DEPENDENCE OF LYUBEZNIK NUMBERS OF CONES OF PROJECTIVE SCHEMES ON PROJECTIVE EMBEDDINGS

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Abstract. We construct complex projective schemes with Lyubeznik numbers of their cones depending on the choices of projective embeddings. This answers a question of G. Lyubeznik in the characteristic 0 case. Note that the situation is quite different in the positive characteristic case using the Frobenius endomorphism. Reducibility of schemes is essential in our argument, and the question is still open in the irreducible singular case.

Introduction

Let X be a projective scheme over $\mathbb{C}$ with L a very ample line bundle. Let C be the cone of X associated with L. Let $x_1,\dots,x_n$ be projective coordinates of $Y:=\mathbb{P}^{n-1}_{\mathbb{C}}$ containing X so that $\mathcal{O}_{\mathbb{P}^{n-1}}(1)|_X=L$. Let I be the ideal of $R:=\mathbb{C}[x_1,\dots,x_n]$ defining the cone $C\subset\mathbb{A}^n_{\mathbb{C}}$. The *Lyubeznik numbers* $\lambda_{k,j}(C)$ are defined by

(1) $$\lambda_{k,j}(C):=\dim_{\mathbb{C}}\mathrm{Ext}^k_R(\mathbb{C},H^{n-j}_I R)\qquad(k,j\in\mathbb{N}),$$

see [Ly1, Ly2, NWZ]. Here the $H^{n-j}_I R$ are the local cohomology groups, and vanish for $j>\dim C$, see Remark (ii) after (1.1) below. The higher extension groups $\mathrm{Ext}^k_R(\mathbb{C},*)$ can be calculated by the Koszul complex for the multiplications of the x_i ($i\in[1,n]$) which gives a free resolution of $\mathbb{C}$ over R. This holds also for the higher torsion groups $\mathrm{Tor}^R_{n-k}(\mathbb{C},*)$. Setting $V:=\mathrm{Spec}\,R=\mathbb{A}^n_{\mathbb{C}}$, we then get the isomorphisms

(2) $$\mathrm{Ext}^k_R(\mathbb{C},H^{n-j}_I R)=\mathrm{Tor}^R_{n-k}(\mathbb{C},H^{n-j}_I R)=H^{k-n}\mathbf{L}i^*_{0,V}(\mathcal{H}^{n-j}_C\mathcal{O}_V).$$

Here $i_{A,B}:A\hookrightarrow B$ denotes an inclusion of a subset $A\subset B$ in general, $\mathbf{L}i^*_{0,V}$ means the derived pull-back functor for $\mathcal{O}$-modules endowed with an integrable connection (that is, for left D-modules), and the $\mathcal{H}^{n-j}_C$ are the algebraic local cohomology functors for the closed subscheme $C\subset V$. Note that $\mathcal{O}_V$ is algebraic so that $\Gamma(V,\mathcal{O}_V)=R$.

Using the Riemann-Hilbert correspondence, we then get the following (see (1.1) below).

Proposition 1. *In the above notation, we have the equalities*

$$\lambda_{k,j}(C)=\dim_{\mathbb{Q}}H^k i^!_{0,C}({}^{\mathbf{p}}\mathcal{H}^{-j}\mathbb{D}\mathbb{Q}_C)\qquad(k,j\in\mathbb{N}).$$

(See also [GS].) Here $\mathbb{Q}_{C^{\mathrm{an}}}$ and its dual $\mathbb{D}\mathbb{Q}_{C^{\mathrm{an}}}$ (see [Ve1]) are respectively denoted by $\mathbb{Q}_C$ and $\mathbb{D}\mathbb{Q}_C$ to simplify the notation (where C^{an} is the analytic space associated with C), and similarly for $\mathbb{Q}_X$, $\mathbb{D}\mathbb{Q}_X$. The ${}^{\mathbf{p}}\mathcal{H}^j$ are the cohomology functors associated with the truncations ${}^{\mathbf{p}}\tau_{\leqslant k}$ constructed in [BBD] (see also [Di, KS]). Note that the usual cohomology functors $\mathcal{H}^j$ for bounded complexes of D-modules having regular holonomic cohomology sheaves correspond to the functors ${}^{\mathbf{p}}\mathcal{H}^j$ by the Riemann-Hilbert correspondence.

Set

(3) $$\mathcal{F}_j:={}^{\mathbf{p}}\mathcal{H}^{-j}\mathbb{D}\mathbb{Q}_C,\quad \mathcal{F}'_j:=\mathcal{F}_j|_{C'}\quad\text{with}\quad C':=C\setminus\{0\}\quad(j\in\mathbb{N}).$$

We have the following (see (1.2) below).

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Proposition 2. *There are isomorphisms*

$$H^k i_{0,C}^! \mathcal{F}_j = H^{k-1} i_{0,C}^* \mathbf{R}(i_{C',C})_* \mathcal{F}'_j = H^{k-1}(C', \mathcal{F}'_j) \quad \text{if} \quad k \geqslant 2.$$

For $k \in \mathbb{Z}$, $j \in \mathbb{N}$, set

$$\begin{aligned} H^k_{(j)}(X) &:= H^k(X, {}^{\mathbf{p}}\mathcal{H}^{-j}\mathbb{D}\mathbb{Q}_X), \\ H^k_{(j)}(X)^L &:= \operatorname{Ker}\bigl(c_1(L) : H^k_{(j)}(X) \to H^{k+2}_{(j)}(X)(1)\bigr), \\ H^k_{(j)}(X)_L &:= \operatorname{Coker}\bigl(c_1(L) : H^{k-2}_{(j)}(X)(-1) \to H^k_{(j)}(X)\bigr), \end{aligned} \tag{4}$$

where (m) denotes a Tate twist for $m \in \mathbb{Z}$, see [De1]. (Note that a subquotient of $H^k_{(j)}(X)$ is identified with $\mathrm{Gr}^j_G H_{j-k}(X)$ by a spectral sequence, where $\{G^j\}$ is a decreasing filtration on $H_{j-k}(X) = H^{k-j}(X, \mathbb{D}\mathbb{Q}_X)$ induced by ${}^{\mathbf{p}}\tau_{\leqslant -j}$ on $\mathbb{D}\mathbb{Q}_X$, see [BBD, Ve2].)

By a generalized Thom-Gysin sequence, we have the following (see (1.3–4) below).

Proposition 3. *There are short exact sequences*

$$0 \to H^k_{(j-1)}(X)_L(1) \to H^{k-1}(C', \mathcal{F}'_j) \to H^{k-1}_{(j-1)}(X)^L \to 0 \quad (k \in \mathbb{Z}).$$

As a corollary of Propositions 1–3, we get the following.

Corollary 1. *The Lyubeznik numbers $\lambda_{k,j}(C)$ of the cone C of a projective scheme X depend on the choice of a very ample line bundle L if the following condition holds:*

$$\dim H^k_{(j-1)}(X)_L + \dim H^{k-1}_{(j-1)}(X)^L \ \text{ depends on } \ L \ \text{ for some } k \geqslant 2,\ j \geqslant 1. \tag{5}$$

Here the converse is true in certain cases, see Corollary 1.7 below. This implies the independence of the Lyubeznik numbers under projective embeddings in the $\mathbb{Q}$-homology manifold case (generalizing [Swl] in the non-singular case), see Corollary 1.8 below. Using Corollary 1, we can prove the following.

