HOMOLOGICAL METHODS FOR HYPERGEOMETRIC FAMILIES

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ABSTRACT. We analyze the behavior of the holonomic rank in families of holonomic systems over complex algebraic varieties by providing homological criteria for rank-jumps in this general setting. Then we investigate rank-jump behavior for hypergeometric systems $H_A(\beta)$ arising from a $d \times n$ integer matrix A and a parameter $\beta \in \mathbb{C}^d$. To do so we introduce an Euler–Koszul functor for hypergeometric families over $\mathbb{C}^d$, whose homology generalizes the notion of a hypergeometric system, and we prove a homology isomorphism with our general homological construction above. We show that a parameter $\beta \in \mathbb{C}^d$ is rank-jumping for $H_A(\beta)$ if and only if β lies in the Zariski closure of the set of $\mathbb{Z}^d$-graded degrees α where the local cohomology $\bigoplus_{i<d} H^i_{\mathfrak{m}}(\mathbb{C}[\mathbb{N}A])_\alpha$ of the semigroup ring $\mathbb{C}[\mathbb{N}A]$ supported at its maximal graded ideal $\mathfrak{m}$ is nonzero. Consequently, $H_A(\beta)$ has no rank-jumps over $\mathbb{C}^d$ if and only if $\mathbb{C}[\mathbb{N}A]$ is Cohen–Macaulay of dimension d.

CONTENTS

1. INTRODUCTION

In the late 1980s, Gelfand, Graev and Zelevinsky introduced a class of systems of linear partial differential equations closely related to toric varieties [GGZ87]. These systems, now called *GKZ systems*, or *A-hypergeometric systems*, are constructed from a $d \times n$ integer matrix A of rank d and a complex parameter vector $\beta \in \mathbb{C}^d$, and are denoted by $H_A(\beta)$. Although A-hypergeometric systems were originally introduced with a view toward representation theory, they are well suited to algebraic geometry in various applications. For example, solutions of A-hypergeometric systems appear as toric residues and as the generating functions for intersection numbers on moduli spaces of curves. In another instance,

Date: 17 June 2004.
LFM was partially supported by postdoctoral fellowships from MSRI and the NSF.
EM was partially supported by the NSF.
UW was partially supported by the NSF, the DfG and the Humboldt foundation.

the Picard–Fuchs equations governing the variation of Hodge structures for Calabi–Yau toric hypersurfaces are very special cases of A-hypergeometric systems.

The first fundamental results about the systems $H_A(\beta)$ were proved by Gelfand, Graev, Kapranov, and Zelevinsky. These results concerned the case where the semigroup $\mathbb{N}A$ generated by the columns of A gives rise to a semigroup ring $\mathbb{C}[\mathbb{N}A]$ that is Cohen–Macaulay and graded in the standard $\mathbb{Z}$-grading [GGZ87, GKZ89]. In geometric terms, the associated toric variety X_A is projective and arithmetically Cohen–Macaulay. The above authors showed that in this case, the system $H_A(\beta)$ gives a holonomic module over the ring D of polynomial $\mathbb{C}$-linear differential operators in n variables, and hence $H_A(\beta)$ has finite *rank*; that is, the dimension of its space of holomorphic solutions is finite. Furthermore, they showed that this dimension can be expressed combinatorially, as the integer $\mathrm{vol}(A)$ that is $d!$ times the Euclidean volume of the convex hull of the columns of A and the origin $0 \in \mathbb{Z}^d$. The remarkable fact is that their rank formula held independently of the parameter β.

Even if $\mathbb{C}[\mathbb{N}A]$ is not Cohen–Macaulay or $\mathbb{Z}$-graded, Adolphson showed in [Ado94] that $H_A(\beta)$ is always a holonomic ideal. He further proved that, for all parameters outside of a closed locally finite arrangement of countably many 'semi-resonant' affine hyperplanes, the characterization of rank through volume is still correct.

It came as quite a surprise when in [ST98] an example was given showing that if $\mathbb{C}[\mathbb{N}A]$ is not Cohen–Macaulay then not all parameters β have to give the same rank. One is hence prompted to introduce $\mathcal{E}_A$, the collection of those *exceptional parameters* $\beta \in \mathbb{C}^d$ for which the rank does not take the expected value. Nearly at the same time the case of projective toric curves was discussed completely in [CDD99]: the set $\mathcal{E}_A$ of exceptional parameters is finite in this case, and empty precisely when $\mathbb{C}[\mathbb{N}A]$ is Cohen–Macaulay; moreover, at each $\beta \in \mathcal{E}_A$ the rank exceeds the volume by exactly 1. It was shown soon after in [SST00] that the rank can never be smaller than the volume as long as $\mathbb{C}[\mathbb{N}A]$ is $\mathbb{Z}$-graded, and it was established in the same book that $\mathcal{E}_A$ is in fact contained in a finite affine subspace arrangement. More recently the much stronger fact emerged that $\mathcal{E}_A$ is a finite union of Zariski locally closed sets by means of Gröbner basis techniques [Mat03]. While rank-jumps can be arbitrarily large [MW04], the absence of rank-jumping parameters is equivalent to the Cohen–Macaulay property for $\mathbb{Z}$-graded $\mathbb{C}[\mathbb{N}A]$ when either $\mathbb{C}[\mathbb{N}A]$ has codimension two [Mat01], or if the convex hull of A is a simplex [Sai02], or if $\mathbb{C}[\mathbb{N}A]$ is a polynomial ring modulo a generic toric ideal [Mat03].

Encouraged by these results, which suggest an algebraic structure on the set of exceptional parameters, it was conjectured in [MM04] that the obstructions to the Cohen–Macaulayness of $\mathbb{C}[\mathbb{N}A]$ and the set of exceptional parameters are identified in an explicit manner. To be precise, let $H^{<d}_{\mathfrak{m}}(\mathbb{C}[\mathbb{N}A])$ be the direct sum of all the local cohomology modules supported at the maximal homogeneous ideal $\mathfrak{m}$ of $\mathbb{C}[\mathbb{N}A]$ in cohomological degrees less than d. Then define the set E_A of *exceptional quasi-degrees* to be the Zariski closure in $\mathbb{C}^d$ of the set of $\mathbb{Z}^d$-graded degrees α such that $H^{<d}_{\mathfrak{m}}(\mathbb{C}[\mathbb{N}A])$ has a nonzero element in degree $-\alpha$. With this notation, the motivating result in this article is the following.

Theorem 1.1. *For any rank d matrix $A \in \mathbb{Z}^{d\times n}$ the set $\mathcal{E}_A$ of exceptional (that is, rank-jumping) parameters is identical to the set E_A of exceptional quasi-degrees.*

We note that there is no assumption on $\mathbb{C}[\mathbb{N}A]$ being $\mathbb{Z}$-graded. The $\mathbb{Z}$-graded simplicial case of this result was proved in [MM04] using results of [Sai02].

Methods and results. The first step in our proof of Theorem 1.1 is to construct a homological theory to systematically detect rank-jumps. In Sections 2 and 3 we study rank variation in any family of holonomic modules over any base B, and not just A-hypergeometric families over $B = \mathbb{C}^d$. The idea is that under a suitable coherence assumption (Definition 2.1), holonomic ranks behave like fiber dimensions in families of algebraic varieties. In particular, rank is constant almost everywhere and can only increase on closed subsets of B (Theorem 2.6). We develop a computational tool to check for rank-jumps at a smooth point $\beta \in B$: since the rank-jump occurs through a failure of flatness at β, ordinary Koszul homology detects it (Theorem 3.1 and Corollary 3.3).

The second step toward Theorem 1.1 is to construct a homological theory for D-modules that reproduces the set E_A of exceptional quasi-degrees, which a priori arises from the commutative notion of local cohomology. Our main observation along these lines is that the *Euler–Koszul complex*, which was already known to Gelfand, Kapranov, and Zelevinsky for Cohen–Macaulay $\mathbb{Z}$-graded semigroup rings [GKZ89], generalizes to fill this need. Adolphson [Ado99] recognized that when the semigroup is not Cohen–Macaulay, certain conditions guarantee that this complex has zero homology. Here, we develop *Euler–Koszul homology* (Definition 4.2) for the class of *toric modules* (Definition 4.5), which are slight generalizations of $\mathbb{Z}^d$-graded modules over the semigroup ring $\mathbb{C}[\mathbb{N}A]$. For any toric module M, we show in Theorem 6.6 that the set of parameters β for which the Euler–Koszul complex has nonzero higher homology is precisely the analogue for M of the exceptional quasi-degree set E_A defined above for $M = \mathbb{C}[\mathbb{N}A]$.

Having now two cohomology theories, one to recover the local cohomology quasi-degrees for hypergeometric families, and another to detect rank-jumping parameters for general holonomic families, we link them in a central result of this article, Theorem 8.2: for toric modules, these two theories coincide. Consequently, we obtain Theorem 1.1 as the special case $M = \mathbb{C}[\mathbb{N}A]$ of Theorem 9.1, which holds for arbitrary toric modules M. We deduce in Corollary 9.2 the equivalence of the Cohen–Macaulay condition on $\mathbb{C}[\mathbb{N}A]$ with the absence of rank-jumps in the GKZ hypergeometric system $H_A(\beta)$.

As a final comment, let us note that we avoid the explicit computation of solutions to hypergeometric systems. This contrasts with the previously cited constructions of exceptional parameters, which rely on combinatorial techniques to produce solutions. It is for this reason that these constructions contained the assumption that the semigroup ring $\mathbb{C}[\mathbb{N}A]$ is graded in the usual $\mathbb{Z}$-grading, for this implies that the corresponding hypergeometric systems are *regular holonomic* and thus have solutions expressible as power series with logarithms, with all the combinatorial control this provides. Our use of homological techniques makes the results in this article valid in both the regular and non-regular cases.

Acknowledgments. The equivalence of hypergeometric rank constancy and toric Cohen–Macaulayness was first conjectured by Bernd Sturmfels, who divulged it in talks and open-problem sessions. He was also the first to mention local cohomology in connection with rank-jumps, although not in the precise form that appears in this article. His encouragement and advice have been invaluable to us throughout this project.

The occasions when all three authors of this article were in the same place at the same time have been rare, and all the more appreciated for that. We would like to thank the organizers of the Joint International Meeting of the AMS and RSME in Sevilla, Spain, for providing us with one such opportunity in the Summer of 2003. LFM and UW are especially grateful to Francisco Castro-Jiménez, Jose María Ucha, and María Isabel Hartillo Hermoso, who were our gracious hosts when we stayed in Sevilla for a week after that conference. All three authors also intersected at MSRI. We thank this institute for the friendly and stimulating research atmosphere it provides.

2. Upper semi-continuity of rank in holonomic families

The results in the first part of this article (Sections 2 and 3) deal with general modules over the Weyl algebra, without restricting to the hypergeometric realm. In this section we define the notion of a *holonomic family* of D-modules, and show that the holonomic rank constitutes an upper semi-continuous function on such a family.

Throughout this article, $\partial = \partial_1, \ldots, \partial_n$ refers to the partial derivation operators with respect to the variables $x = x_1, \ldots, x_n$. Writing δ_{ij} for the Kronecker delta, so that $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ otherwise, the Weyl algebra D is the quotient of the free associative $\mathbb{C}$-algebra on $\{x_i, \partial_i\}_{i=1}^n$ by the two-sided ideal $\langle x_i x_j - x_j x_i,\ \partial_i\partial_j - \partial_j\partial_i,\ \partial_i x_j - x_j\partial_i - \delta_{ij}\rangle_{i,j=1}^n$.

Every left D-module is also a module over the commutative subalgebra $\mathbb{C}[x]$ of D. If N is any $\mathbb{C}[x]$-module or sheaf of $\mathbb{C}[x]$-modules on some space, let $N(x) = \mathbb{C}(x) \otimes_{\mathbb{C}[x]} N$ denote the localization by inverting all polynomials in the x-variables. We shall also have occasion to consider modules, sheaves of modules, and schemes defined over $\mathbb{C}$, and if N is such an object then $N(x)$ denotes its base extension to the field $\mathbb{C}(x)$ of rational functions.

