K in L. For each index i, the ring $L[[x_1, \ldots, x_d]]$ is a faithfully flat module over $K_i[[x_1, \ldots, x_d]]$, and hence $L[[x_1, \ldots, x_d]]$ is a faithfully flat module over $\varinjlim K_i[[x_1, \ldots, x_d]] \cong L \otimes_K R$. Since $L[[x_1, \ldots, x_d]]$ is Noetherian, it follows that $L \otimes_K R$ is Noetherian as well. Every element of $L \otimes_K M$ is killed by a power of the maximal ideal of $L \otimes_K R$, and $\mathrm{soc}(L \otimes_K M) = L \otimes_K (\mathrm{soc}\, M)$ is a finite dimensional L-vector space, so $L \otimes_K M$ is an Artinian module over $L \otimes_K R$ and hence also over its completion $L \widehat{\otimes}_K R \cong L[[x_1, \ldots, x_d]]$. Since $L \otimes_K M$ is F-torsion, [HS, Proposition 1.11] implies that there exists e such that $F^e(L \otimes_K M) = 0$, but then $F^e(M) = 0$ as well. $\square$

If R is a ring containing a perfect field K of positive characteristic and M is an R-module with a Frobenius action, then $F(M)$ is a K-vector space, and we have a descending sequence of K-vector spaces

$$F(M) \supseteq F^2(M) \supseteq F^3(M) \supseteq \cdots .$$

The *F-stable* part of M is the vector space $M_{\mathrm{st}} = \bigcap_{e\in\mathbb{N}} F^e(M)$.

Proposition 5.2. *Let $(R, \mathfrak{m}, K)$ be a local ring of dimension d which contains a field of positive characteristic.*

1. *$H^0_{\mathfrak{m}}(R)$ is F-torsion if and only if $d > 0$.*
2. *$H^d_{\mathfrak{m}}(R)$ is not F-torsion.*
3. *If $d = 0$ and K is perfect, then $H^0_{\mathfrak{m}}(R)_{\mathrm{st}} = R_{\mathrm{st}} = K$.*

Proof. (1) If $d = 0$ then $H^0_{\mathfrak{m}}(R) = R$, which is not F-torsion. If $d > 0$ then $H^0_{\mathfrak{m}}(R) \subset \mathfrak{m}$ and every element of $H^0_{\mathfrak{m}}(R)$ is nilpotent. (See also [Ly, Corollary 4.6(a)].)

(2) View $H^d_{\mathfrak{m}}(R)$ as the cohomology of a Čech complex on a system of parameters $x_1, \dots, x_d$ for R, and let $\eta = [1 + (x_1, \dots, x_d)] \in H^d_{\mathfrak{m}}(R)$. For all $e_0 \in \mathbb{N}$, the collection of elements $F^e(\eta)$ with $e > e_0$ generates $H^d_{\mathfrak{m}}(R)$ as an R-module. Hence $F^{e_0}(\eta)$ cannot be zero by Grothendieck's nonvanishing theorem.

(3) Since $\mathfrak{m}$ is nilpotent in this case, for integers $e \gg 0$ we have

$$F^e(H^0_{\mathfrak{m}}(R)) = F^e(R) = \{x^{p^e} : x \in R\} = \{(y+z)^{p^e} : y \in K, z \in \mathfrak{m}\} = K.$$

$\square$

Theorem 5.3. *Let $(R, \mathfrak{m})$ be a local ring with a perfect coefficient field K of positive characteristic. Let M be an Artinian R-module with a Frobenius action. Then M_{st} is a finite dimensional K-vector space, and $F : M_{\mathrm{st}} \longrightarrow M_{\mathrm{st}}$ is an automorphism of the Abelian group M_{st}.*

If K is algebraically closed, then there exists a K-basis $e_1, \dots, e_n$ for M_{st} such that $F(e_i) = e_i$ for all $1 \le i \le n$.

Proof. The finiteness result is [HS, Theorem 1.12], and the existence of the special basis when K is algebraically closed follows from [Di, Proposition 5, page 233]. $\square$

Theorem 5.4. [HS, Theorem 1.13] *Let $(R, \mathfrak{m})$ be a complete local ring with an algebraically closed coefficient field of positive characteristic. Let L, M, N be R-modules with Frobenius actions, such that we have a commutative diagram*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & L & \xrightarrow{\alpha} & M & \xrightarrow{\beta} & N & \longrightarrow & 0 \\
 & & {\scriptstyle F}\big\downarrow & & {\scriptstyle F}\big\downarrow & & {\scriptstyle F}\big\downarrow & & \\
0 & \longrightarrow & L & \xrightarrow{\alpha} & M & \xrightarrow{\beta} & N & \longrightarrow & 0
\end{array}$$

with exact rows. If L is Noetherian and N is Artinian, then the F-stable parts form a short exact sequence

$$0 \longrightarrow L_{\mathrm{st}} \longrightarrow M_{\mathrm{st}} \longrightarrow N_{\mathrm{st}} \longrightarrow 0.$$

Proposition 5.5. *Let $(R, \mathfrak{m}, K)$ be a complete local ring with an algebraically closed coefficient field of positive characteristic. Let $\mathfrak{n}$ denote the nilradical of R. Then for all $i \ge 0$, the natural map $H^i_{\mathfrak{m}}(R) \longrightarrow H^i_{\mathfrak{m}}(R/\mathfrak{n})$, when restricted to F-stable subspaces, gives an isomorphism*

$$H^i_{\mathfrak{m}}(R)_{\mathrm{st}} \xrightarrow{\cong} H^i_{\mathfrak{m}}(R/\mathfrak{n})_{\mathrm{st}}.$$