Theorem 1. *There are projective schemes over $\mathbb{C}$ such that their irreducible components are smooth and the Lyubeznik numbers $\lambda_{k,j}(C)$ of their cones C depend on the choices of projective embeddings for some $k \geqslant 2$. Here j coincides with the dimension of the lowest dimensional irreducible component of C, and X can be equidimensional.*

This answers a question of G. Lyubeznik [Ly2] in the characteristic 0 case, see [NWZ, Zh] for the positive characteristic case and [Swl] for the X non-singular case. Theorem 1 may be rather unexpected, since the situation is entirely different in the positive characteristic case where the Frobenius endomorphism can be used. In the case of schemes over $\mathbb{C}$, the proof of Theorem 1 is reduced by Corollary 1 to finding complex projective schemes X satisfying condition (5). It is rather easy to construct such schemes if the condition $k \geqslant 2$ is omitted (where k may be negative), see (2.1) below. To satisfy the last condition, we need some more construction, where the argument is much easier in the non-equidimensional case (see (2.2) below), and we have to use Hodge theory in the equidimensional case (see (2.3) below). In these arguments, *reducibility* of schemes is essential, and the question is still open in the X irreducible singular case.

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In Section 1 we review generalized Thom-Gysin sequences, study the Lyubeznik numbers in the X $\mathbb{Q}$-homology manifold case, and prove Propositions 1–3. In Section 2 we prove Theorem 1 by constructing desired examples.

1. Preliminaries

In this section we review generalized Thom-Gysin sequences, study the Lyubeznik numbers in the X $\mathbb{Q}$-homology manifold case, and prove Propositions 1–3.

1.1 Proof of Proposition 1. By the Riemann-Hilbert correspondence for algebraic D-modules using the de Rham functor DR (see for instance [Bo]), the derived pull-back functor $\mathbf{L}i_{0,V}^*[-n]$ (explained before Proposition 1) corresponds to $i_{0,V}^!$, that is,

$$\mathrm{DR}\circ\mathbf{L}i_{0,V}^*[-n]=i_{0,V}^!\circ\mathrm{DR}, \tag{1.1.1}$$

see also Remark (iii) below (and [Sa2, Remark after Corollary 2.24]). We have moreover

$$\mathrm{DR}(\mathcal{O}_V[n])=\mathbb{D}\mathbb{C}_V\ (=\mathbb{C}_V[2n]), \tag{1.1.2}$$

since $\mathrm{DR}(\mathcal{O}_V)=\mathbb{C}_V[n]$. Here $\mathbb{D}$ denotes the dual functor, see [Ve1]. Note that the functor $\mathbf{L}i_{0,V}^*[-n]$ corresponds to $i_{0,V}^*$ under the *contravariant* functor $\mathrm{Sol}=\mathbb{D}\circ\mathrm{DR}$, see Remark (i) below and also [Ka], [KK], [Me], [Sa1, Remark 2.4.15 (3)]. (The equivalence of categories itself is not really needed here.)

The derived local cohomology functor $\mathbf{R}\Gamma_C$ corresponds to $(i_{C,V})_*i_{C,V}^!$ (see [Bo]), and we have

$$i_{C,V}^!\mathbb{D}\mathbb{Q}_V=\mathbb{D}i_{C,V}^*\mathbb{Q}_V=\mathbb{D}\mathbb{Q}_C. \tag{1.1.3}$$

So Proposition 1 follows.

Remarks. (i) Let X be a complex manifold (or a smooth complex algebraic variety) of dimension n. For a bounded complex $M^\bullet$ of left $\mathcal{D}_X$-modules having regular holonomic cohomology sheaves, the de Rham and solution functors can be defined by

$$\begin{aligned}\mathrm{DR}(M^\bullet)&:=\mathbf{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{O}_X,M^\bullet)[n],\\ \mathrm{Sol}(M^\bullet)&:=\mathbf{R}\mathcal{H}om_{\mathcal{D}_X}(M^\bullet,\mathcal{O}_X)[n].\end{aligned} \tag{1.1.4}$$

(If X is a smooth complex algebraic variety, X and $M^\bullet$ on the right-hand side are respectively replaced by X^{an} and $M^{\mathrm{an}\,\bullet}$.) Taking the composition, we can get a perfect pairing

$$\mathrm{DR}(M^\bullet)\otimes_{\mathbb{C}}\mathrm{Sol}(M^\bullet)\to\mathbf{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{O}_X,\mathcal{O}_X)[2n]=\mathbb{C}_X[2n]=\mathbb{D}\mathbb{C}_X, \tag{1.1.5}$$

that is, the corresponding morphism

$$\mathrm{DR}(M^\bullet)\to\mathbb{D}\big(\mathrm{Sol}(M^\bullet)\big):=\mathbf{R}\mathcal{H}om_{\mathbb{C}_X}\big(\mathrm{Sol}(M^\bullet),\mathbb{D}\mathbb{C}_X\big) \tag{1.1.6}$$

is an isomorphism. It is also known that DR commutes with $\mathbb{D}$. (This follows, for instance, from [Sa1, Proposition 2.4.12].)

(ii) In the notation of the introduction, we have by [Ha, Theorem 3.8] (see also [BS, Iy])

$$H_I^{n-j}R=0\quad\text{for}\quad n-j<\operatorname{codim}C. \tag{1.1.7}$$

In our case this follows by taking a minimal dimensional complete intersection containing $C\subset\mathbb{C}^n$ and using the composition of derived local cohomology functors. Note also that, by the Riemann-Hilbert correspondences and (1.1.2), this vanishing is equivalent to

$$^{\mathbf{p}}\mathcal{H}^{-j}\mathbf{R}\Gamma_C\mathbb{D}\mathbb{Q}_X=0\ \ (\text{that is, }{}^{\mathbf{p}}\mathcal{H}^j\mathbb{Q}_C=0\,)\quad\text{for}\quad j>\dim C, \tag{1.1.8}$$

and the assertion for $\mathbb{Q}_C$ follows easily from the definition of the t-structure in [BBD].

(iii) Let X be a closed submanifold of a complex manifold Y. For a regular holonomic right $\mathcal{D}_Y$-module M, there are natural inclusions (as $\mathcal{O}_Y$-modules)

$$\mathcal{E}xt^j_{\mathcal{O}_Y}(\mathcal{O}_X, M) \hookrightarrow \mathcal{H}^j_{[X]}M \quad (j \in \mathbb{N}), \tag{1.1.9}$$

inducing isomorphisms of right $\mathcal{D}_Y$-modules

$$\mathcal{E}xt^j_{\mathcal{O}_Y}(\mathcal{O}_X, M) \otimes_{\mathcal{D}_X} \mathcal{D}_{X\to Y} \xrightarrow{\sim} \mathcal{H}^j_{[X]}M \quad (j \in \mathbb{N}). \tag{1.1.10}$$

Here $\mathcal{D}_{X\hookrightarrow Y} := \mathcal{O}_X \otimes_{\mathcal{O}_Y} \mathcal{D}_Y$, and the $\mathcal{H}^j_{[X]}M$ are the algebraic local cohomology sheaves defined by

$$\mathcal{H}^j_{[X]}M := \varinjlim_k \mathcal{E}xt^j_{\mathcal{O}_Y}(\mathcal{O}_Y/\mathcal{I}^k_X, M) \quad (j \in \mathbb{N}), \tag{1.1.11}$$

with $\mathcal{I}_X \subset \mathcal{O}_Y$ the ideal sheaf of $X \subset Y$, see [KK]. Note that the sources of (1.1.9) and (1.1.10) are respectively the cohomological pull-back of M as right $\mathcal{D}$-module by the inclusion $i_{X,Y} : X \hookrightarrow Y$ and its direct image as right $\mathcal{D}$-module by $i_{X,Y}$. (Using a spectral sequence, the proof of (1.1.10) can be reduced to the codimension 1 case.)