Our focus is on *holonomic* D-modules, which are by definition finitely generated left D-modules N such that $\mathrm{Ext}^j_D(N, D) = 0$ whenever $j \neq n$. The holonomic modules form a full subcategory of the category of D-modules that is closed under taking extensions, submodules, and quotient modules. When N is a holonomic module, the $\mathbb{C}(x)$-vector space $N(x)$ has finite dimension, and this dimension equals the *holonomic rank* $\mathrm{rank}(N)$ by a celebrated theorem of Kashiwara, see [SST00, Thm. 1.4.19, Cor. 1.4.14]. We note that rank is hence additive in short exact sequences of holonomic modules.

We are interested in families of D-modules parameterized by a Noetherian complex algebraic variety B with structure sheaf $\mathcal{O}_B$. If $\beta \in B$ we denote by $\mathfrak{p}_\beta$ the prime ideal (sheaf) of β, and by κ_β the residue field of the stalk $\mathcal{O}_{B,\beta}$, so $\kappa_\beta = \mathcal{O}_{B,\beta}/\mathfrak{p}_\beta\mathcal{O}_{B,\beta}$.

Consider the sheaf $D \otimes_{\mathbb{C}} \mathcal{O}_B$ of noncommutative $\mathcal{O}_B$-algebras on B. By a *coherent* sheaf of left $(D \otimes_{\mathbb{C}} \mathcal{O}_B)$-modules we mean a quasi-coherent sheaf of $\mathcal{O}_B$-modules on B whose sections over each open affine subset $U \subseteq B$ are finitely generated over the ring of global sections of $D \otimes_{\mathbb{C}} \mathcal{O}_U$. The sheaf $D \otimes_{\mathbb{C}} \mathcal{O}_B$ contains the subsheaf $\mathcal{O}_B[x] = \mathbb{C}[x] \otimes_{\mathbb{C}} \mathcal{O}_B$ of commutative polynomials over $\mathcal{O}_B$, whose localization at $\langle 0\rangle \in \mathrm{Spec}(\mathbb{C}[x])$ is by our conventions $\mathcal{O}_B(x)$. The sheaf-spectrum of $\mathcal{O}_B(x)$ is the base-extended scheme $B(x) = \mathrm{Spec}\,\mathbb{C}(x) \times_{\mathrm{Spec}\,\mathbb{C}} B$, which can be considered as a scheme over $\mathbb{C}(x)$ or as fibered over B.

Definition 2.1. A *holonomic family* over a complex scheme B is a coherent sheaf $\tilde{\mathcal{M}}$ of left $(D \otimes_{\mathbb{C}} \mathcal{O}_B)$-modules whose fibers $\mathcal{M}_\beta = \tilde{\mathcal{M}} \otimes_{\mathcal{O}_B} \kappa_\beta$ are holonomic D-modules for all $\beta \in B$, and whose *rank sheaf* $\tilde{\mathcal{M}}(x)$ is coherent on the scheme $B(x)$.

Note that every holonomic family over B is generated by its global sections if B is affine.

Example 2.2. When B equals the complex vector space $\mathbb{C}^d$ of dimension d, the structure sheaf $\mathcal{O}_B$ has global sections $\mathbb{C}[b]$, the commutative polynomial ring in variables

$b = b_1, \ldots, b_d$. A holonomic family $\tilde{\mathcal{M}}$ over $\mathbb{C}^d$ can be represented by its global sections, a left $D[b]$-module $\mathcal{M}$ with finitely many generators and relations. In order to be a holonomic family in our sense, $\mathcal{M}_\beta = \mathcal{M} \otimes_{\mathcal{O}_B} \kappa_\beta$ has to be a holonomic D-module for all $\beta \in \mathbb{C}^d$ while the global section module $\mathcal{M}(x) = \mathbb{C}(x) \otimes_{\mathbb{C}[x]} \mathcal{M}$ of the rank sheaf has to be a finitely generated $\mathbb{C}[b](x)$-module.

The rank sheaf $\tilde{\mathcal{M}}(x)$ is a sheaf of $\mathcal{O}_B(x)$-modules on $B(x)$, but ignoring the process of pushing $\tilde{\mathcal{M}}(x)$ down to B we abuse notation and speak of the fiber of $\tilde{\mathcal{M}}(x)$ over $\beta \in B$. Let $B_\mathbb{C} \subseteq B$ be the $\mathbb{C}$-valued closed points of B (that is, points with residue field $\mathbb{C}$).

Proposition 2.3. *For a holonomic family $\tilde{\mathcal{M}}$ over a scheme B, the fiber of $\tilde{\mathcal{M}}(x)$ over each point $\beta \in B$ is $\mathcal{M}_\beta(x)$, which is a $\mathbb{C}(x)$-vector space of dimension* $\mathrm{rank}(\mathcal{M}_\beta)$ *if $\beta \in B_\mathbb{C}$.*

Proof. Tensoring with the rational functions $\mathbb{C}(x)$ over $\mathbb{C}[x]$ commutes with the passage from $\tilde{\mathcal{M}}$ to $\mathcal{M}_\beta = \tilde{\mathcal{M}} \otimes_{\mathcal{O}_B} \kappa_\beta$. □

The next lemma says that Zariski closed subsets of the base-extended scheme $B(x)$ descend to Zariski closed subsets of the original scheme B. We use the term *prime ideal* to refer to the kernel of the morphism of structure sheaves associated to any map from the spectrum of a field to B.

Lemma 2.4. *For a scheme B defined over $\mathbb{C}$, the map taking each prime ideal $\mathfrak{p} \subset \mathcal{O}_B$ to its extension $\mathfrak{p}\mathcal{O}_B(x)$ constitutes a continuous injection $B \to B(x)$ of topological spaces.*

Proof. Note first that if $\mathfrak{p} \subset \mathcal{O}_B$ is prime, then the extension $\mathfrak{p}\mathcal{O}_B(x)$ is prime. The lemma is equivalent to saying that, for every subset $Y \subseteq B(x)$ that is Zariski closed, the set of points $\beta \in B$ whose prime ideal sheaves $\mathfrak{p}_\beta$ in $\mathcal{O}_B$ extend to prime ideals $\mathfrak{p}_\beta\mathcal{O}_B(x) \in Y$ is Zariski closed in B. This statement is local on B, so we may assume that B is affine.

Suppose that Y is the variety of a set $\mathcal{F}$ of global sections of $\mathcal{O}_B(x)$. Any free $\mathbb{C}$-basis for $\mathbb{C}(x)$ is also a free $\mathcal{O}_B$-basis for $\mathcal{O}_B(x)$; choose such a basis. For each global section $f \in \mathcal{F}$, let $\mathcal{G}_f \subseteq \mathcal{O}_B$ be its set of nonzero coefficients in the chosen basis. Then a prime $\mathfrak{p} \subset \mathcal{O}_B$ satisfies $\mathfrak{p}\mathcal{O}_B(x) \in Y$ if and only if $\mathfrak{p}$ contains the sets $\mathcal{G}_f$ for all $f \in \mathcal{F}$. □

Example 2.5. Lemma 2.4 does not say that B is closed in $B(x)$; indeed, the closure of the image is all of $B(x)$, whereas there are always points in $B(x) \smallsetminus B$. Neither does Lemma 2.4 say that the morphism $B(x) \to B$ takes closed sets to closed sets. For example, if $n = d = 1$ in Example 2.2, then the variety of the ideal $\langle b - x \rangle \subset \mathbb{C}[b](x)$ is a Zariski closed point in $B(x)$, whereas the image of this point in $B = \mathbb{C}^1$ is the generic point. Note that the set of prime ideals $\mathfrak{p} \subset \mathbb{C}[b]$ whose extensions $\mathfrak{p}\mathbb{C}[b](x)$ contain $\langle b - x \rangle$ is empty, and therefore Zariski closed.

We now come to the main result of this section. To state it, we call $\beta \in B$ *rank-jumping* if (i) $\beta \in B_\mathbb{C}$ is a $\mathbb{C}$-valued point, and (ii) for all open sets U containing β, the rank of $\mathcal{M}_\beta$ is strictly greater than the minimal holonomic rank attained by any fiber $\mathcal{M}_{\beta'}$ for $\beta' \in U_\mathbb{C}$.

Theorem 2.6. *If $\mathcal{M}$ is a holonomic family over a scheme B, then the function $\beta \mapsto$* $\mathrm{rank}(\mathcal{M}_\beta)$ *is upper semi-continuous on both B and on $B_\mathbb{C}$ (endowed with the subspace topology). In particular, the locus of rank-jumping points $\beta \in B_\mathbb{C}$ is closed in $B_\mathbb{C}$.*

Proof. The function sending each point $\beta(x) \in B(x)$ to the $\mathbb{C}(x)$-dimension of the fiber $\tilde{\mathcal{M}}(x) \otimes \kappa_{\beta(x)}$ is upper-semicontinuous on $B(x)$ because $\tilde{\mathcal{M}}(x)$ is coherent on $B(x)$; see [Har77, III.12.7.2]. Therefore, given an integer i, the subset of points in $B(x)$ on which this fiber dimension is at least i is Zariski closed. Lemma 2.4 shows that the corresponding subset of points in B is Zariski closed in B. Proposition 2.3, which says that the fiber dimensions over $\beta \in B_{\mathbb{C}}$ are holonomic ranks, completes the proof. □

Remark 2.7. Without the coherence hypothesis on $\tilde{\mathcal{M}}(x)$ over $B(x)$, the conclusion of Theorem 2.6 can be false, even if B is of finite type over $\mathbb{C}$ and all of the fibers $\mathcal{M}_\beta$ for $\beta \in B_{\mathbb{C}}$ are holonomic. For an example, consider the setup in Example 2.2 with $n = d = 1$, and take $\mathcal{M} = D[b]/\langle b\partial - 1\rangle$. When $\beta \neq \langle b - 0\rangle$, the fiber over β is the rank 1 holonomic module corresponding to the solution $x^{1/\beta}$. But when $\beta = \langle b - 0\rangle$ the fiber of $\tilde{\mathcal{M}}$ is zero. Hence the rank actually drops on the closed subset $\{0\} \subset \mathbb{C}^1$. See Example 3.5 for further details.

Remark 2.8. Fix a holonomic family $\tilde{\mathcal{M}}$. The semicontinuity of holonomic rank in Theorem 2.6 suggests that a "solution sheaf" $\mathcal{S}ol(\tilde{\mathcal{M}})$, constructed as below, might be an algebraic coherent sheaf on B; a priori, it can only be expected to be an analytic sheaf. See Remark 9.5 for further comments on this issue, as it relates to hypergeometric systems.

For the construction, suppose there is a vector $v \in \mathbb{C}^n$ such that the singular locus of $\mathcal{M}_\beta$ does not contain v for any $\beta \in B_{\mathbb{C}}$ (this is certainly possible locally on B). Let the point v have maximal ideal $\langle x_1 - v_1, \ldots, x_n - v_n\rangle$ in $\mathbb{C}[x]$. The D-module restriction of $\tilde{\mathcal{M}}_\beta$ to v is the derived tensor product over D with the right module $D/\langle x_1 - v_1, \ldots, x_n - v_n\rangle D$. This restriction is naturally dual to the space of formal power series solutions of $\tilde{\mathcal{M}}_\beta$ at v. But as v is a regular point, formal and convergent solutions of $\tilde{\mathcal{M}}_\beta$ are identical. Since tensor products commute in x and b, restricting the family $\tilde{\mathcal{M}}$ gives rise to an $\mathcal{O}_B$-module whose fiber over each point $\beta \in B_{\mathbb{C}}$ is naturally dual to the solution space of $\tilde{\mathcal{M}}_\beta$. Taking duals therefore yields the analytic $\mathcal{O}_B$-module $\mathcal{S}ol(\tilde{\mathcal{M}})$.

3. Rank-jumps as failures of flatness

For applications to hypergeometric systems, we need some concrete criteria to help us apply the results of the previous section. First we characterize rank-jumps in holonomic families over reduced schemes using Koszul homology in a commutative setting (Corollary 3.3). Then we provide a criterion for a family of D-modules to satisfy the coherence property required of a holonomic family (Proposition 3.4); we will appeal to it in Section 7.