Proof. Let k an integer such that $\mathfrak{n}^{p^k} = 0$. The short exact sequence

$$0 \longrightarrow \mathfrak{n} \longrightarrow R \longrightarrow R/\mathfrak{n} \longrightarrow 0$$

induces a long exact sequence of local cohomology modules

$$\longrightarrow H^i_{\mathfrak{m}}(\mathfrak{n}) \xrightarrow{\alpha} H^i_{\mathfrak{m}}(R) \xrightarrow{\beta} H^i_{\mathfrak{m}}(R/\mathfrak{n}) \xrightarrow{\gamma} H^{i+1}_{\mathfrak{m}}(\mathfrak{n}) \longrightarrow .$$

Consider an element $\mu \in \ker(\beta) \cap H^i_{\mathfrak{m}}(R/\mathfrak{n})_{\mathrm{st}}$. Then $\mu \in \mathrm{image}(\alpha)$, so $F^k(\mu) = 0$. The Frobenius action on $H^i_{\mathfrak{m}}(R/\mathfrak{n})_{\mathrm{st}}$ is an automorphism so $\mu = 0$, and hence $H^i_{\mathfrak{m}}(R)_{\mathrm{st}} \longrightarrow H^i_{\mathfrak{m}}(R/\mathfrak{n})_{\mathrm{st}}$ is injective.

To complete the proof, by Theorem 5.3 it suffices to consider $\eta \in H^i_{\mathfrak{m}}(R/\mathfrak{n})_{\mathrm{st}}$ with $F(\eta) = \eta$ and prove that it lies in the image of $H^i_{\mathfrak{m}}(R)_{\mathrm{st}}$. Now $\gamma(\eta) \in H^{i+1}_{\mathfrak{m}}(\mathfrak{n})$ so $F^k(\gamma(\eta)) = 0$, and therefore $F^k(\eta) = \eta \in \ker(\gamma)$.

Let $\eta = \beta(\mu)$ for some element $\mu \in H^i_\mathfrak{m}(R)$. Then $\beta(F(\mu)-\mu) = 0$ which implies that $F(\mu) - \mu \in \operatorname{image}(\alpha)$. Consequently $F^k(F(\mu) - \mu) = 0$, which shows that $F^{k+1}(\mu) = F^k(\mu) \in H^i_\mathfrak{m}(R)_{\mathrm{st}}$. Since

$$\beta(F^k(\mu)) = F^k(\beta(\mu)) = F^k(\eta) = \eta,$$

we are done. □

References

[Di] J. Dieudonné, *Lie groups and Lie hyperalgebras over a field of characteristic $p > 0$. II*, Amer. J. Math. **77** (1955), 218–244.

[Fa] G. Faltings, *Über lokale Kohomologiegruppen hoher Ordnung*, J. Reine Angew. Math. **313** (1980), 43–51.

[GW] S. Goto and K.-i. Watanabe, *The structure of one-dimensional F-pure rings*, J. Algebra **49** (1977), 415–421.

[HaN] N. Hara, *Classification of two-dimensional F-regular and F-pure singularities* Adv. Math. **133** (1998), 33–53.

[HaR] R. Hartshorne, *Cohomological dimension of algebraic varieties*, Ann. of Math. (2) **88** (1968), 403–450.

[HS] R. Hartshorne and R. Speiser, *Local cohomological dimension in characteristic p*, Ann. of Math. (2) **105** (1977), 45–79.

[HR1] M. Hochster and J. Roberts, *Rings of invariants of reductive groups acting on regular rings are Cohen-Macaulay*, Advances in Math. **13** (1974), 115–175.

[HR2] M. Hochster and J. Roberts, *The purity of the Frobenius and local cohomology*, Advances in Math. **21** (1976), 117–172.

[HL] C. Huneke and G. Lyubeznik, *On the vanishing of local cohomology modules*, Invent. Math. **102** (1990), 73–93.

[Ly] G. Lyubeznik, *On the vanishing of local cohomology in characteristic $p > 0$*, Preprint (2004).

[MS] V. B. Mehta and V. Srinivas, *Normal F-pure surface singularities*, J. Algebra **143** (1991), 130–143.

[Og] A. Ogus, *Local cohomological dimension of algebraic varieties*, Ann. of Math. (2) **98** (1973), 327–365.

[PS] C. Peskine and L. Szpiro, *Dimension projective finie et cohomologie locale*, Inst. Hautes Études Sci. Publ. Math. **42** (1973), 47–119.

[Ra] M. Raynaud, *Anneaux locaux henséliens*, Lecture Notes in Mathematics, Vol. 169, Springer-Verlag, Berlin-New York, 1970.

[Wa1] K.-i. Watanabe, *Study of F-purity in dimension two*, in: Algebraic Geometry and Commutative Algebra in honor of Masayoshi Nagata, Vol. II, Kinokuniya, Tokyo, 1988, 791–800.

[Wa2] K.-i. Watanabe, *F-regular and F-pure normal graded rings*, J. Pure Appl. Algebra **71** (1991), 341–350.

School of Mathematics, Georgia Institute of Technology, Atlanta, GA 30332.
E-mail address: `singh@math.gatech.edu`

Department of Mathematics, Purdue University, West Lafayette, IN 47907.
E-mail address: `walther@math.purdue.edu`