Set $r := \mathrm{codim}_Y X$. The formula corresponding to (1.1.10) for a regular holonomic *left* $\mathcal{D}$-module M is as follows (see, for instance, the proof of [KK, Corollary 5.4.6]):

$$\mathcal{D}_{Y\leftarrow X} \otimes_{\mathcal{D}_X} \mathcal{T}or^{\mathcal{O}_Y}_{r-j}(\mathcal{O}_X, M) = \mathcal{H}^j_{[X]}M \quad (j \in \mathbb{N}). \tag{1.1.12}$$

1.2 Proof of Proposition 2. The last isomorphism in Proposition 2 holds, since 0 is the vertex of the cone C. The first isomorphism follows from the long exact sequence associated with the distinguished triangle

$$i^!_{0,C}\mathcal{F}_j \to i^*_{0,C}\mathcal{F}_j \to i^*_{0,C}\mathbf{R}(i_{C',C})_* i^*_{C',C}\mathcal{F}_j \xrightarrow{+1}, \tag{1.2.1}$$

since $H^k i^*_{0,C}\mathcal{F}_j = 0$ for $k > 0$, see [BBD] (and also [Sa2, Remark after Corollary 2.24]). The last triangle can be obtained by applying the functor $i^*_{0,C}$ to the triangle

$$(i_{0,C})_* i^!_{0,C} \to id \to \mathbf{R}(i_{C',C})_* i^*_{C',C} \xrightarrow{+1} .$$

This finishes the proof of Proposition 2.

1.3. Generalized Thom-Gysin sequences (see also [Ko, Swz]). Let $\pi : E \to X$ be a vector bundle of rank r on a complex analytic space X which is assumed connected. Set $E' := E \setminus X$, where X is identified with the zero section of E. There are natural morphisms

$$i_X : X \hookrightarrow E, \quad j_{E'} : E' \hookrightarrow E, \quad \pi' := \pi|_{E'} : E' \to X.$$

For $\mathcal{F}^\bullet \in D^b_c(X, \mathbb{Q})$, we have the distinguished triangle

$$\mathcal{F}^\bullet \xrightarrow{\xi} \mathcal{F}^\bullet(r)[2r] \to \mathbf{R}\pi'_*\pi'^!\mathcal{F}^\bullet \xrightarrow{+1}, \tag{1.3.1}$$

inducing a long exact sequence called a *generalized Thom-Gysin sequence*:

$$\to H^k(X, \mathcal{F}^\bullet) \to H^{k+2r}(X, \mathcal{F}^\bullet)(r) \to H^k(E', \pi'^!\mathcal{F}^\bullet) \to H^{k+1}(X, \mathcal{F}^\bullet) \to . \tag{1.3.2}$$

Indeed, the triangle (1.3.1) is identified with the distinguished triangle

$$i^!_X\pi^!\mathcal{F}^\bullet \to \mathbf{R}\pi_*\pi^!\mathcal{F}^\bullet \to \mathbf{R}\pi'_*\pi'^!\mathcal{F}^\bullet \xrightarrow{+1}, \tag{1.3.3}$$

since $\pi^!\mathcal{F}^\bullet = \pi^{-1}\mathcal{F}^\bullet(r)[2r]$. The last triangle is obtained by applying $\mathbf{R}\pi_*$ to

$$(i_X)_* i^!_X\pi^!\mathcal{F}^\bullet \to \pi^!\mathcal{F}^\bullet \to \mathbf{R}(j_{E'})_* j^*_{E'}\pi^!\mathcal{F}^\bullet \xrightarrow{+1} . \tag{1.3.4}$$

The morphism ξ in (1.3.1) is induced by the Euler class of E via the adjunction isomorphism for a^*_X and $(a_X)_*$:

$$\mathrm{Hom}(\mathbb{Q}_X, \mathbb{Q}_X(r)[2r]) = \mathrm{Hom}\big(\mathbb{Q}, \mathbf{R}\Gamma(\mathbb{Q}_X(r)[2r])\big) = H^{2r}(X, \mathbb{Q})(r). \tag{1.3.5}$$

Here Hom denotes the group of morphisms in the derived categories, and the *Euler class* of E is the image of 1 by the following morphism which is induced by ξ for $\mathcal{F}^\bullet = \mathbb{Q}_X$:

$$\mathbb{Q} = H^0(X,\mathbb{Q}) \to H^{2r}(X,\mathbb{Q})(r), \tag{1.3.6}$$

see also [KS, Ex. III.7]. Note that the Euler class of E (denoted by e) induces the morphism ξ in (1.3.1) by using its tensor product with the identity on $\mathcal{F}^\bullet$ as follows:

$$\mathcal{F}^\bullet = \mathbb{Q}_X \otimes_{\mathbb{Q}} \mathcal{F}^\bullet \xrightarrow{e\otimes id} \mathbb{Q}_X(r)[2r] \otimes_{\mathbb{Q}} \mathcal{F}^\bullet = \mathcal{F}^\bullet(r)[2r]. \tag{1.3.7}$$

Indeed, ξ coincides with the tensor product of ξ for $\mathcal{F}^\bullet = \mathbb{Q}_X$ (that is, the Euler class of E) with the identity on $\mathcal{F}^\bullet$. This is shown by taking π_* of the commutative diagram

$$\begin{array}{ccc} (\Gamma_X I^\bullet) \otimes_{\mathbb{Q}} \pi^{-1}\mathcal{F}^\bullet & \hookrightarrow & I^\bullet \otimes_{\mathbb{Q}} \pi^{-1}\mathcal{F}^\bullet \\ \downarrow & & \downarrow \\ \Gamma_X J^\bullet & \hookrightarrow & J^\bullet \end{array} \tag{1.3.8}$$

where $I^\bullet$ is a flasque resolution of $\mathbb{Q}_E = \pi^{-1}\mathbb{Q}_X$, and $J^\bullet$ is a flasque resolution of $I^\bullet \otimes \pi^{-1}\mathcal{F}^\bullet$.

Remark. The long exact sequence (1.3.2) holds in the category of mixed $\mathbb{Q}$-Hodge structures if $\mathcal{F}^\bullet$ underlies a bounded complex of mixed Hodge modules. Indeed, the above construction can be lifted naturally in the category of mixed Hodge modules, see [Sa2].

1.4. Proof of Proposition 3. In the notation of (1.3) and the introduction, we have

$$\pi'^{\,!}\mathcal{F}^\bullet = \mathcal{F}'^\bullet_j[1] \quad \text{by setting} \quad \mathcal{F}^\bullet := {}^{\mathbf{p}}\mathcal{H}^{-j+1}\mathbb{D}\mathbb{Q}_X, \tag{1.4.1}$$

where $E' = C'$ and $r = 1$. So Proposition 3 follows from (1.3.2).