In the following statement, we use the standard notion of *reduced* for a scheme to mean that the coordinate rings of its affine open subschemes have no nilpotent elements.

Theorem 3.1. *Let $\tilde{\mathcal{M}}$ be a holonomic family over a reduced scheme B of finite type over $\mathbb{C}$, and fix a $\mathbb{C}$-valued closed point $\beta \in B_{\mathbb{C}}$. The parameter β is rank-jumping if and only if $\operatorname{Tor}_i^{\mathcal{O}_B}(\tilde{\mathcal{M}}(x), \kappa_\beta)$ is nonzero for some homological degree $i > 0$.*

Proof. Since B is of finite type, the set $B_{\mathbb{C}}$ is dense in B via the inclusion of Lemma 2.4. As B is always dense in $B(x)$, we deduce that $B_{\mathbb{C}}$ is also dense in $B(x)$. Therefore

$$(*) \qquad \min_{\beta \in B_{\mathbb{C}}}\{\operatorname{rank}(\mathcal{M}_\beta)\} \;=\; \min_{\beta \in B(x)}\{\dim_{\mathbb{C}(x)} \tilde{\mathcal{M}}_\beta(x)\}$$

by Proposition 2.3 and the upper-semicontinuity of fiber dimension [Har77, III.12.7.2]. By [Har77, Exercise II.5.8], the points $\beta(x) \in B(x)$ over which the fiber dimension equals the quantity given in $(*)$ coincide with the points $\beta(x)$ near which $\tilde{\mathcal{M}}(x)$ is locally free. But local freeness and flatness agree for coherent sheaves by [Har77, Proposition III.9.2]. Flatness in turn is characterized by the vanishing of $\operatorname{Tor}_i^{\mathcal{O}_{B(x)}}(\tilde{\mathcal{M}}(x), \kappa_{\beta(x)})$ for all $i > 0$ [Eis95, Theorem 6.8], and for $\beta \in B_{\mathbb{C}}$, $\operatorname{Tor}_i^{\mathcal{O}_{B(x)}}(\tilde{\mathcal{M}}(x), \kappa_{\beta(x)}) \cong \operatorname{Tor}_i^{\mathcal{O}_B}(\tilde{\mathcal{M}}(x), \kappa_\beta)$. □

Remark 3.2. The result of Theorem 3.1 is false if B is not reduced. Consider for example $B = \operatorname{Spec}(\mathbb{C}[b]/\langle b^2 \rangle)$, and let $\tilde{\mathcal{M}}$ be the sheaf induced by the module $D[b]/\langle \partial, b \rangle$. Since there is only one point in B, there is no possibility of a rank-jump. On the other hand, $\operatorname{Tor}_i^{\mathcal{O}_B}(\tilde{\mathcal{M}}(x), \kappa_\beta) = D(x)/\langle \partial \rangle$ for all i.

Theorem 3.1 has an interpretation via Koszul homology. For notation, suppose T is a commutative ring. For a sequence $\mathbf{y} = y_1, \dots, y_d$ of elements in T, we write the Koszul complex $K_\bullet(\mathbf{y})$ [BH93, Chapter 1.6] with lowered indices decreasing from d to 0. For any T-module N set $K_\bullet(\mathbf{y}; N) = K_\bullet(\mathbf{y}) \otimes_T N$, and abbreviate the i^{th} Koszul homology of N as $H_i(\mathbf{y}; N) = H_i(K_\bullet(\mathbf{y}; N))$. Call $\mathbf{y}$ a *regular sequence* in T if $T/\langle \mathbf{y} \rangle$ is a nonzero module and y_i is a non-zerodivisor on $T/\langle y_1, \dots, y_{i-1} \rangle$ for $i = 1, \dots, d$. Koszul cohomology for regular sequences gives Tor groups [Eis95, Exercise 17.10], so we get the following.

Corollary 3.3. *Let $\mathcal{M}$ be the global sections of a holonomic family over a reduced affine variety B with finitely generated coordinate ring $\mathbb{C}[B]$. If a regular sequence $\mathbf{y}$ in $\mathbb{C}[B]$ generates the ideal of a $\mathbb{C}$-valued closed point $\beta \in B_{\mathbb{C}}$, then β is rank-jumping if and only if the Koszul complex $K_\bullet(\mathbf{y}; \mathcal{M}(x))$ has nonzero homology $H_i(\mathbf{y}; \mathcal{M}(x))$ for some $i > 0$.* □

Note that $K_\bullet(\mathbf{y}; \mathcal{M}(x))$ can be obtained either by tensoring $K_\bullet(\mathbf{y})$ with $\mathcal{M}(x)$ over $\mathbb{C}[B]$, or by viewing $\mathbf{y}$ as a sequence in $\mathbb{C}[B](x)$ and tensoring over $\mathbb{C}[B](x)$.

We now turn to the coherence criterion. We refer to [Bjö79, SST00] for more information about filtrations on D-modules and their associated graded objects. On D define the *order filtration* by taking its k^{th} level to be the vector space of all expressions $\sum_\nu p_\nu(x)\partial^\nu$ in which the monomials ∂^ν for $\nu \in \mathbb{N}^n$ have total degree $|\nu| \leq k$. Note that each filtration level is a finitely generated $\mathbb{C}[x]$-module. The associated graded object is the commutative polynomial ring $\operatorname{gr}(D) = \mathbb{C}[x, \xi]$ in the variables $x = x_1, \dots, x_n$ and $\xi = \xi_1, \dots, \xi_n$.

The order filtration extends to any free module D^r with a fixed basis, and to left submodules $K \subseteq D^r$, by letting the k^{th} level be spanned by elements whose coefficients in the given basis have total degree at most k in ∂. Given a presentation $N = D^r/K$, the left module N has an induced filtration with associated graded module $\operatorname{gr}(N) = \operatorname{gr}(D^r)/\operatorname{gr}(K)$. This naturally graded $\operatorname{gr}(D)$-module depends on the choice of presentation.

Note that since $\mathbb{C}(x)$ is flat over $\mathbb{C}[x]$, the module $\operatorname{gr}(N)(x)$ is isomorphic to $\operatorname{gr}(N(x))$ if we extend the filtration above in the obvious way to $N(x) = (D^r/K) \otimes_{\mathbb{C}[x]} \mathbb{C}(x)$. It is a fundamental fact of D-module theory, going back to Kashiwara, that for a holonomic D-module N, the number $\dim_{\mathbb{C}(x)}(\operatorname{gr}(N)(x))$ is independent of the presentation of N and equals the holonomic rank of N [SST00, Theorem 1.4.19 and Definition 1.4.8].

Our primary use of order filtrations will be on *families* of D-modules over an affine scheme B—that is, on left modules over the ring $D[B] = D \otimes_{\mathbb{C}} \mathbb{C}[B]$. The order filtration generalizes naturally to a filtration on $D[B]$ with the property that each level is a finitely generated $\mathbb{C}[B][x]$-module. The associated graded ring of $D[B]$ is the polynomial ring $\mathbb{C}[B][x, \xi]$ in $2n$ variables over the coordinate ring $\mathbb{C}[B]$. A choice of presentation for a $D[B]$-module $\mathcal{M}$ determines an associated graded $\mathbb{C}[B][x, \xi]$-module $\mathrm{gr}(\mathcal{M})$. The formation of such associated graded structures commutes with the tensor product with the flat $\mathbb{C}[x]$-module $\mathbb{C}(x)$.

Proposition 3.4. *For an affine scheme B and a finitely generated $D[B]$-module $\mathcal{M}$, the module $\mathcal{M}(x)$ is finitely generated over $\mathbb{C}[B](x)$ provided that $(\mathrm{gr}(\mathcal{M}))(x)$ is.*

Proof. As B is affine, the modules $\mathcal{M}$, $\mathcal{M}(x)$ and $\mathrm{gr}(\mathcal{M}(x))$ are generated by global sections. Any lift of set of generators for the $\mathbb{C}[B](x)$-module $(\mathrm{gr}\,\mathcal{M})(x)$ is a set of generators for $\mathcal{M}(x)$, because the order filtration of $\mathcal{M}(x)$ does not descend infinitely. □

We will apply the above result when all fibers $\mathcal{M}_\beta$ for $\beta \in B_\mathbb{C}$ are holonomic D-modules to conclude that $\mathcal{M}$ constitutes (the global sections of) a holonomic family over B. It should be pointed out that the finiteness condition on $\mathcal{M}$ is necessary: even if the fibers $\mathcal{M}_\beta$ are holonomic, the fiber $(\mathrm{gr}\,\mathcal{M})(x)_\beta$ of the associated graded module $(\mathrm{gr}\,\mathcal{M})(x)$ over $\beta \in B_\mathbb{C}$ is a $\mathbb{C}(x)$-vector space whose dimension need not be equal to $\mathrm{rank}(\mathcal{M}_\beta)$.

Example 3.5. Continue with $\mathcal{M} = D[b]/\langle b\partial - 1\rangle$ as in Remark 2.7. Let $\mathcal{M}(x)_0 \subseteq \mathcal{M}(x)_1 \subseteq \cdots$ denote the order filtration of $\mathcal{M}(x)$. Then $\mathcal{M}(x)_1 = \mathcal{M}(x)$ locally near every parameter $\beta \in \mathbb{C}$ except for $\beta = 0$. At $\beta = 0$, in contrast, $\mathcal{M}(x)_k/\mathcal{M}(x)_{k-1}$ is minimally generated by ξ^k, even though $\mathrm{rank}(\mathcal{M}_0) = 0$. In fact, $(\mathrm{gr}\,\mathcal{M})(x)$ is the direct sum of the rank 1 free module $\mathbb{C}[b](x)$ with $\bigoplus_{k\geq 1} \mathbb{C}[b](x)/\langle b\rangle$.

4. EULER–KOSZUL HOMOLOGY OF TORIC MODULES

In this section we introduce generalized A-hypergeometric systems in the sense indicated in the introduction. After reviewing some basic facts of GKZ hypergeometric systems, we provide foundations for Euler–Koszul homology of what we call *toric $\mathbb{Z}^d$-graded modules*.

For the rest of this paper, fix a $d \times n$ integer matrix $A = (a_{ij})$ of rank d. We emphasize that we do not assume that the columns $a_1, \ldots, a_n$ of A lie in an affine hyperplane. However, we do assume that A is *pointed*, meaning that $a_1, \ldots, a_n$ lie in a single open half-space of $\mathbb{R}^d$. This guarantees that the semigroup

$$\mathbb{N}A \;=\; \Big\{\sum_{i=1}^{n} \gamma_i a_i \mid \gamma_1, \ldots, \gamma_n \in \mathbb{N}\Big\}$$

has no units. (Pointedness will come into play in the proof of Theorem 6.6, where it is used to ensure that local duality holds.) The *semigroup ring* associated to the $d \times n$ matrix A is $S_A = \mathbb{C}[\mathbb{N}A] \cong R/I_A$, where $R = \mathbb{C}[\partial_1, \ldots, \partial_n]$ and

$$I_A \;=\; \langle \partial^\mu - \partial^\nu \mid \mu, \nu \in \mathbb{Z}^n, A\cdot\mu = A\cdot\nu\rangle$$

is the *toric ideal* of A. Notice that S_A and R are naturally graded by $\mathbb{Z}^d$ if we define $\deg(\partial_j) = -a_j$, the negative of the j^{th} column of A.

Our choice of signs in the $\mathbb{Z}^d$-grading of R is compatible with a $\mathbb{Z}^d$-grading on the Weyl algebra D in which $\deg(x_j) = a_j$ and $\deg(\partial_j) = -a_j$. Under this $\mathbb{Z}^d$-grading, the i^{th} *Euler operator* $E_i = \sum_{j=1}^n a_{ij}x_j\partial_j \in D$ is homogeneous of degree 0 for $i = 1, \ldots, d$. Given a vector $\beta \in \mathbb{C}^d$, we write $E - \beta$ for the sequence $E_1 - \beta_1, \ldots, E_d - \beta_d$.