1.5. $\mathbb{Q}$-homology manifold case (see [GS] for the X smooth case). Let X be a projective scheme such that

$${}^{\mathbf{p}}\mathcal{H}^j\mathbb{Q}_X = 0 \quad (j \ne d), \tag{1.5.1}$$

where $d \in \mathbb{Z}_{>0}$. This condition is satisfied for instance if X^{an} is purely d-dimensional, and is a $\mathbb{Q}$-homology manifold or analytic-locally a complete intersection. (The proof of the last assertion follows, for instance, by using the Riemann-Hilbert correspondence and the local cohomology sheaves defined as in (1.1.11).)

In the notation of the introduction, the assumption (1.5.1) implies that

$$\mathrm{Supp}\,{}^{\mathbf{p}}\mathcal{H}^k\mathbb{D}\mathbb{Q}_C,\ \mathrm{Supp}\,{}^{\mathbf{p}}\mathcal{H}^k\mathbf{R}(j_{C'})_*\mathbb{D}\mathbb{Q}_{C'} \subset \{0\} \quad (k \ne -d-1). \tag{1.5.2}$$

We have the distinguished triangle

$$\mathbb{Q}_{\{0\}} \to \mathbb{D}\mathbb{Q}_C \to \mathbf{R}(j_{C'})_*\mathbb{D}\mathbb{Q}_{C'} \xrightarrow{+1}, \tag{1.5.3}$$

which is the dual of the short exact sequence

$$0 \to (j_{C'})_!\mathbb{Q}_{C'} \to \mathbb{Q}_C \to \mathbb{Q}_{\{0\}} \to 0.$$

In this section $j_{C',C}$ and $i_{0,C}$ are denoted respectively by $j_{C'}$ and i_0 to simplify the notation.

We have the Leray-type spectral sequences

$${}^*E_2^{p,q} = H^p i_0^*\,{}^{\mathbf{p}}\mathcal{H}^q\mathbf{R}(j_{C'})_*\mathbb{D}\mathbb{Q}_{C'} \Longrightarrow H^{p+q}i_0^*\mathbf{R}(j_{C'})_*\mathbb{D}\mathbb{Q}_{C'}, \tag{1.5.4}$$

$${}^!E_2^{p,q} = H^p i_0^!\,{}^{\mathbf{p}}\mathcal{H}^q\mathbf{R}(j_{C'})_*\mathbb{D}\mathbb{Q}_{C'} \Longrightarrow H^{p+q}i_0^!\mathbf{R}(j_{C'})_*\mathbb{D}\mathbb{Q}_{C'} = 0, \tag{1.5.5}$$

where the last vanishing follows from $i_0^!\mathbf{R}(j_{C'})_* = 0$ (the latter is the dual of $i_0^*\mathbf{R}(j_{C'})_! = 0$). These can be constructed by using spectral objects in [Ve2] together with an argument similar to [De1, Example 1.4.8], or we can use the Riemann-Hilbert correspondence after the scalar extension by $\mathbb{Q} \hookrightarrow \mathbb{C}$.

By (1.1.8), (1.5.2) together with properties of i_0^*, $i_0^!$ in [BBD] (see also [Sa2, Remark after Corollary 2.24]), we get

$$(1.5.6) \qquad {}^*E_2^{-p,q} = {}^!E_2^{p,q} = 0 \quad \text{unless} \quad p=0,\ q \geqslant -d-1 \ \text{ or } \ q=-d-1,\ p \geqslant 0.$$

The generalized Thom-Gysin sequence (1.3.2) together with an isomorphism similar to the last isomorphism of Proposition 2 implies that

$$(1.5.7) \qquad H^{-k} i_0^* \mathbf{R}(j_{C'})_* \mathbb{D}\mathbb{Q}_{C'} = 0 \quad \text{unless} \quad k \in [1, 2d+2].$$

(This can be shown also by using the link $L_{C,0}$ of C at 0. It is the intersection of C with a sphere S^{2n-1} around $0 \in \mathbb{C}^n$, and is a $(2d{+}1)$-dimensional real analytic space, see [DS]. Its dualizing complex $\mathbb{D}\mathbb{Q}_{L_{C,0}}$ (see [Ve1]) is isomorphic to the restriction of $\mathbb{D}\mathbb{Q}_{C'}[-1]$ to $L_{C,0}$.)

The spectral sequence (1.5.4) degenerates at E_2 by (1.5.6), and it follows from (1.1.8), (1.5.2), (1.5.7) that

$$(1.5.8) \qquad {}^{\mathbf{p}}\mathcal{H}^{-j} \mathbf{R}(j_{C'})_* \mathbb{D}\mathbb{Q}_{C'} = 0 \quad \text{unless} \quad j \in [1, d+1].$$

Using (1.5.3) and (1.5.8), we can prove the isomorphisms

$$(1.5.9) \qquad {}^{\mathbf{p}}\mathcal{H}^{k} \mathbb{D}\mathbb{Q}_{C} \xrightarrow{\sim} {}^{\mathbf{p}}\mathcal{H}^{k} \mathbf{R}(j_{C'})_* \mathbb{D}\mathbb{Q}_{C'} \quad (k \neq -1),$$

together with the short exact sequence

$$(1.5.10) \qquad 0 \to {}^{\mathbf{p}}\mathcal{H}^{-1} \mathbb{D}\mathbb{Q}_{C} \to {}^{\mathbf{p}}\mathcal{H}^{-1} \mathbf{R}(j_{C'})_* \mathbb{D}\mathbb{Q}_{C'} \to \mathbb{Q}_{\{0\}} \to 0\,.$$

Here the vanishing of ${}^{\mathbf{p}}\mathcal{H}^{0} \mathbb{D}\mathbb{Q}_{C}$ is rather nontrivial. If we have ${}^{\mathbf{p}}\mathcal{H}^{0} \mathbb{D}\mathbb{Q}_{C} \neq 0$, then (1.5.8) and the long exact sequence associated with (1.5.3) imply the surjectivity of the composition

$$\mathbb{Q}_{\{0\}} \to \mathbb{D}\mathbb{Q}_{C} \to {}^{\mathbf{p}}\mathcal{H}^{0} \mathbb{D}\mathbb{Q}_{C}.$$

We then get a splitting of the first morphism, but this is a contradiction. So the vanishing of ${}^{\mathbf{p}}\mathcal{H}^{0} \mathbb{D}\mathbb{Q}_{C}$ follows.

Combined with (1.5.6), the spectral sequence (1.5.5) implies the isomorphisms

$$(1.5.11) \qquad H^p i_0^!\, {}^{\mathbf{p}}\mathcal{H}^{-d-1} \mathbf{R}(j_{C'})_* \mathbb{D}\mathbb{Q}_{C'} = \begin{cases} {}^{\mathbf{p}}\mathcal{H}^{p-d-2} \mathbf{R}(j_{C'})_* \mathbb{D}\mathbb{Q}_{C'} & \text{if} \quad p \geqslant 2, \\ 0 & \text{if} \quad p = 0, 1. \end{cases}$$

Here the direct image $(i_0)_*$ is omitted on the left-hand side to simplify the notation.

By (1.5.9–11), we get the following (see [GS] for the X smooth case).

Proposition 1.6. *Under the assumption* (1.5.1)*, we have*

$$(1.6.1) \qquad \lambda_{k,j}(C) = 0 \quad \textit{unless} \quad j = d+1,\ k \in [2, d+1] \quad \textit{or} \quad k = 0,\ j \in [1, d],$$

and moreover the following relations among the Lyubeznik numbers hold:

$$(1.6.2) \qquad \lambda_{k,d+1}(C) = \lambda_{0,d+2-k}(C) + \delta_{k,d+1} \quad (k \in [2, d+1]),$$

where $\delta_{k,d+1} = 1$ *if* $k = d+1$*, and* 0 *otherwise.*

This implies the following.

Corollary 1.7. *Under the assumption* (1.5.1)*, the converse of Corollary* 1 *holds.*

We then get the following generalization of [Swl] in the X non-singular case.

Corollary 1.8. *The Lyubeznik numbers* $\lambda_{k,j}(C)$ *of the cone* C *of a complex projective scheme* X *are independent of the choice of a projective embedding of* X*, if the associated analytic space* X^{an} *is a* $\mathbb{Q}$*-homology manifold.*

Proof. By the definition of $\mathbb{Q}$-homology manifold, we have $H^j_{\{x\}}\mathbb{Q}_X = \mathbb{Q}$ if $j = 2\dim X$, and 0 otherwise ($\forall\, x \in X^{\rm an}$). By induction on strata, we see that the composition of the following two canonical morphisms is an isomorphism (see [GM, BBD]):

$$\mathbb{Q}_X[\dim X] \to \mathrm{IC}_X\mathbb{Q} \to \mathbb{D}\mathbb{Q}_X(-\dim X)[-\dim X], \tag{1.8.1}$$

where the last morphism is the dual of the first. This implies that ${}^{\mathbf{p}}\mathcal{H}^j(\mathbb{Q}_X[\dim X]) = 0$ ($j \neq 0$), and the above two morphisms are both isomorphisms by [GM] or using the simplicity of the intersection complex $\mathrm{IC}_X\mathbb{Q}$. Moreover the hard Lefschetz theorem holds for the intersection cohomology of the complex projective variety X, see [BBD] (and [Sa1]). The dimensions of the kernel and cokernel of the action of $c_1(L)$ on the (intersection) cohomology are then read off from the Betti numbers of X by using the primitive decomposition. So the assertion follows from Corollary 1.7. This finishes the proof of Corollary 1.8.

Remark. The independence of the Lyubeznik numbers of cones also holds if X has only isolated singularities and (1.5.1) is satisfied.

2. Construction of examples

In this section we prove Theorem 1 by constructing desired examples.

2.1. Projective schemes with condition (5) satisfied for some $k \in \mathbb{Z}$. Let Y be a smooth complex projective variety of dimension $d_1 \geqslant 2$ having very ample divisors D, D' such that their Chern classes $c_1(D)$, $c_1(D')$ are linearly independent. (The last condition can be satisfied in case the Picard number of Y is at least 2, for instance, if Y is a one-point blow-up of $\mathbb{P}^2$ or $\mathbb{P}^1 \times \mathbb{P}^1$.) Consider the line bundle L corresponding to D. We have the section Y_1 of L corresponding to D. Here we may assume D smooth (hence reduced). Then Y_1 intersects the zero section Y_0 of L *transversally* along D. Set $X_1 := Y_0 \cup Y_1$. We have

$${}^{\mathbf{p}}\mathcal{H}^{d_1}\mathbb{Q}_{X_1} = \mathbb{Q}_{X_1}[d_1], \quad {}^{\mathbf{p}}\mathcal{H}^{-d_1}\mathbb{D}\mathbb{Q}_{X_1} = \mathbb{D}\mathbb{Q}_{X_1}[-d_1], \tag{2.1.1}$$

where $\mathbb{D}\mathbb{Q}_{X_1}$ is denoted by $\mathbb{D}\mathbb{Q}_{X_1}$ as in the introduction. Using the dual of the short exact sequence

$$0 \to \mathbb{Q}_{X_1} \to \mathbb{Q}_{Y_0} \oplus \mathbb{Q}_{Y_1} \to \mathbb{Q}_D \to 0, \tag{2.1.2}$$

where the latter is identified with a distinguished triangle, we get the isomorphisms

$$\begin{aligned} \mathrm{Gr}^W_2 H^{2-2d_1}\big(X_1, \mathbb{D}\mathbb{Q}_{X_1}(-d_1)\big) &= \mathrm{Coker}\big(H^0(D)(-1) \xrightarrow{i_*} H^2(Y) \oplus H^2(Y)\big), \\ H^{-2d_1}\big(X_1, \mathbb{D}\mathbb{Q}_{X_1}(-d_1)\big) &= H^0(Y) \oplus H^0(Y). \end{aligned} \tag{2.1.3}$$

Here i_* denotes the Gysin morphism for the inclusion $i : D \hookrightarrow Y$. Since X_1 is finite over Y, and a finite morphism is ample in the sense of Grothendieck with relatively ample line bundle trivial, the ample divisors D, D' on Y give ample line bundles on X_1 via the pull-back by the morphism $X_1 \to Y$, see [Gr, Propositions 4.4.10 and Corollary 6.1.11].

By (2.1.3) (together with (2.1.1)), we get a difference in the dimension of the images of

$$c_1(D),\, c_1(D') : H^{-d_1}(X_1, {}^{\mathbf{p}}\mathcal{H}^{-d_1}\mathbb{D}\mathbb{Q}_{X_1})(-1) \to H^{2-d_1}(X_1, {}^{\mathbf{p}}\mathcal{H}^{-d_1}\mathbb{D}\mathbb{Q}_{X_1}). \tag{2.1.4}$$

So condition (5) is satisfied for

$$j - 1 = d_1, \quad k = 2 - d_1. \tag{2.1.5}$$

by assuming further $H^1(Y) = 0$.

2.2. Non-equidimensional projective schemes with condition (5) satisfied. Let $\widetilde{Z}$ be the blow-up of $\mathbb{P}^{d_2+2}$ along a point $P\in\mathbb{P}^{d_2+2}$, where $d_2\geqslant 2$. This is identified with a $\mathbb{P}^1$-bundle over $\mathbb{P}^{d_2+1}$, and we have the projection

$$\rho:\widetilde{Z}\to\mathbb{P}^{d_2+1},$$

where the target is identified with the set of lines of $\mathbb{P}^{d_2+2}$ passing through P. The projection ρ has the zero-section given by the exceptional divisor of the blow-up. It has another section which is disjoint with the zero-section, and is given by the inverse image of a hyperplane of $\mathbb{P}^{d_2+2}$ not containing the center of the blow-up P. Let Z_2 be the disjoint union of these two sections. Let Z_1 be the inverse image of the intersection of two different hyperplanes of $\mathbb{P}^{d_2+1}$ by ρ. We have

$$\dim Z_1=d_2,\quad \dim Z_2=d_2+1.$$

Set

$$X_2:=Z_1\cup Z_2\subset\widetilde{Z},\quad Z_1':=Z_1\setminus Z_2,$$

with $j_{Z_1'}:Z_1'\hookrightarrow Z_1$ the natural inclusion. We have the short exact sequence

$$0\to(j_{Z_1'})_!\mathbb{Q}_{Z_1'}\to\mathbb{Q}_{X_2}\to\mathbb{Q}_{Z_2}\to 0, \tag{2.2.1}$$

where the direct images by closed embeddings are omitted to simplify the notation. Taking the dual, we get the distinguished triangle

$$\mathbb{D}\mathbb{Q}_{Z_2}\to\mathbb{D}\mathbb{Q}_{X_2}\to\mathbf{R}(j_{Z_1'})_*\mathbb{D}\mathbb{Q}_{Z_1'}\xrightarrow{+1}. \tag{2.2.2}$$