Definition 4.1. The *A-hypergeometric GKZ system* with *parameter β* is the left ideal

$$H_A(\beta) \quad = \quad D \cdot \langle I_A, E - \beta \rangle$$

in the Weyl algebra D. The *A-hypergeometric D-module* with *parameter β* is

$$\mathcal{M}_\beta^A \quad = \quad D/H_A(\beta).$$

Results of [GKZ89, Ado94, Hot98, SST00] imply that the module $\mathcal{M}_\beta^A$ is holonomic of nonzero rank. A-hypergeometric systems constitute an important class of D-modules, playing a role similar to that of toric varieties in algebraic geometry, since they possess enough combinatorial underpinning to make calculations feasible, and enough diversity of behavior to make them interesting as a test class for conjectures and computer experimentation.

If $y \in N_\alpha$ is homogeneous of degree α in a $\mathbb{Z}^d$-graded D-module N, write $\deg_i(y) = \alpha_i$. In particular, for any homogeneous element $P \in D$,

$$E_i P - P E_i \quad = \quad \deg_i(P) P.$$

The displayed equation shows that if N is any $\mathbb{Z}^d$-graded left D-module, then the map $E_i - \beta_i : N \to N$ sending the homogeneous element $y \in N$ to

$$(E_i - \beta_i) \circ y \quad = \quad (E_i - \beta_i - \deg_i(y))y$$

determines a D-linear endomorphism of N when extended $\mathbb{C}$-linearly to inhomogeneous elements of N. This endomorphism is functorial, in the sense that it commutes with degree 0 homomorphisms $N \to N'$ of $\mathbb{Z}^d$-graded left D-modules. Moreover, the endomorphisms for the various Euler operators commute: $[E_i - \beta_i, E_j - \beta_j]$ is the zero endomorphism on any $\mathbb{Z}^d$-graded left D-module for all $i, j = 1, \ldots, d$ and all complex numbers β_i, β_j.

Definition 4.2. Fix $\beta \in \mathbb{C}^d$ and a $\mathbb{Z}^d$-graded R-module N. The *Euler–Koszul complex* $\mathcal{K}_\bullet(E - \beta; N)$ is the Koszul complex of left D-modules defined by the sequence $E - \beta$ of commuting endomorphisms on the left D-module $D \otimes_R N$ concentrated in homological degrees d to 0. The i^{th} *Euler–Koszul homology* of N is $\mathcal{H}_i(E - \beta; N) = H_i(\mathcal{K}_\bullet(E - \beta; N))$.

This complex was used before in the special case $N = S_A$ in [GKZ89, Ado94, Ado99] in order to study the solutions to $H_A(\beta)$ for special parameters β. We use the script '$\mathcal{K}$' for Euler–Koszul complexes instead of the usual 'K' for ordinary Koszul complexes to emphasize that the maps in $\mathcal{K}_\bullet(E - \beta; N)$ are homomorphisms of D-modules. The action of an endomorphism in the sequence $E - \beta$ on a homogeneous element depends on the degree of that element, in contrast to maps in Koszul complexes over commutative graded rings. Nonetheless, Euler–Koszul complexes behave much like ordinary Koszul complexes. To see why, let $\theta_i = x_i \partial_i \in D$. In the $\mathbb{Z}^d$-graded commutative subalgebra $\Theta = \mathbb{C}[\theta_1, \ldots, \theta_n]$ of D every element has degree $0 \in \mathbb{Z}^d$. Consequently, N_α is a left Θ-module whenever N is a $\mathbb{Z}^d$-graded D-module. We hence obtain the following.

Lemma 4.3. *Let N be a $\mathbb{Z}^d$-graded R-module and $\alpha \in \mathbb{Z}^d$. The $\mathbb{Z}^d$-graded degree α part $\mathcal{K}_\bullet(E - \beta; N)_\alpha$ of the Euler–Koszul complex agrees with the ordinary Koszul complex $K_\bullet(E - \beta - \alpha; (D \otimes_R N)_\alpha)$ constructed over the polynomial subalgebra $\Theta \subseteq D$. That is,*

$$\mathcal{K}_\bullet(E - \beta; N) \quad = \quad \bigoplus_{\alpha \in \mathbb{Z}^d} K_\bullet(E - \beta - \alpha; (D \otimes_R N)_\alpha),$$

the right hand side being a direct sum of ordinary Koszul complexes of Θ-modules. $\square$

Since D is a free R-module, the Euler–Koszul complex constitutes an exact functor from the category of $\mathbb{Z}^d$-graded R-modules with $\mathbb{Z}^d$-graded morphisms of degree 0 to the category of bounded complexes of $\mathbb{Z}^n$-graded left D-modules with $\mathbb{Z}^d$-graded morphisms of degree 0. Consequently, short exact sequences of $\mathbb{Z}^d$-graded modules over R give rise to long exact sequences of Euler–Koszul homology.

We now specify a subcategory of the category of R-modules that is central to what follows. This subcategory will contain the semigroup ring S_A as well as its monomial (that is, $\mathbb{Z}^d$-graded) ideals, and it will be closed under taking $\mathbb{Z}^d$-graded submodules, quotient modules, and extensions. In the following section we then study the Euler–Koszul homology of objects in this category and establish strong (non)vanishing conditions.

Notation 4.4. By a *face of* A we mean a set F of columns of A minimizing some linear functional on $\mathbb{N}A$. We interpret F as a submatrix of A. The *dimension* $\dim(F)$ of a face F equals the rank of the subgroup $\mathbb{Z}F \subseteq \mathbb{Z}^d$ it generates. Let S_F be the semigroup ring generated by F. Thus $S_F \cong R/I_F$, where $I_F = I_A^F + \langle \partial_j \mid a_j \notin F\rangle$ is the prime ideal obtained by starting with the toric ideal I_A^F (defined as the kernel of the surjection $\mathbb{C}[\partial_j \mid a_j \in F] \to S_F$) and adding the variables corresponding to the columns of A outside of F.

Definition 4.5. A $\mathbb{Z}^d$-graded R-module M is *toric* if it has a *toric filtration*

$$0 = M_0 \subset M_1 \subset \cdots \subset M_{\ell-1} \subset M_\ell = M,$$

meaning that M_k/M_{k-1} is, for each k, a $\mathbb{Z}^d$-graded translate of S_{F_k} for some face F_k of $\mathbb{N}A$. We say that M has *toric length* ℓ if the minimal length of a toric filtration for M is ℓ.

Remark 4.6. Most toric modules have many different toric filtrations, and usually more than one of these has minimal length. A toric module of toric length 1 is simply a $\mathbb{Z}^d$-graded translate $S_F(-\alpha)$ of S_F generated in degree $\alpha \in \mathbb{Z}^d$, for some face F of A.

Example 4.7. If N is a finitely generated $\mathbb{Z}^d$-graded S_F-module for some face F of A, then N is toric. To see this, argue by induction on $\dim(N)$, the Artinian case $\dim(N) = 0$ being trivial. If $\dim(N) > 0$, then N has a submodule of the form $S_{F'}(-\alpha)$ with $F' \subseteq F$ being some face of A of dimension $\dim(N)$. If we replace N by $N/S_{F'}(\alpha)$ then a finite number of such steps will reduce the dimension of N by at least one.

Example 4.7 implies that $\mathbb{Z}^d$-graded submodules, quotients, and extensions of toric modules are toric, since they have a composition chain by shifts of modules of the form S_F.

If F is a face of A, then we write E_i^F for the Euler operator obtained by erasing all terms $a_{ij}x_j\partial_j$ from E_i such that $a_j \notin F$. We will now consider the actions of Euler endomorphisms $E_i - \beta_i$ on $D \otimes_R M$ for an S_F-module M. To describe such actions, let D_F and $D_{\overline{F}}$ be the Weyl algebras on the variables $\{x_F, \partial_F\}$ and $\{x_{\overline{F}}, \partial_{\overline{F}}\}$ corresponding to the columns $a_j \in F$ and $a_j \notin F$, respectively. Then for each S_F-module M we have

$$(\dagger) \qquad D \otimes_R M \quad \cong \quad \mathbb{C}[x_{\overline{F}}] \otimes_{\mathbb{C}} (D_F \otimes_{\mathbb{C}[\partial_F]} M)$$

as left modules over $D = D_{\overline{F}} \otimes_{\mathbb{C}} D_F$, identifying $x_F^\mu x_{\overline{F}}^{\overline{\mu}} \partial_F^\nu \partial_{\overline{F}}^{\overline{\nu}} \otimes m$ with $x_{\overline{F}}^{\overline{\mu}} \otimes (x_F^\mu \partial_F^\nu \otimes \partial_{\overline{F}}^{\overline{\nu}} m)$, where of course both sides are zero if $|\overline{\nu}| > 0$.

Suppose that M is a $\mathbb{Z}^d$-graded S_F-module and interpret it as a $\mathbb{C}[\partial_F]$-module (rather than as a module over a quotient of R). Note that E_i^F induces an Euler endomorphism on

$D_F \otimes_{\mathbb{C}[\partial_F]} M$ by setting $E_i^F \circ (P \otimes m) = \sum_{j \in F}(a_{ij}x_j\partial_j - \deg_i(P \otimes m))P \otimes m$ for all homogeneous elements $P \in D_F$ and $m \in M$.

Lemma 4.8. *If M is a finitely generated $\mathbb{Z}^d$-graded S_F-module, then under the isomorphism (†) above, the action of the endomorphism $E_i - \beta_i$ on $D \otimes_R M$ coincides with the action of the endomorphism $E_i^F - \beta_i$ on the right hand factor of $\mathbb{C}[x_{\overline{F}}] \otimes_{\mathbb{C}} (D_F \otimes_{\mathbb{C}[\partial_F]} M)$:*

$$(E_i - \beta_i) \circ (x_F^\mu x_{\overline{F}}^{\overline{\mu}} \partial_F^\nu \partial_{\overline{F}}^{\overline{\nu}} \otimes m) \quad = \quad x_{\overline{F}}^{\overline{\mu}} \otimes (E_i^F - \beta_i) \circ (x_F^\mu \partial_F^\nu \otimes \partial_{\overline{F}}^{\overline{\nu}} m).$$

Proof. Let $P = x_F^\mu x_{\overline{F}}^{\overline{\mu}} \partial_F^\nu \partial_{\overline{F}}^{\overline{\nu}}$ and $P' = x_F^\mu \partial_F^\nu \partial_{\overline{F}}^{\overline{\nu}}$. Note that $[E_i^F, x_{\overline{F}}^{\overline{\mu}}] = 0$ and therefore $[E_i - E_i^F, x_{\overline{F}}^{\overline{\mu}}] = \deg_i(x_{\overline{F}}^{\overline{\mu}}) x_{\overline{F}}^{\overline{\mu}}$. We then compute

$$\begin{aligned}
(E_i - \beta_i) \circ (P \otimes m) &= (E_i - \beta_i - \deg_i(P \otimes m)) \cdot P \otimes m \\
&= \big((E_i - E_i^F) - \deg_i(x_{\overline{F}}^{\overline{\mu}})\big) P \otimes m \\
&\qquad + \big(E_i^F - \beta_i - \deg_i(P' \otimes m)\big) P \otimes m \\
&= 0 + x_{\overline{F}}^{\overline{\mu}} \big(E_i^F - \beta_i - \deg_i(P' \otimes m)\big) P' \otimes m
\end{aligned}$$

which is identified with $x_{\overline{F}}^{\overline{\mu}} \otimes (E_i^F - \beta_i - \deg_i(P' \otimes m))P' \otimes m$. □

In analogy with the construction of the A-hypergeometric system $\mathcal{M}_\beta^A$ from A, the faces of A give rise to holonomic D-modules, but sometimes these modules can be zero.

Lemma 4.9. *Fix a vector $\beta \in \mathbb{C}^d$ and a face F of A. The following are equivalent for the quotient $\mathcal{M}_\beta^F = D/\langle I_F, E - \beta \rangle$.*

1. *The module $\mathcal{M}_\beta^F$ is nonzero.*
2. *$\mathcal{M}_\beta^F$ is a holonomic D-module of nonzero rank.*
3. *The parameter β (or, equivalently, $-\beta$) lies in the span (over $\mathbb{C}$) of the columns of F.*

Proof. If β lies outside the column-span of F, then the $\mathbb{C}$-linear span of $E - \beta$ and the products $\{x_j\partial_j \mid a_j \notin F\} \subset D \cdot I_F$ contains a nonzero scalar, so $\mathcal{M}_\beta^F$ is zero, as is its rank.