Note that $Z_1\cap Z_2$ is a divisor on Z_1, and $j_{Z_1'}:Z_1'\hookrightarrow Z_1$ is an affine open embedding so that

$${}^{\mathbf{p}}\mathcal{H}^{-j}\mathbf{R}(j_{Z_1'})_*\mathbb{D}\mathbb{Q}_{Z_1'}=0\quad(j\neq d_2). \tag{2.2.3}$$

Since Z_1, Z_2 are smooth, we then get

$${}^{\mathbf{p}}\mathcal{H}^{-j}\mathbb{D}\mathbb{Q}_{X_2}=\begin{cases}\mathbf{R}(j_{Z_1'})_*\mathbb{D}\mathbb{Q}_{Z_1'}[-d_2] & \text{if } j=d_2\,,\\ \mathbb{D}\mathbb{Q}_{Z_2}[-d_2-1] & \text{if } j=d_2+1\,,\\ 0 & \text{otherwise.}\end{cases} \tag{2.2.4}$$

This implies that

$$H^k(X_2,{}^{\mathbf{p}}\mathcal{H}^{-d_2}\mathbb{D}\mathbb{Q}_{X_2})=H^{k-d_2}(Z_1',\mathbb{D}\mathbb{Q}_{Z_1'})=\begin{cases}\mathbb{Q}(d_2) & \text{if } k=-d_2,\\ \mathbb{Q} & \text{if } k=d_2-1,\\ 0 & \text{otherwise.}\end{cases} \tag{2.2.5}$$

Indeed, $\mathbb{D}\mathbb{Q}_{Z_1'}=\mathbb{Q}_{Z_1'}(d_2)[2d_2]$, and $Z_1'=\mathbb{C}^{d_2}\setminus\{0\}$, since $\widetilde{Z}\setminus Z_2=\mathbb{C}^{d_2+2}\setminus\{0\}$ and Z_1' is obtained by taking the intersection of $\widetilde{Z}\setminus Z_2$ with two different hyperplanes successively.

Set

$$X:=X_1\times X_2,\quad d:=d_1+d_2+1\,(=\dim X),$$

where X_1, d_1 are as in (2.1). Then

$${}^{\mathbf{p}}\mathcal{H}^{-j}\mathbb{D}\mathbb{Q}_X=\begin{cases}\mathbb{D}\mathbb{Q}_{X_1}[-d_1]\boxtimes\mathbf{R}(j_{Z_1'})_*\mathbb{D}\mathbb{Q}_{Z_1'}[-d_2] & \text{if } j=d-1\,,\\ \mathbb{D}\mathbb{Q}_{X_1}[-d_1]\boxtimes\mathbb{D}\mathbb{Q}_{Z_2}[-d_2-1] & \text{if } j=d\,,\\ 0 & \text{otherwise.}\end{cases} \tag{2.2.6}$$

Here we use the Segre embedding so that a very ample line bundle is given by the tensor product of the pull-backs of very ample line bundles by the first and second projections from X. Condition (5) then holds by (2.1.4) and (2.2.4–6) for

$$j-1=d-1,\quad k=(2-d_1)+(d_2-1)=d_2-d_1+1. \tag{2.2.7}$$

Here $k\geqslant 2$ if $d_2>d_1$.

Remark. A similar argument holds if we replace Z_1 with the inverse image of a higher codimensional linear subspace of $\mathbb{P}^{d_2+1}$.

2.3. Equidimensional projective schemes with condition (5) satisfied. For an integer $d_2 > 2$, set

$$B := \mathbb{P}^2 \times \mathbb{P}^{d_2-2}.$$

Let $\rho' : Z' \to B$ be the pull-back of the very ample line bundle on $\mathbb{P}^{d_2-2}$ corresponding to $\mathcal{O}_{\mathbb{P}^{d_2-2}}(1)$. We have the associated $\mathbb{P}^1$-bundle

$$\rho : Z \to B.$$

This is a compactification of Z' such that $Z \setminus Z'$ is the section at infinity of ρ. Let Z_2 be the union of the zero-section and the section at infinity of ρ so that

$$Z_2 \cong \mathbb{P}^2 \times \mathbb{P}^{d_2-2} \sqcup \mathbb{P}^2 \times \mathbb{P}^{d_2-2} \subset Z.$$

Take any smooth curve $E \subset \mathbb{P}^2$ of degree $d_E \geqslant 3$. The genus g_E of E is given by

$$g_E = (d_E - 1)(d_E - 2)/2.$$

(Note that the vector bundle E in (1.3) is not used in this section.) Set

$$Z_1 := \rho^{-1}(E \times \mathbb{P}^{d_2-2}) \subset Z$$

Here $\dim Z_1 = \dim Z_2 = d_2$. Put

$$X_2 := Z_1 \cup Z_2 \subset Z, \quad Z_1' := Z_1 \setminus Z_2,$$

with $j_{Z_1'} : Z_1' \hookrightarrow Z_1$ the natural inclusion. As in (2.2.2), we have the distinguished triangle

$$\mathbb{D}\mathbb{Q}_{Z_2} \to \mathbb{D}\mathbb{Q}_{X_2} \to \mathbf{R}(j_{Z_1'})_*\mathbb{D}\mathbb{Q}_{Z_1'} \xrightarrow{+1} . \tag{2.3.1}$$

Then

$$ {}^{\mathbf{p}}\mathcal{H}^{-d_2}\mathbb{D}\mathbb{Q}_{X_2} = \mathbb{D}\mathbb{Q}_{X_2}[-d_2]. \tag{2.3.2}$$

and we have the long exact sequence of mixed $\mathbb{Q}$-Hodge structures (see [De2], [Sa2]):

$$\to H_k(Z_2) \to H_k(X_2) \to H_k^{\mathrm{BM}}(Z_1') \to H_{k-1}(Z_2) \to, \tag{2.3.3}$$

where $H_\bullet^{\mathrm{BM}}$ denotes the Borel-Moore homology.

By the Thom-Gysin sequence (1.3.2) for $\mathcal{F}^\bullet = \mathbb{D}\mathbb{Q}_{B_0}$ with $B_0 := E \times \mathbb{P}^{d_2-2}$, we have the long exact sequence of mixed $\mathbb{Q}$-Hodge structures (see Remark after (1.3)):

$$H_k(B_0) \xrightarrow{c'} H_{k-2}(B_0)(1) \to H_k^{\mathrm{BM}}(Z_1') \to H_{k-1}(B_0) \xrightarrow{c'} H_{k-3}(B_0)(1), \tag{2.3.4}$$

where c' is the first Chern class of the line bundle $\rho' : Z' \to B$.

Using (2.3.3–4), we can show the isomorphisms of W-graded mixed Hodge structures of *odd* weights

$$\mathrm{Gr}^W_{\mathrm{odd}} H_k(X_2) = \mathrm{Gr}^W_{\mathrm{odd}} H_k^{\mathrm{BM}}(Z_1') = \begin{cases} H_1(E)(d_2-1) & \text{if } k = 2d_2 - 1\,, \\ H_1(E) & \text{if } k = 2\,, \\ 0 & \text{otherwise,} \end{cases} \tag{2.3.5}$$

where

$$\mathrm{Gr}^W_{\mathrm{odd}} H_k(X_2) := \textstyle\bigoplus_{i\in 2\mathbb{Z}+1} \mathrm{Gr}^W_i H_k(X_2), \text{ etc.}$$

Indeed, the first isomorphism of (2.3.5) follows from (2.3.3). For the second, we have

$$\mathrm{Gr}^W_{\mathrm{odd}} H_k(B_0) = \begin{cases} H_1(E)(j) & \text{if } k = 2j+1,\ j \in [0, d_2-2], \\ 0 & \text{otherwise,} \end{cases}$$

and (2.3.4) implies the isomorphisms

$$\begin{aligned}\mathrm{Gr}^W_{\mathrm{odd}} H^{\mathrm{BM}}_{2d_2-1}(Z'_1) &= \mathrm{Gr}^W_{\mathrm{odd}} H_{2d_2-3}(B_0)(1) = H_1(E)(d_2-1),\\ \mathrm{Gr}^W_{\mathrm{odd}} H^{\mathrm{BM}}_{2}(Z'_1) &= \mathrm{Gr}^W_{\mathrm{odd}} H_1(B_0) = H_1(E),\end{aligned}$$

where the other $\mathrm{Gr}^W_{\mathrm{odd}} H^{\mathrm{BM}}_k(Z'_1)$ vanish. So (2.3.5) follows.

Set

$$X := X_1 \times X_2, \quad d := d_1 + d_2 \,(= \dim X),$$

where X_1, d_1 are as in (2.1). Then

$$ {}^{\mathrm{p}}\mathcal{H}^{-j}\mathbb{D}\mathbb{Q}_X = \begin{cases} \mathbb{D}\mathbb{Q}_{X_1}[-d_1] \boxtimes \mathbb{D}\mathbb{Q}_{X_2}[-d_2] & \text{if } j = d\,,\\ 0 & \text{otherwise.}\end{cases}$$

To simplify the argument, we assume the following:

$$d_1 = 2.$$

By the last assumption in (2.1), we have

$$H_1(X_1) = H_3(X_1) = 0.$$

The following morphisms are surjective (using an extension of (2.1.3)):

$$c_1(D), c_1(D') : H_2(X_1) \twoheadrightarrow H_0(X_1).$$

So there is only a *difference in the rank* for

$$c_1(D), c_1(D') : H_4(X_1) \to H_2(X_1). \tag{2.3.6}$$

We have very ample line bundles L, L' on X defined by

$$L := \mathrm{pr}_1^*L(D) \otimes \mathrm{pr}_2^*L_2, \quad L' := \mathrm{pr}_1^*L(D') \otimes \mathrm{pr}_2^*L_2.$$

Here pr_i denotes the ith projection from $X_1 \times X_2$, $L(D)$ is the very ample line bundle on X_1 corresponding to D (similarly for $L(D')$ with D replaced by D'), and L_2 is a very ample line bundle on X_2.

We now show that condition (5) holds if $j = j_0$, $k = k_0$ with

$$j_0 := d+1, \quad k_0 := (2-d_1) + (d_2-2) = d_2 - d_1 \;(= d_2 - 2), \tag{2.3.7}$$

where $k_0 \geqslant 2$ if $d_2 \geqslant d_1 + 2 \;(= 4)$. The number $d_2 - 2$ appears here, since we have in (2.3.5)

$$H_2(X_2) = H^{-2}(X_2, \mathbb{D}\mathbb{Q}_{X_2}) = H^{d_2-2}(X_2, {}^{\mathrm{p}}\mathcal{H}^{-d_2}\mathbb{D}\mathbb{Q}_{X_2}).$$

We will show that the *even* weight part can be neglected effectively in condition (5) if $g_E \gg 0$. In the notation of (4), set

$$\mathrm{Gr}^W_{\mathrm{even}} H^k_{(d)}(X)^L := \textstyle\bigoplus_{i\in 2\mathbb{Z}} \mathrm{Gr}^W_i H^k_{(d)}(X)^L \qquad (k \in \mathbb{Z}),$$

and similarly for $\mathrm{Gr}^W_{\mathrm{odd}} H^k_{(d)}(X)^L$, $\mathrm{Gr}^W_{\mathrm{even}} H^k_{(d)}(X)_L$, $\mathrm{Gr}^W_{\mathrm{odd}} H^k_{(d)}(X)_L$. Here W is the weight filtration of the canonical mixed Hodge structure on

$$H^k_{(d)}(X) := H^k(X, {}^{\mathrm{p}}\mathcal{H}^{-d}\mathbb{D}\mathbb{Q}_X) = H_{d-k}(X).$$

Put

$$\mu^k_{\mathrm{odd}}(X, L) := \mu^k_{\mathrm{odd}}(X)_L + \mu^{k-1}_{\mathrm{odd}}(X)^L \qquad \text{with}$$

$$\mu^k_{\mathrm{odd}}(X)_L := \dim \mathrm{Gr}^W_{\mathrm{odd}} H^k_{(d)}(X)_L, \quad \mu^{k-1}_{\mathrm{odd}}(X)^L := \dim \mathrm{Gr}^W_{\mathrm{odd}} H^{k-1}_{(d)}(X)^L,$$

and similarly for $\mu^k_{\mathrm{even}}(X, L)$, $\mu^k_{\mathrm{even}}(X)_L$, $\mu^{k-1}_{\mathrm{even}}(X)^L$. For L, L', k_0 as above, we then get

$$\delta^{k_0}_{\mathrm{odd}} := \big|\mu^{k_0}_{\mathrm{odd}}(X, L) - \mu^{k_0}_{\mathrm{odd}}(X, L')\big| > \delta^{k_0}_{\mathrm{even}} := \big|\mu^{k_0}_{\mathrm{even}}(X, L) - \mu^{k_0}_{\mathrm{even}}(X, L')\big|, \tag{2.3.8}$$

if $g_E \gg 0$. Indeed, $\delta^{k_0}_{\rm odd}$ is strictly positive by (2.1.4), (2.3.5), and is proportional to g_E by Lemma below via the inclusion (using the Künneth formula):

$$H^{2-d_1}(X_1, {}^{\mathbf{p}}\mathcal{H}^{-d_1}\mathbb{D}\mathbb{Q}_{X_1}) \otimes H^{d_2-2}(X_2, {}^{\mathbf{p}}\mathcal{H}^{-d_2}\mathbb{D}\mathbb{Q}_{X_2}) \hookrightarrow H^{k_0}(X, {}^{\mathbf{p}}\mathcal{H}^{-d}\mathbb{D}\mathbb{Q}_X), \tag{2.3.9}$$

when the curve E is changed. On the other hand, $\delta^{k_0}_{\rm even}$ in (2.3.8) is independent of g_E. So condition (5) holds if $g_E \gg 0$, see also Remarks (i–iii) below for more precise arguments. This finishes the proof of Theorem 1.