Suppose now that β lies in the column-span of F. Let A_F' be a submatrix of A composed of $\dim(F)$ many linearly independent rows of A_F. Then $\mathcal{M}_\beta^{A_F} = D_F/\langle I_A^F, E^F - \beta \rangle$ equals $\mathcal{M}_\beta^{A_F'}$. Hence $\mathcal{M}_\beta^{A_F}$ is a holonomic D_F-module of nonzero rank, by [GKZ89, SST00]. Since $\mathbb{C}[x_{\overline{F}}] \otimes_{\mathbb{C}} \mathcal{M}_\beta^{A_F} \cong \mathcal{M}_\beta^F$ by Lemma 4.8, $\mathcal{M}_\beta^F$ is holonomic over $D = D_{\overline{F}} \otimes_{\mathbb{C}} D_F$ and has nonzero rank. □

We close this section by defining the main object of study of the remainder of the paper.

Definition 4.10. Let M be a toric R-module and $\beta \in \mathbb{C}^d$. The *generalized hypergeometric system* associated to M and β is the zeroth Euler–Koszul homology $\mathcal{H}_0(E - \beta; M)$.

If $M = S_A$ then $\mathcal{H}_0(E - \beta; M)$ is the holonomic GKZ-module $\mathcal{M}_\beta^A$ from Definition 4.1.

5. Rigidity and holonomicity of Euler–Koszul homology

We begin our treatment of generalized hypergeometric systems by showing that they are holonomic, as are all of the higher Euler–Koszul homology modules of toric modules.

Proposition 5.1. *The generalized hypergeometric system $\mathcal{H}_0(E-\beta;M)$ is holonomic for every toric module M and every parameter $\beta \in \mathbb{C}^d$. Consequently, $\mathcal{H}_i(E-\beta;M)$ is holonomic for all $i>0$ as well.*

Proof. Consider first the case of $\mathcal{H}_0(E-\beta;M)$. The proof is by induction on the toric length ℓ of M. When $M = S_F$ for some face F, use Lemma 4.9. The general $\ell = 1$ case follows because $\mathcal{H}_0(E-\beta;S_F(\alpha)) \cong \mathcal{H}_0(E-\beta+\alpha;S_F)(\alpha)$. For $\ell > 1$, the Euler–Koszul functor applied to a toric short exact sequence $M_1 \hookrightarrow M \twoheadrightarrow M/M_1$ induces a sequence

$$\mathcal{H}_0(E-\beta;M_1) \to \mathcal{H}_0(E-\beta;M) \to \mathcal{H}_0(E-\beta;M/M_1)$$

that is exact and has holonomic modules at both ends by induction.

Now consider $i>0$. Let $m \in \ker(\mathcal{K}_i(E-\beta;M) \to \mathcal{K}_{i-1}(E-\beta;M))$, so its coset $\bar{m}$ is an element of $\mathcal{H}_i(E-\beta;M)$. Since the Euler–Koszul complex is $\mathbb{Z}^d$-graded, we may assume that m is homogeneous of degree $\alpha \in \mathbb{Z}^d$. Note that $\mathcal{H}_i(E-\beta;M)$ is generated by finitely many such $\bar{m}$, since $\mathcal{K}_i(E-\beta;M)$ is a direct sum of $\binom{d}{i}$ copies of $(D \otimes_R M)$, which is a Noetherian D-module. It is hence sufficient to prove that $D \cdot \bar{m}$ is holonomic.

Consider the Koszul complex $K_{\bullet}(E-\beta-\alpha,(D\otimes_R M)_\alpha)$ on the (left) Θ-module $(D\otimes_R M)_\alpha$. Within this complex, m descends to an element $\bar{m}$ of the i^{th} homology module $H_i(E-\beta-\alpha,(D\otimes_R M)_\alpha)$. We remark that the two usages of $\bar{m}$ agree in the sense of the isomorphism in Lemma 4.3. Since Θ is commutative, left multiplication by $E_j-\beta_j-\alpha_j$ annihilates $\bar{m}$ by [BH93, Proposition 1.6.5]. On the other hand, m lies in the direct sum of $\binom{d}{i}$ copies of $(D\otimes_R M)$. Since $I_A \cdot M = 0$, there is some integer $k \in \mathbb{N}$ such that $(I_A)^k m = 0$. It follows that the D-module $D\cdot\bar{m}$ is a $\mathbb{Z}^d$-graded quotient of $D/\langle E-\beta-\alpha,(I_A)^k\rangle$. In particular, $D\cdot\bar{m}$ has a finite composition series such that each composition factor is a quotient of $D/\langle E-\beta-\alpha, I_A\rangle \cong \mathcal{H}_0(E-\beta-\alpha;S_A)$ and hence is holonomic by the first sentence of the proposition. □

Definition 5.2. Let N be any $\mathbb{Z}^d$-graded R-module. A vector $\alpha \in \mathbb{Z}^d$ is a *true degree* of N, written $\alpha \in \operatorname{tdeg}(N)$, if the graded piece N_α is nonzero. A vector $\alpha \in \mathbb{C}^d$ is a *quasi-degree* of N, written $\alpha \in \operatorname{qdeg}(N)$, if α lies in the complex Zariski closure $\operatorname{qdeg}(N)$ of the true degrees of N via the natural embedding $\mathbb{Z}^d \hookrightarrow \mathbb{C}^d$.

We now prove a rigidity property of the Euler–Koszul complex.

Proposition 5.3. *For a toric R-module M and $\beta \in \mathbb{C}^d$ the following are equivalent.*
1. *$\mathcal{H}_0(E-\beta;M)$ has holonomic rank 0.*
2. *$\mathcal{H}_0(E-\beta;M) = 0$.*
3. *$\mathcal{H}_i(E-\beta;M) = 0$ for all $i \geq 0$.*
4. *$-\beta \notin \operatorname{qdeg}(M)$.*

Proof. $2 \Leftrightarrow 3$: The failure of the last condition in Lemma 4.9 is equivalent to the $\mathbb{C}$-span of $E-\beta$ containing an endomorphism whose action on $D/D\cdot I_F$ is multiplication by a nonzero scalar. This observation, together with the isomorphism $\mathcal{H}_0(E-\beta;S_F(\alpha)) \cong \mathcal{H}_0(E-\beta+\alpha;S_F)(\alpha)$, shows that $2 \Leftrightarrow 3$ when M has toric length 1. Now suppose that M has toric length $\ell > 1$, and assume the result for modules of smaller toric length. For any toric filtration of M, the long exact sequence of Euler–Koszul homology for $M_1 \hookrightarrow M \twoheadrightarrow M/M_1$ shows that $\mathcal{H}_0(E-\beta;M)$ surjects onto $\mathcal{H}_0(E-\beta;M/M_1)$. Assuming that the former is zero, so is the latter. By induction, we find that $\mathcal{H}_i(E-\beta; M/M_1)$ vanishes for all $i \geq 0$, and hence that $\mathcal{H}_i(E-\beta; M) \cong \mathcal{H}_i(E-\beta; M_1)$ for all $i \geq 0$. The result now holds for M because it holds by induction for M_1.

$1 \Leftrightarrow 2$: We need that $\mathcal{H}_0(E-\beta; M) \neq 0$ implies that its rank is nonzero. Again use induction on the toric length ℓ. For $\ell = 1$ this is Lemma 4.9 plus the isomorphism $\mathcal{H}_0(E-\beta; S_F(\alpha)) \cong \mathcal{H}_0(E-\beta+\alpha; S_F)(\alpha)$, so assume $\ell > 1$. Make again the observation that $\mathcal{H}_0(E-\beta; M)$ surjects onto $\mathcal{H}_0(E-\beta; M/M_1)$. If this latter module is nonzero, then it has nonzero rank by the $\ell = 1$ case, so $\mathcal{H}_0(E-\beta; M)$ has nonzero rank, too. Hence we assume that $\mathcal{H}_0(E-\beta; M/M_1) = 0$. Now, using the equivalence $2 \Leftrightarrow 3$, we find that $\mathcal{H}_0(E-\beta; M) \cong \mathcal{H}_0(E-\beta; M_1)$, so again we are done by induction.

$2 \Leftrightarrow 4$: For the semigroup ring S_F of a face, $\mathcal{H}_0(E-\beta; S_F) = \mathcal{M}_\beta^F$ is the D-module from Lemma 4.9. Therefore $2 \Leftrightarrow 4$ for $M = S_F$. For $\mathbb{Z}^d$-graded translates of S_F,

$$[\mathcal{H}_0(E-\beta; S_F) \neq 0] \quad \Leftrightarrow \quad [-\beta \in \operatorname{qdeg}(S_F)] \quad \Leftrightarrow \quad [-\beta-\alpha \in \operatorname{qdeg}(S_F(\alpha))]$$

for $\alpha \in \mathbb{Z}^d$, but also $[\mathcal{H}_0(E-\beta; S_F) \neq 0] \Leftrightarrow [\mathcal{H}_0(E-\beta-\alpha; S_F(\alpha)) \neq 0]$ by definition of the endomorphisms $E_i - \beta_i$. This proves $2 \Leftrightarrow 4$ when M has toric length 1. To treat modules of toric length $\ell > 1$, consider a toric sequence $M_1 \hookrightarrow M \twoheadrightarrow M/M_1$. Since $\operatorname{qdeg}(M) = \operatorname{qdeg}(M_1) \cup \operatorname{qdeg}(M/M_1)$, we find that $-\beta \in \operatorname{qdeg}(M)$ if and only if

$$[-\beta \in \operatorname{qdeg}(M_1)] \quad \text{or} \quad [-\beta \in \operatorname{qdeg}(M/M_1)]\,.$$

This condition is equivalent by induction on ℓ to

$$[\mathcal{H}_0(E-\beta; M_1) \neq 0] \quad \text{or} \quad [\mathcal{H}_0(E-\beta; M/M_1) \neq 0]\,.$$

Now $\mathcal{H}_0(E-\beta; M)$ always surjects onto $\mathcal{H}_0(E-\beta; M/M_1)$, and if $\mathcal{H}_0(E-\beta; M/M_1)$ vanishes then by the equivalence $2 \Leftrightarrow 3$, $\mathcal{H}_0(E-\beta; M_1) \cong \mathcal{H}_0(E-\beta; M)$. Therefore, this last displayed condition is equivalent to $\mathcal{H}_0(E-\beta; M) \neq 0$. □

6. Euler–Koszul homology detects local cohomology

In this section we describe the set of parameters $\beta \in \mathbb{C}^d$ for which the Euler–Koszul complex $\mathcal{K}_\bullet(E-\beta; N)$ has nonzero higher homology. Namely, we identify this set with the quasi-degrees of the local cohomology of the toric module N. Our proof of this result in Theorem 6.6 uses a spectral sequence, constructed in Theorem 6.3, that arises from the holonomic duality functor. For background on holonomic duality we refer to [Bjö79].

Duality $\mathbb{D}$ for (complexes of) D-modules is the combination of the derived functor $\mathbb{R}\mathrm{Hom}_D(\_, D)$ of homomorphisms to D followed by the involution τ taking $x^\mu\partial^\nu$ to

$$\tau(x^\mu\partial^\nu) \;=\; (-\partial)^\nu x^\mu.$$

Hence, in order to compute the dual $\mathbb{D}(N)$ of the module N placed in homological degree zero, let $F_\bullet$ be a D-free resolution of N and apply τ to $\mathrm{Hom}_D(F_\bullet, D)$. Holonomicity of N is equivalent to $\mathbb{D}(N)$ being exact in all cohomological degrees but n. If N is a module in *homological* degree k, then the *co*homology $H^i\mathbb{D}(N)$ is nonzero only for $i = k+n$. Therefore $\mathbb{D}$ constitutes an exact functor from the category of complexes of D-modules with holonomic homology to itself, and $\mathbb{D}(\mathbb{D}(N)) = N$ for all N.

The rank of a holonomic module N equals that of its dual: $\operatorname{rank}(N) = \operatorname{rank}(\mathbb{D}(N))$. To be more precise, locally near a nonsingular point of the complex analytic manifold $\mathbb{C}^n_{an}$, N

is a connection: $N_{an} = {D_{an}}^r/\langle \partial_j - C_j \mid j = 1, \ldots, n\rangle$ where $C_j \in {\mathcal{O}_{an}}^{r,r}$ are $r\times r$ matrices of holomorphic functions and one has vanishing commutators $[\partial_i - C_i, \partial_j - C_j]$. Hence the right Koszul complex $K_\bullet(\partial - C, D_{an})$ on the operators $\partial_i - C_i$ is a free resolution of N_{an} and the holonomic dual is computed from this Koszul complex. In particular, $\mathbb{D}(N_{an}) = {D_{an}}^r/\langle \partial_j + C_j \mid j = 1, \ldots, n\rangle$ and so $\operatorname{rank}(N) = \operatorname{rank}(\mathbb{D}(N)) = r$.

The automorphism $x \mapsto -x$ on $\mathbb{C}^n$ induces, via $\partial \mapsto -\partial$, auto-equivalences $N \mapsto N^-$ on the categories of R-modules and of D-modules (but not of the category of $\mathbb{Z}^n$-graded S_A-modules, since I_A is not preserved under $(\_)^-$ unless it is projective). The formation of Euler–Koszul complexes is equivariant under this sign change since $E_i - \beta_i = (E_i - \beta_i)^-$. Moreover, for $\mathbb{Z}^n$-graded R-modules N we have $D \otimes_R N \cong \tau(N \otimes_R D)^-$ as D-modules, where the tensor products exploit the two different R-structures on D.

The ordinary Koszul complex $K_\bullet(\mathbf{y}; T)$ on a sequence $\mathbf{y} = y_1, \ldots, y_d$ in a commutative ring T is isomorphic to its dual $K^\bullet(\mathbf{y}; T) = \operatorname{Hom}_T(K_\bullet(\mathbf{y}; T), T)$. In fact, $K_\bullet(\mathbf{y}; T)$ equals $K^\bullet(\mathbf{y}; T)$ after replacing each lowered homological index i by the raised cohomological index $d - i$ and a suitable sign change in the differentials. Administering this sign- and index-change to $\mathcal{K}_\bullet(E - \beta; N)$ in Definition 4.2 yields $\mathcal{K}^\bullet(E - \beta; N)$, whose cohomology we call the *Euler–Koszul cohomology* of N; we have $\mathcal{H}^i(E - \beta; N) \cong \mathcal{H}_{d-i}(E - \beta; N)$.

We shall apply $\mathcal{K}_\bullet(E - \beta; \_)$ and $\mathcal{K}^\bullet(E - \beta; \_)$ to $\mathbb{Z}^d$-graded complexes of R-modules. Our conventions for indexing the resulting double complexes are set up as follows, so that all homological and cohomological indices are positive. If $\mathcal{F}_\bullet\colon \mathcal{F}_0 \leftarrow \mathcal{F}_1 \leftarrow \cdots \leftarrow \mathcal{F}_n$ is a complex of R-modules with decreasing lowered indices, then we write the differentials of the double complex $\mathcal{K}_\bullet(E - \beta; \mathcal{F}_\bullet)$ pointing downward (the $\mathcal{K}_\bullet$ direction) and to the left (the $\mathcal{F}_\bullet$ direction). On the other hand, if $\mathcal{F}^\bullet\colon \mathcal{F}^0 \to \mathcal{F}^1 \to \cdots \to \mathcal{F}^n$ has increasing raised indices, then we write the double complex $\mathcal{K}^\bullet(E - \beta; \mathcal{F}^\bullet)$ with differentials pointing upward and to the right. Applying $\operatorname{Hom}_T(\_, T)$ to a complex of free T-modules with decreasing lowered indices (with $T = R$ or $T = D$) yields a complex with increasing raised indices.

Our theory of toric modules applies to the modules $\operatorname{Ext}^i_R(M, R)$ whenever M is toric.

Lemma 6.1. *If M is a toric R-module then $\operatorname{Ext}^i_R(M, R)$ is toric for all i.*

Proof. The module M has a composition chain by $\mathbb{Z}^d$-graded S_A-modules. If M is of toric length 1 then the finitely generated S_A-module $\operatorname{Ext}^i_R(M, R)$ is toric by Example 4.7.

In the general case, argue by induction on the toric length of M. The $\mathbb{Z}^d$-graded long exact sequence of $\operatorname{Ext}_R(\_, R)$ arising from $M_1 \hookrightarrow M \twoheadrightarrow M/M_1$ places $\operatorname{Ext}^i_R(M, R)$ between two modules both of which are toric by induction. □

Definition 6.2. Let ε_A be the sum $\sum_{j=1}^n a_j$ of the columns of A. Using our convention that $\deg(\partial_j) = -a_j$, the *canonical module* of R is $\omega_R = R(\varepsilon_A)$.

Theorem 6.3. *If M is a toric R-module, then there is a spectral sequence*

$$E_2^{p,q} = \mathcal{H}^q\big(E + \beta; \operatorname{Ext}^p_R(M, \omega_R)\big)(-\varepsilon_A) \implies H^{p+q}\mathbb{D}\big(\mathcal{H}_{p+q-n}(E - \beta; M)\big)^-.$$

Proof. Let $\mathcal{F}_\bullet$ be a minimal $\mathbb{Z}^d$-graded R-free resolution of M. Consider the double complex $\tau\operatorname{Hom}_D(\mathcal{K}_\bullet(E - \beta; \mathcal{F}_\bullet), D)^-$. With $\mathcal{F}^\bullet = \operatorname{Hom}_R(\mathcal{F}_\bullet, R)$, we claim that

$$\tau\operatorname{Hom}_D\big(\mathcal{K}_\bullet(E - \beta; \mathcal{F}_\bullet), D\big)^- \;\cong\; \mathcal{K}^\bullet(-E - \beta - \varepsilon_A; \mathcal{F}^\bullet).$$

To see why, begin by noting that $\tau(\_)^-$ is an isomorphism when D-modules are regarded as R-modules, but identifies (for example) the left D-module $D \otimes_R S_A$ with the right D-module $S_A \otimes_R D$. Hence each row of $\tau\mathrm{Hom}_D(\mathcal{K}_\bullet(E-\beta;\mathcal{F}_\bullet), D)^-$ is a direct sum of complexes of the form $\tau(\mathcal{F}^\bullet \otimes_R D)^-$ with cohomology $D \otimes_R \mathrm{Ext}^i_R(M,R)$.

On the other hand, note that $\tau(E_i - \beta_i)^- = -E_i - \beta_i - (\varepsilon_A)_i$. Therefore each column of $\tau\mathrm{Hom}_D(\mathcal{K}_\bullet(E-\beta;\mathcal{F}_\bullet), D)^-$ is an Euler–Koszul cocomplex induced by $-E-\beta-\varepsilon_A$ on $\tau(\mathcal{F}^i \otimes_R D)^-$. We consider the spectral sequences associated to this double complex.

Taking first the horizontal and then the vertical cohomology of the double complex, we obtain $\mathcal{H}^q(-E-\beta-\varepsilon_A; \mathrm{Ext}^p_R(M,R)) \cong \mathcal{H}^q\big(-E-\beta; \mathrm{Ext}^p_R(M,\omega_R)\big)(-\varepsilon_A)$. We now determine the abutment by reversing the order of taking horizontal and vertical cohomology.

The natural projection from the total complex $\mathrm{Tot}\,\mathcal{K}_\bullet(E-\beta;\mathcal{F}_\bullet)$ to the Euler–Koszul complex $\mathcal{K}_\bullet(E-\beta;M)$ is a quasi-isomorphism because the rows of $\mathcal{K}_\bullet(E-\beta;\mathcal{F}_\bullet)$ are resolutions for M positioned in homological degree 0. Since $\mathcal{H}_k(E-\beta;M)$ is holonomic for all k by Proposition 5.1, the complex $\mathrm{Hom}_D(\mathrm{Tot}\,\mathcal{K}_\bullet(E-\beta;\mathcal{F}_\bullet), D)$ has cohomology

$$H^{p+q}\,\mathrm{Hom}_D\big(\mathrm{Tot}\,\mathcal{K}_\bullet(E-\beta;\mathcal{F}_\bullet), D\big) \;=\; \mathrm{Ext}^n_D\big(\mathcal{H}_{p+q-n}(E-\beta;M), D\big).$$

Applying the standard involution τ and the auto-equivalence $(\_)^-$ yields

$$H^{p+q}\mathbb{D}\big(\mathrm{Tot}\,\mathcal{K}_\bullet(E-\beta;\mathcal{F}_\bullet)\big)^- \;=\; H^{p+q}\mathbb{D}\big(\mathcal{H}_{p+q-n}(E-\beta;M)\big)^-. \qquad \square$$

Remark 6.4. It follows from this spectral sequence that if M is a Cohen–Macaulay toric module of dimension d, then $\mathcal{H}_i(E-\beta;M) = 0$ for all $i > 0$ since then $\mathrm{Ext}^{n-d+i}_R(M,\omega_R) = 0$ for all $i > 0$. In the sequel we link the failure of M to be Cohen–Macaulay to the appearance of non-vanishing $\mathcal{H}_i(E-\beta;M)$ for suitable β and $i > 0$.

Let $\mathfrak{m} = \langle \partial_1, \dots \partial_n \rangle$ be the $\mathbb{Z}^d$-graded maximal ideal of R. Given a $\mathbb{Z}^d$-graded R-module N, its *local cohomology modules*

$$H^i_{\mathfrak{m}}(N) \;=\; \varinjlim_t \mathrm{Ext}^i_R(R/\mathfrak{m}^t, N)$$

supported at $\mathfrak{m}$ are $\mathbb{Z}^d$-graded. We refer to [MS04] for details on the $\mathbb{Z}^d$-graded aspects of local cohomology. By [BH93, Section 3.5] there is a natural vector space isomorphism

$$\mathrm{Ext}^i_R(N,R)_\alpha \;\cong\; \mathrm{Hom}_{\mathbb{C}}(H^{d-i}_{\mathfrak{m}}(N)_{-\alpha+\varepsilon_A}, \mathbb{C})$$

called $\mathbb{Z}^d$*-graded local duality* (see also [Mil02]).

Definition 6.5. Fix a toric module M. A degree $\alpha \in \mathbb{Z}^d$ such that $H^i_{\mathfrak{m}}(M)_{-\alpha} \neq 0$ (note the minus sign in the subscript) for some $i \leq d-1$ is called a *true exceptional degree* of M. If $\beta \in \mathbb{C}^d$ lies in the Zariski closure of the set of true exceptional degrees of M, then β is an *exceptional quasi-degree* of M.

Exactness of the Euler–Koszul complex can be expressed in terms of local cohomology.

Theorem 6.6. *The Euler–Koszul homology $\mathcal{H}_i(E-\beta;M)$ of a toric module M over R is nonzero for some $i \geq 1$ if and only if $\beta \in \mathbb{C}^d$ is an exceptional quasi-degree of M. More precisely, if k equals the smallest homological degree i for which $-\beta \in \mathrm{qdeg}(H^i_{\mathfrak{m}}(M))$ then $\mathcal{H}_{d-k}(E-\beta;M)$ is holonomic of nonzero rank while $\mathcal{H}_i(E-\beta;M) = 0$ for $i > d-k$.*

Proof. The module M has dimension at most d, because this is true of every successive quotient in any toric filtration. Therefore, using notation as in Theorem 6.3, the modules $\operatorname{Ext}^p_R(M, \omega_R))$ can only be nonzero when $p \geq n - d$.

First suppose that β is not an exceptional quasi-degree. This means precisely that β lies outside of $\operatorname{qdeg}(\operatorname{Ext}^p_R(M, \omega_R))$ for all $p \neq n - d$ by local duality. Proposition 5.3 therefore implies that the only column of the spectral sequence page $E_2^{p,q}$ in Proposition 6.3 that can possibly be nonzero is column $p = n - d$. Furthermore, the highest possible row of a nonzero entry in this column is row $q = d$. Since the cohomology of the abutment is only nonzero when $p+q \geq n$, it follows that $H^{p+q}\mathbb{D}(\mathcal{H}_{p+q-n}(E - \beta; M))^-$ can only be nonzero when $p + q = n$. Applying the auto-equivalence $(\_)^-$ and taking holonomic duals, we find that $\mathcal{H}_i(E - \beta; M)$ can only be nonzero when $i = 0$.