Lemma. *In the above notation,* $\delta^{k_0}_{\rm odd}$ *in* (2.3.8) *is proportional to* g_E*, and* $\delta^{k_0}_{\rm even}$ *remains invariant under the change of the plane curve* $E \subset \mathbb{P}^2$.

Proof. The *odd* weight part of $H_\bullet(X_1)$ is

$$\mathrm{Gr}^W_{-1} H_2(X_1) = H_1(D),$$

and the actions of $c_1(D), c_1(D')$ on it vanish. We get an *even* weight part of $H_\bullet(X)$ via the Künneth formula by taking the tensor product of this *odd* weight part with the *odd* weight part of $H_\bullet(X_2)$ which is isomorphic by (2.3.5) to

$$H_1(E) \oplus H_1(E)(d_2-1). \tag{2.3.10}$$

The action of $c_1(L_2)$ on the latter odd weight part vanishes. Hence the actions of $c_1(L), c_1(L')$ vanish on the above tensor product, which is called the *odd-odd* weight part. (Here we consider the actions on the graded pieces of the weight filtration W. Note that any morphism of mixed Hodge structures is *strictly compatible* with the weight filtration, and the kernel and cokernel commute with the passage to the *graded quotients* of the weight filtration, see [De1].) So the contribution of this *odd-odd* weight part vanishes by taking the difference between $\mu^{k_0}_{\rm even}(X, L)$ and $\mu^{k_0}_{\rm even}(X, L')$. This shows the invariance of $\delta^{k_0}_{\rm even}$ under the change of the curve E, since the *even-even* weight part is clearly independent of E.

As for $\delta^{k_0}_{\rm odd}$ in (2.3.8), we see that the contribution of the tensor product of the even weight part of $H_\bullet(X_1)$ with (2.3.10), that is, the *even-odd* weight part, is proportional to g_E (where the action of $c_1(L_2)$ vanish on (2.3.10)). Note that only $H_1(E)$ in (2.3.10) contributes here, and there is no contribution of $H_1(E)(d_2{-}1)$ for a reason of degree (since $k_0 = d_2 - d_1$ in (2.3.7)), see also (2.3.5–6). So it remains to consider the *odd-even* weight part. We see that the odd weight part of $H_\bullet(X_1)$ does not contribute to $\delta^{k_0}_{\rm odd}$ in (2.3.8) by taking the tensor product with the even weight part of $H_\bullet(X_2)$, since there is a *difference of actions* only on the *even* weight part of $H_\bullet(X_1)$ as is explained above, see also (2.3.6). This finishes the proof of Lemma.

Remarks. (i) Some part of the above Lemma can be avoided if we assume, for instance,

$$Y = \mathbb{P}^1 \times \mathbb{P}^1, \;\; D = \{x_1x_2 = y_1y_2\}, \quad \text{so that} \quad H^1(D) = 0.$$

Here x_j, y_j are the projective coordinates of the jth factor of $\mathbb{P}^1 \times \mathbb{P}^1$ $(j = 1, 2)$.

(ii) Setting $\delta_2 := d_2 - 4 \geqslant 0$, we have

$$\dim H^k(Z_2, {}^{\mathbf{p}}\mathcal{H}^{-d_2}\mathbb{D}\mathbb{Q}_{Z_2}) = \begin{cases} 6 - |k| + \delta_2 & \text{if} \;\; |k| - \delta_2 = 2 \;\; \text{or} \;\; 4, \\ 6 & \text{if} \;\; |k| \leqslant \delta_2, \, k - \delta_2 \in 2\mathbb{Z}, \\ 0 & \text{otherwise}, \end{cases}$$

$$\dim H^k(Z'_1, {}^{\mathbf{p}}\mathcal{H}^{-d_2}\mathbb{D}\mathbb{Q}_{Z'_1}) = \begin{cases} 2g_E & \text{if} \;\; k = -\delta_2{-}3 \;\; \text{or} \;\; \delta_2{+}2, \\ 1 & \text{if} \;\; k = -\delta_2{-}3 \pm 1 \;\; \text{or} \;\; \delta_2{+}2 \pm 1, \\ 0 & \text{otherwise}. \end{cases}$$

Moreover the following morphisms appearing essentially in (2.3.3) are injective:

$$H^k(Z'_1, {}^{\mathbf{p}}\mathcal{H}^{-d_2}\mathbb{D}\mathbb{Q}_{Z'_1}) \to H^{k+1}(Z_2, {}^{\mathbf{p}}\mathcal{H}^{-d_2}\mathbb{D}\mathbb{Q}_{Z_2}) \qquad (k = d_2{-}2 \pm 1).$$

(Note that $d_2-2\pm 1=\delta_2+2\pm 1$.) This can be proved by using a long exact sequence like (2.3.3) with Z_2, X_2 respectively replaced by $Z_1\setminus Z_1'$, Z_1, together with a morphism from this sequence to (2.3.3). (Here we also study a similar sequence for the $\mathbb{P}^1$-bundle over $\mathbb{P}^{d_2-2}$.) We then get

$$H^{d_2-1}(X_2,{}^{\mathrm{p}}\mathcal{H}^{-d_2}\mathbb{D}\mathbb{Q}_{X_2})=0.$$

So the contribution to $\delta^{k_0}_{\mathrm{even}}$ in (2.3.8) via the following inclusion vanishes:

$$H^{-d_1}(X_1,{}^{\mathrm{p}}\mathcal{H}^{-d_1}\mathbb{D}\mathbb{Q}_{X_1})\otimes H^{d_2-1}(X_2,{}^{\mathrm{p}}\mathcal{H}^{-d_2}\mathbb{D}\mathbb{Q}_{X_2})\hookrightarrow H^{k_0-1}(X,{}^{\mathrm{p}}\mathcal{H}^{-d}\mathbb{D}\mathbb{Q}_X),$$

where $\mu^{k_0-1}_{\mathrm{even}}(X)^L-\mu^{k_0-1}_{\mathrm{even}}(X)^{L'}$ is involved.

We have also the vanishing of the contribution to $\delta^{k_0}_{\mathrm{even}}$ via the inclusion (2.3.9), where $\mu^{k_0}_{\mathrm{even}}(X)_L-\mu^{k_0}_{\mathrm{even}}(X)_{L'}$ is involved. This follows from Remark (iii) below together with the hard Lefschetz property of the action of $c_1(L_2)$ on $H^\bullet(Z_2)$ and the commutativity of the morphisms in (2.3.3) with the action of $c_1(L_2)$. (The argument is rather delicate in the case $d_2=4$, where we need also the primitive decomposition together with (2.3.6).)

For (2.3.8) it is then sufficient to assume $d_E\geqslant 3$ so that $g_E\geqslant 1$, since $\dim H^1(E)=2g_E$.

(iii) Let $A_\bullet$, $B_\bullet$ be graded vector spaces having the actions $c':A_i\to A_{i+1}$, $c'':B_i\to B_{i+1}$ ($i\in\mathbb{Z}$), and satisfying the following conditions for some integers p,q :

$$A_i=0\ (\forall\, i\notin[p,p+2]),\quad c''(B_i)=B_{i+1}\ (\forall\, i\in[q,q+2]).$$

Set $c:=c'\otimes\mathrm{id}+\mathrm{id}\otimes c''$ on $C_\bullet:=A_\bullet\otimes B_\bullet$. Then $c(C_{p+q+2})=C_{p+q+3}$.

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