Now suppose that β is an exceptional quasi-degree. Let k be the smallest cohomological degree i such that $\beta \in -\operatorname{qdeg}(H^i_{\mathfrak{m}}(M)) = \operatorname{qdeg}(\operatorname{Ext}^{n-i}_R(M, \omega_R))$. By Proposition 5.3 and Lemma 6.1, $\mathcal{H}^d(E + \beta; \operatorname{Ext}^{n-k}_R(M, \omega_R))(-\varepsilon_A)$ is nonzero. Moreover, all columns to the right of column $n - k$ and all rows above row d in the spectral sequence page $E_2^{p,q}$ of Theorem 6.3 are zero. Hence

$$\mathcal{H}^d(E + \beta; \operatorname{Ext}^{n-k}_R(M, \omega_R))(-\varepsilon_A) \quad = \quad H^{d-k+n}\mathbb{D}\big(\mathcal{H}_{d-k}(E - \beta; M)\big)^-$$

is by Proposition 5.1 a holonomic module of nonzero rank. After applying the auto-equivalence $(\_)^-$ and taking holonomic duals, we find that $\mathcal{H}_{d-k}(E - \beta; M)$ is a holonomic module of nonzero rank, and the highest nonzero Euler–Koszul homology of M. □

7. Global Euler–Koszul homology as a holonomic family

In this section we explore the interactions of the Euler–Koszul functor with our notion of holonomic family. The main result, Theorem 7.5, is that the family of D-modules $\mathcal{H}_0(E - \beta; \mathcal{M})$ for varying $\beta \in \mathbb{C}^d$ constitutes a holonomic family over $\mathbb{C}^d$.

Resume the notation from Example 2.2 and Definition 4.2. If we give all of the variables $b = b_1, \ldots, b_d$ degree zero, then the polynomial ring $D[b]$ over the Weyl algebra D is $\mathbb{Z}^d$-graded, as is its commutative subalgebra $R[b]$.

Definition 7.1. Let $\mathcal{M}$ be a $\mathbb{Z}^d$-graded $D[b]$-module. The *global Euler endomorphisms* of $\mathcal{M}$ are the commuting endomorphisms $E_1 - b_1, \ldots, E_d - b_d$, the i^{th} of which acts by

$$E_i - b_i : \; m \mapsto (E_i - b_i - \alpha_i)m \quad \text{whenever} \quad m \in \mathcal{M}_\alpha.$$

Each $\mathbb{Z}^d$-graded R-module N yields a *global Euler–Koszul complex* of left $D[b]$-modules,

$$\mathcal{K}_\bullet(E - b; N) \quad = \quad \mathcal{K}_\bullet(E - b; D[b] \otimes_R N),$$

with $D[b]$-linear homomorphisms. Write the homology as $\mathcal{H}_i(E-b; N) = H_i\mathcal{K}_\bullet(E-b; N)$.

For any fixed $\beta \in \mathbb{C}^d$, the zeroth Euler–Koszul homology $\mathcal{H}_0(E - \beta; N)$ of N can be recovered as a fiber (in the sense of Section 2) of the zeroth global Euler–Koszul homology $\mathcal{H}_0(E - b; N)$ of N. The precise statement, as follows, is immediate from the definitions.

Lemma 7.2. *Suppose that N is a $\mathbb{Z}^d$-graded R-module, and that $\beta \in \mathbb{C}^d$. The fiber of $\mathcal{M} = \mathcal{H}_0(E - b; N)$ over β is $\mathcal{M}_\beta = \mathcal{H}_0(E - \beta; N)$.* □

In view of Lemma 7.2, to prove holonomicity for the family $\mathcal{H}_0(E-b;M)$ determined by a toric R-module M we need to establish the coherence condition in Definition 2.1. For this, we shall use the criterion in Proposition 3.4. But before we can do so, we need to know that Euler operators form sequences of parameters on the quotients of polynomial rings by certain initial ideals of toric ideals. We will get a handle on these initial ideals via their minimal primes. For notation, consider the polynomial ring $\mathbb{C}[\xi](x)$ with its usual $\mathbb{Z}$-grading, so that each variable in the list $\xi = \xi_1, \dots, \xi_n$ has degree 1. By a *minimal prime* of a $\mathbb{C}[\xi](x)$-module N, we mean a prime minimal among all of those containing the annihilator of N.

Lemma 7.3. *Let N be a finitely generated $\mathbb{Z}$-graded $\mathbb{C}[\xi](x)$-module, and let $\mathbf{y}$ be a sequence of $\mathbb{Z}$-graded elements in $\mathbb{C}[\xi](x)$. If $\mathbb{C}[\xi](x)/(\langle \mathbf{y}\rangle + \mathfrak{p})$ is finite-dimensional as a vector space over $\mathbb{C}(x)$ for every minimal prime $\mathfrak{p}$ of N, then so is $N/\mathbf{y}N$.*

Proof. Let $I \subseteq \mathbb{C}[\xi](x)$ be the annihilator of N. As N is finitely generated over $\mathbb{C}[\xi](x)/I$, it is enough to show that $\langle \mathbf{y}\rangle + I$ contains a power of the $\mathbb{Z}$-graded maximal ideal $\mathfrak{m} = \langle \xi\rangle$.

Suppose that I has radical J. Then if $\mathfrak{m}^s \subseteq \langle \mathbf{y}\rangle + J$ then $\mathfrak{m}^{rs} \subseteq (\langle \mathbf{y}\rangle + J)^r$, and so $\mathfrak{m}^{rs} \subseteq (\langle \mathbf{y}\rangle + J)^r \subseteq \langle \mathbf{y}\rangle + J^r \subseteq \langle \mathbf{y}\rangle + I$ for some r. Since the minimal primes of I and J are identical it is sufficient to consider J instead of I.

Now let $J = \mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_m$ be a primary decomposition where of course each $\mathfrak{p}_j$ is prime. Then $\mathbf{y} + J = \mathbf{y} + (\mathfrak{p}_1 \cap \mathfrak{p}_2 \cdots \cap \mathfrak{p}_m)$ contains the product $(\mathbf{y}+\mathfrak{p}_1)\cdots(\mathbf{y}+\mathfrak{p}_m)$. Each of the factors $(\mathbf{y}+\mathfrak{p}_j)$ contains a power of $\mathfrak{m}$ by the hypothesis, so $J+\mathbf{y}$ contains the product of these powers of $\mathfrak{m}$. □

We now return to the order filtration on the Weyl algebra D, and the associated graded ring $\mathbb{C}[x,\xi]$ (see Section 3). This filtration, when restricted to the subring R of D, gives an associated graded ring $\mathbb{C}[\xi]$. It induces on R the partial ordering by total degree in which the *initial form* of a polynomial $f \in R$ is the sum $\mathsf{in}(f) \in \mathbb{C}[\xi]$ of all terms of highest total degree, with the variables ∂_i changed to ξ_i. The *initial ideal* of any ideal $J \subseteq R$ is the ideal $\mathsf{in}(J) = \langle \mathsf{in}(f) \mid f \in J\rangle$ generated by the initial forms of all polynomials in J.

In the coming statement, we consider the ideal I_F from Notation 4.4. Let $Ax\xi \subset \mathbb{C}[\xi](x)$ be the sequence $\mathsf{in}(E_1), \dots, \mathsf{in}(E_d)$ obtained from $E_1, \dots, E_d$ by replacing each ∂_i with ξ_i.

Proposition 7.4. *If $\mathsf{in}(I_F) \subseteq \mathbb{C}[\xi]$ is the initial ideal of I_F in the above sense, then the $\mathbb{C}(x)$-vector space $\mathbb{C}[\xi](x)/\langle \mathsf{in}(I_F), Ax\xi\rangle$ is finite-dimensional over $\mathbb{C}(x)$.*

Proof. This statement immediately reduces to the case where $I_F = I_A$, after first replacing $\mathbb{C}[\xi](x)$ with $\mathbb{C}[\xi_j \mid a_j \in F](x_j \mid a_j \in F)$, and then using any matrix A'_F composed of $\dim(F)$ many linearly independent rows of F to play the role of A. We wish to apply Lemma 7.3, so we need to describe the minimal primes of $\mathbb{C}[\xi]/\mathsf{in}(I_A)$.

Define the $(d+1)\times(n+1)$ matrix $\hat{A}$ by placing a row $(1,\dots,1)$ across the top of A, and subsequently adding a leftmost column $(1,0,0,\dots,0)$. If ξ_0 is a new variable and $\hat{\xi} = \{\xi_0\}\cup\xi$, then $\mathbb{C}[\xi]/\mathsf{in}(I_A) \cong \mathbb{C}[\hat{\xi}]/\langle I_{\hat{A}}, \xi_0\rangle$. Since $\langle I_{\hat{A}}, \xi_0\rangle$ is $\mathbb{Z}\hat{A}$-graded, every minimal prime of $\mathbb{C}[\xi]/\mathsf{in}(I_A)$ is the image in $\mathbb{C}[\xi] = \mathbb{C}[\hat{\xi}]/\langle \xi_0\rangle$ of a prime ideal $I_{\hat{F}} \subset \mathbb{C}[\hat{\xi}]$ for some face $\hat{F}$ of $\mathbb{N}\hat{A}$ with $\xi_0 \in I_{\hat{F}}$.

Thus we only need that for all faces $\hat{F}$ of $\mathbb{N}\hat{A}$ with $\xi_0 \in I_{\hat{F}}$, the ring $\mathbb{C}[\hat{\xi}](\hat{x})/\langle I_{\hat{F}}, Ax\xi\rangle$ has finite dimension as a $\mathbb{C}(\hat{x})$-vector space. Pick one such $\hat{F}$. Then $\xi_0 \notin \hat{F}$, since $\xi_0 \in I_{\hat{F}}$.

Hence the column indices of $\hat{F}$ form a face F of A whose $\mathbb{C}$-linear span does not contain the origin; indeed, there is a bijection between the faces of $\hat{A}$ not containing the column $\hat{a_0}$ and the faces of A not containing the origin in their span.

We infer that the columns of A constituting F lie on a hyperplane in $\mathbb{C}^d$ off the origin. Therefore the vector $(1,\dots,1) \in \mathbb{C}^{|F|}$ lies in the rowspan of F. This implies that

$$\begin{aligned}\mathbb{C}[\hat{\xi}](\hat{x})/\langle I_{\hat{F}}, Ax\xi\rangle &= \mathbb{C}[\hat{\xi}](\hat{x})/\langle I_{\hat{F}}, Fx\xi\rangle \\ &= \mathbb{C}[\hat{\xi}](\hat{x})/\langle I_{\hat{F}}, \hat{F}x\xi\rangle \\ &= \mathbb{C}[\hat{\xi}](\hat{x})/\langle I_{\hat{F}}, \hat{A}\hat{x}\hat{\xi}\rangle,\end{aligned}$$

where we write $Fx\xi = \{\mathsf{in}(E_1^F),\dots,\mathsf{in}(E_d^F)\}$, $\hat{F}x\xi = \{\sum_{j\in F} x_j\xi_j, \mathsf{in}(E_1^F),\dots,\mathsf{in}(E_d^F)\}$, and $\hat{A}\hat{x}\hat{\xi} = \{\sum_{j=0}^n x_j\xi_j, \mathsf{in}(E_1),\dots,\mathsf{in}(E_d)\}$.

We have thus reduced to the projective situation. This case of the proposition follows from the proof of [Ado94, Theorem 3.9], which proceeds by showing precisely that in the projective situation the Krull dimension of $\mathbb{C}[x,\xi]/(\mathsf{in}(I_A) + \langle Ax\xi\rangle)$ equals n. □

Theorem 7.5. *For any toric R-module M, the sheaf $\tilde{\mathcal{M}}$ on $\mathbb{C}^d$ whose global section module is $\mathcal{M} = \mathcal{H}_0(E-b;M)$ constitutes a holonomic family over $\mathbb{C}^d$; in other words, $\mathcal{M}_\beta = \mathcal{H}_0(E-\beta;M)$ is holonomic for all $\beta \in \mathbb{C}^d$, and $\mathcal{M}(x)$ is finitely generated over $\mathbb{C}[b](x)$.*

Proof. That the fibers $\mathcal{M}_\beta = \mathcal{H}_0(E-\beta;M)$ are holonomic modules is a consequence of Lemma 7.2 and Proposition 5.1.

For the coherence condition, suppose first that $M = S_F$ for some face F. Using order filtrations as above, the graded $\mathbb{C}[b](x)$-module associated to $\mathcal{M}(x)$ is a quotient of $N = \mathbb{C}[\xi][b](x)/\langle \mathsf{in}(I_F), Ax\xi\rangle$, since $\mathsf{in}(I_F) + \langle Ax\xi\rangle = \mathsf{in}(I_F) + \mathsf{in}(E-b) \subseteq \mathsf{in}(\langle I_F, E-b\rangle)$. As the generators for $\mathsf{in}(I_F)$ and $Ax\xi$ do not involve b, Proposition 7.4 implies that N is finitely generated over $\mathbb{C}[b](x)$. Hence $\mathcal{M}(x)$ is finitely generated by Proposition 3.4.

Now let M be any toric R-module. For $\alpha \in \mathbb{Z}^d$ and with $b_i' = b_i - \alpha_i$ the isomorphism

$$\mathcal{H}_0(E-b';S_F)(\alpha) \;=\; \mathcal{H}_0(E-b+\alpha;S_F)(\alpha) \;=\; \mathcal{H}_0(E-b;S_F(\alpha))$$

together with the independence of $\mathsf{in}(I_F)$ and $Ax\xi$ of b proves the theorem if M has toric length $\ell = 1$. For $\ell > 1$, use the short exact sequence $M_1 \hookrightarrow M \twoheadrightarrow M/M_1$ from a toric filtration. The resulting long exact sequence of global Euler–Koszul homology $\mathcal{H}_\bullet(E-b;\_)$ tensored with the flat module $\mathbb{C}(x)$ places the module $\mathcal{H}_0(E-b;M)(x)$ between the two modules $\mathcal{H}_0(E-b;M_1)(x)$ and $\mathcal{H}_0(E-b;M/M_1)(x)$, both of which are finitely generated by induction on ℓ. □

8. Isomorphism of the two homology theories

Let B be a reduced parameter scheme and suppose $\beta \in B_{\mathbb{C}}$ is determined by a regular sequence $\mathbf{y}$ in $\Gamma(\mathcal{O}_B, B)$. In Corollary 3.3 we found that the commutative Koszul complex $K_\bullet(\mathbf{y}, \mathcal{M}(x))$ detects rank-jumps in holonomic families $\mathcal{M}$ of D-modules at β. In the special case of generalized hypergeometric systems associated to toric modules, where in particular $B = \mathbb{C}^d$, we also found in Theorem 6.6 that the noncommutative Euler–Koszul complex $\mathcal{K}_\bullet(E-\beta;M)$ detects the quasi-degrees where local cohomology of the toric R-module M is non-vanishing in cohomological degree less than d. In this section we prove that if $\mathcal{M} = \mathcal{H}_0(E-b; M)$ is defined through the toric module M then these two complexes are isomorphic in the derived category, and in particular have the same homology.

We begin by noting that the global Euler–Koszul complex can be interpreted as a collection of commutative Koszul complexes, similarly to Lemma 4.3.

Lemma 8.1. *Let N be a $\mathbb{Z}^d$-graded R-module and $\alpha \in \mathbb{Z}^d$. The $\mathbb{Z}^d$-graded degree α part $\mathcal{K}_\bullet(E-b; N)_\alpha$ of the global Euler–Koszul complex agrees with the ordinary Koszul complex $K_\bullet(E-b-\alpha; (D[b] \otimes_R N)_\alpha)$ constructed over the polynomial ring $\Theta[b] \subseteq D[b]$. That is,*

$$\mathcal{K}_\bullet(E-\beta; N) \quad = \quad \bigoplus_{\alpha \in \mathbb{Z}^d} K_\bullet(E-\beta-\alpha; (D[b] \otimes_R N)_\alpha),$$

the right hand side being a direct sum of ordinary Koszul complexes of $\Theta[b]$-modules. $\square$

Recall from Theorem 7.5 that $\mathcal{H}_0(E-b; M)$ represents a holonomic family whenever M is a toric R-module. The following statement links the two homological theories that we have studied: Euler–Koszul homology and Koszul homology of holonomic families.

Theorem 8.2. *Let M be a toric R-module, and consider the holonomic family on $\mathbb{C}^d$ with sections $\mathcal{M} = \mathcal{H}_0(E-b; M)$. For each parameter vector $\beta \in \mathbb{C}^d$, the Euler–Koszul homology of M and the ordinary Koszul homology of $\mathcal{M}$ over $\mathbb{C}[b]$ are isomorphic:*

$$\mathcal{H}_i(E-\beta; M) \quad \cong \quad H_i(b-\beta; \mathcal{M}).$$

Proof. Write $\mathcal{K}^E_\bullet = \mathcal{K}_\bullet(E-b; M)$ for the global Euler–Koszul complex, thought of as a column pointing downward with its bottom at row 0, and let $K^b_\bullet = K_\bullet(b-\beta, \mathbb{C}[b])$ be the ordinary Koszul complex on $b-\beta$, thought of as a row pointing leftward toward its end at column 0. Consider the double complex $K^b_\bullet \otimes_{\mathbb{C}[b]} \mathcal{K}^E_\bullet$.

Taking horizontal $(K^b_\bullet)$-homology first leaves only one column, namely the leftmost column, and that column is the Euler–Koszul complex $\mathcal{K}_\bullet(E-\beta; M)$. It follows that $H_\bullet \operatorname{Tot}(K^b_\bullet \otimes \mathcal{K}^E_\bullet) \cong \mathcal{H}_\bullet(E-\beta; M)$.

Now we need to check that $H_\bullet \operatorname{Tot}(K^b_\bullet \otimes \mathcal{K}^E_\bullet) \cong H_\bullet(b-\beta; \mathcal{M})$. For this it is enough to show that taking the vertical $(\mathcal{K}^E_\bullet)$-homology of the double complex leaves only one row (i.e., the bottom row), for then that row is the Koszul complex $K^b_\bullet$ tensored over $\mathbb{C}[b]$ with the global Euler–Koszul homology $\mathcal{M} = \mathcal{H}_0(E-b; M)$.

All the homomorphisms in $K^b_\bullet \otimes \mathcal{K}^E_\bullet$ are $\mathbb{Z}^d$-graded, so it suffices to check the acyclicity of the vertical differential separately on each $\mathbb{Z}^d$-graded component. But $(D[b] \otimes_R M)_\alpha = (D \otimes_R M)_\alpha \otimes_{\mathbb{C}} \mathbb{C}[b]$, so $\big(b_1 - (E_1 - \alpha_1), \ldots, b_d - (E_d - \alpha_d)\big) = -(E-b-\alpha)$ is a regular sequence in $\Theta[b]$ on the $\Theta[b]$-module $(D[b] \otimes_R M)_\alpha$. Now use Lemma 8.1. $\square$

9. Combinatorics of hypergeometric ranks

This section contains our results on our motivating problem: a geometric description of the rank-jumping locus for generalized hypergeometric systems (Theorem 9.1 and Corollary 9.3), and a characterization of Cohen–Macaulayness for semigroup rings through the absence of rank-jumps (Corollary 9.2).

Theorem 9.1. *The rank-jumping parameters $\beta \in \mathbb{C}^d$ for the hypergeometric holonomic family $\mathcal{H}_0(E-b; M)$ of a toric module M are the exceptional quasi-degrees of M.*

Proof. By Theorem 6.6, a parameter β is an exceptional quasi-degree of M if and only if the Euler–Koszul homology $\mathcal{H}_i(E-\beta;M)$ is of nonzero rank for some $i>0$. This occurs if and only if the homology $H_i(b-\beta;\mathcal{M})$ of the hypergeometric holonomic family $\mathcal{M}=\mathcal{H}_0(E-b;M)$ is holonomic of nonzero rank for some $i>0$, by Theorem 8.2. The nonzero rank condition ensures that the non-vanishing of $H_i(b-\beta;\mathcal{M})$ is equivalent to its non-vanishing after tensoring with $\mathbb{C}(x)$ over $\mathbb{C}[x]$. But $\mathbb{C}(x)\otimes_{\mathbb{C}[x]} H_i(b-\beta;\mathcal{M}) = H_i(b-\beta;\mathcal{M}(x))$ by exactness of localization, and this Koszul homology is nonzero if and only if β is rank-jumping by Corollary 3.3. □

For a finitely generated R-module, the vanishing of $H^i_{\mathfrak{m}}(M)$ for all $i<d$ is equivalent to M having depth d. This immediately implies the following.

Corollary 9.2. *The holonomic family $\mathcal{M}^A_\beta=\mathcal{H}_0(E-\beta;S_A)$ has a nonempty set of rank-jumping parameters $\beta\in\mathbb{C}^d$ if and only if $S_A=R/I_A$ fails to be Cohen–Macaulay.* □

Not only is the rank-jump locus closed in $\mathbb{C}^d$ but it has a nice geometric structure.

Corollary 9.3. *The set of rank-jumping parameters is a finite union of translates of linear subspaces $\mathbb{C}F$ generated by faces F of A. A translate of $\mathbb{C}F$ appears if and only if the prime ideal I_F lies in the support of $\mathrm{Ext}^{n-i}_R(M,R)$ for some $i\le d-1$.*

Proof. The graded Matlis dual of the local cohomology module $H^i_{\mathfrak{m}}(M)$ is, by local duality, a $\mathbb{Z}^d$-graded translate of $\mathrm{Ext}^{n-i}_R(M,R)$. Hence the rank-jumping parameters form a $\mathbb{Z}^d$-graded translate of the negative of the set $\mathrm{qdeg}(\bigoplus_{i=0}^{d-1}\mathrm{Ext}^{n-i}_R(M,R))$. □

Porism 9.4. *The set of exceptional parameters for a GKZ hypergeometric system $\mathcal{M}^A_\beta=\mathcal{H}_0(E-\beta;S_A)$ has codimension at least 2 in $\mathbb{C}^d$.*

Proof. Set $S=S_A$. By Corollary 9.3, we need $\mathrm{Ext}^{n-i}_R(S,R)_{\mathfrak{p}}=0$ for all primes $\mathfrak{p}$ of dimension $d-1$ and all $i\le d-1$. For such primes, $R_{\mathfrak{p}}$ is regular and local of dimension $n-d+1$, so $\mathrm{Ext}^{n-i}_R(S,R)_{\mathfrak{p}}=\mathrm{Ext}^{n-i}_{R_{\mathfrak{p}}}(S_{\mathfrak{p}},R_{\mathfrak{p}})=0$ unless $i\ge d-1$. As $\mathrm{Ext}^{n-d+1}_R(S,R)_{\mathfrak{p}}$ is Matlis dual over $R_{\mathfrak{p}}$ to $H^0_{\mathfrak{p}R_{\mathfrak{p}}}(S_{\mathfrak{p}})$, which vanishes because S is a domain, we are done. □

Remark 9.5. Corollary 9.3, and more generally Theorem 2.6, indicates that the dimensions of the holomorphic solution spaces of hypergeometric systems $\mathcal{M}^A_\beta$ behave quite tamely as functions of β. However, the methods in this article do not directly involve a study of the variation of the holomorphic solutions themselves (cf. Remark 2.8), which could in principle be much worse. In fact, work of Saito [Sai01] implies that the isomorphism classes of A-hypergeometric systems $\mathcal{M}^A_\beta$ do not vary at all algebraically with β. The question therefore remains whether the holomorphic solution space of a hypergeometric system $\mathcal{M}^A_\beta$ varies as a function of β more like the rank of $\mathcal{M}^A_\beta$ or more like the isomorphism class of $\mathcal{M}^A_\beta$. This problem is particularly important for research toward constructing explicit solutions to A-hypergeometric systems.

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