A LOCALIZATION ALGORITHM FOR D-MODULES

TOSHINORU OAKU[1], NOBUKI TAKAYAMA[2] AND ULI WALTHER[3]

ABSTRACT. We present a method to compute the holonomic extension of a D-module from a Zariski open set in affine space to the whole space. A particular application is the localization of coherent D-modules which are holonomic on the complement of an affine variety.

Throughout this article let $R = K[x_1, \dots, x_n]$ be the ring of polynomials in n variables over the field K of characteristic zero and $D = R\langle\partial_1, \dots, \partial_n\rangle$ the n-th Weyl algebra over K. Here, $\partial_i = \frac{\partial}{\partial x_i}$. Suppose $f \in R$, let J be a left ideal of D and set $M = D/J$. Let $X = K^n$, $Y = \mathrm{V}(f) \subseteq X$, $U = X \setminus Y$ and $j : U \hookrightarrow X$ the natural inclusion.

In the landmark paper [K2] M. Kashiwara proved that $\int_j j^{-1}M$ is holonomic provided that M is holonomic on U, but not necessarily on all of X. In this note we make this important result algorithmic in the sense that we provide an algorithm that computes the module structure of $\int_j j^{-1}M$ over D.

We remark that U is affine and $X = K^n$, and so $\int_j j^{-1}M = R_f \otimes_R M$ as D-module. An algorithm to compute $R_f \otimes_R M$ as a D-module has been given in [Oa3] under the assumption that M is holonomic on K^n and f-torsion free, and in [O-T2] under the assumption that M is holonomic on K^n. The advantage of our new algorithm, besides removing the requirement of holonomicity on $\mathrm{V}(f)$, is that the natural map $M \to R_f \otimes_R M$ can be traced, which is an important property for other computations as local cohomology ([Oa3], [W1]) and de Rham cohomology ([O-T1], [W2]).

1. Description of the Algorithm

We shall start with a motivating construction which will also provide a skeleton of the proof of the correctness of our algorithm.

Assume that K is in fact the field $\mathbb{C}$ of complex numbers. Let s, v be two new variables and let $D_v = D\langle v, \partial_v\rangle$.

Notation 1. Throughout we will denote $\partial f/\partial x_i$ by f_i. If $P \in D_v$ is an operator interpreted as a polynomial in $\partial_1, \dots, \partial_n$, we will simply write $P(\partial_x)$. Similarly we will abbreviate $P(\partial_1 - v^2 f_1 \partial_v, \dots, \partial_n - v^2 f_n \partial_v)$ by $P(\partial_x - v^2 f_x \partial_v)$.

Consider the ring D_v and the $\mathbb{C}$-vectorspace $R_f[s] \otimes f^{-s}$. Here, f^{-s} should be thought of as a symbol that behaves like the complex valued function f^{-s}

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under differentiation and the tensor is over $\mathbb{C}$. We shall turn this vectorspace into a module over D_v by means of the action $\bullet$ defined as follows. We require that all x_i act as multiplication on the left factor, while ∂_i acts by the "product rule" $\partial_i \bullet (\frac{g(x,s)}{f^m} \otimes f^{-s}) = \partial_i(\frac{g(x,s)}{f^m}) \otimes f^{-s} + \frac{g(x,s)}{f^{m+1}}(-s)f_i \otimes f^{-s}$. Thus, we only need to define the action of v and ∂_v. We set

$$\begin{aligned} v \bullet \left(\frac{g(x,s)}{f^m} \otimes f^{-s}\right) &= \frac{g(x,s+1)}{f^{m+1}} \otimes f^{-s}, \\ \partial_v \bullet \left(\frac{g(x,s)}{f^m} \otimes f^{-s}\right) &= -(s-2)\frac{f \cdot g(x,s-1)}{f^m} \otimes f^{-s}. \end{aligned}$$

We observe that v is simply shifting s up by one (in particular, the extra f in the denominator comes from shifting the exponent in f^{-s}) and ∂_v is shifting s down by one, together with the "differential" $2 - s$ of the shift operator (cf. the action defined in [M]).

Armed with this action on $R_f[s] \otimes f^{-s}$ we can define an action of D_v on $D/J \otimes_R (R_f[s] \otimes f^{-s})$ as follows. v and ∂_v commute with the first factor in the tensor product, x_i acts by left multiplication on the first factor and ∂_i acts by the product rule.

We detect a certain set of trivial relations for this action. Namely, let us ask which operators on D_v are candidates for annihilating the element $\overline{1} \otimes 1 \otimes f^{-s}$, the bar denoting cosets in D/J. Independently of J, $1 - vf$ is such an annihilator. Assume that J is generated by $\{P_j(\partial_x)\}_1^r$ where we think of P_j as a polynomial in ∂_x with left coefficients in R. Let us define a ring map ϕ from D to D_v by mapping $P \in D$ to $\phi(P_j(\partial_x)) = Q_j(\partial_x) := P_j(\partial_x - v^2 f_x \partial_v) \in D_v$ (recall $f_i = \partial_i(f)$). It is easy to see that this assignment preserves commutators and hence is an actual ring map. We remark that ϕ represents the map $U \xrightarrow{\cong} \mathrm{V}(1 - vf) \subseteq \mathbb{C}^{n+1}$ given by $x \to (1/f(x), x)$ on the level of differential operators, while the inclusion $\phi(D) \hookrightarrow D_v$ corresponds to $\mathrm{V}(1 - vf) \hookrightarrow \mathbb{C}^{n+1}$.

We claim that $Q_j(\partial_x)$ annihilates $\overline{1} \otimes 1 \otimes f^{-s}$. To prove this, we observe that $(\partial_i - v^2 f_i \partial_v) \bullet (\overline{P} \otimes 1 \otimes f^{-s})$ equals $\overline{\partial_i P} \otimes 1 \otimes f^{-s}$ for all $P \in D$. Thus $Q_i(\partial_x) \bullet (\overline{1} \otimes 1 \otimes f^{-s}) = \overline{P_i(\partial_x)} \otimes 1 \otimes f^{-s} = 0$. We point out that $-v\partial_v + 1$ acts as multiplication by s and that this construction is (formally) independent of the base field.

Our main statement is the following

Theorem 2. *Let $M = D/J$. Consider the left ideal $D_v \cdot \{1 - xv, \phi(J)\}$ and the right ideal $\{\partial_v\} \cdot D_v$ in D_v. The quotient $D_v/(D_v \cdot \{1 - xv, \phi(J)\} + \{\partial_v\} \cdot D_v)$ is isomorphic to $M \otimes R_f$ as D-module. Moreover, the natural map $M \to M \otimes_R R_f$ sends $\overline{1} \in M$ to $\overline{1} \otimes \frac{1}{f^2}$.*

It is worth noticing that this is in fact a left D-module due to the fact that D and ∂_v commute.

In the general context in which the theorem is stated, the D-module $D_v/(D_v \cdot \{1 - xv, \phi(J)\} + \{\partial_v\} \cdot D_v)$ will not be a finitely generated D-module. If however M is holonomic on $U = K^n \setminus \mathrm{V}(f)$, then $M \otimes_R R_f = D_v/(D_v \cdot \{1-$

$xv, \phi(J)\} + \{\partial_v\} \cdot D_v)$ will be holonomic, and in particular finitely generated. In order to compute the structure of the localized module, we observe that it is the integration of the left D_v-module $D_v/D_v \cdot \{1 - xv, \phi(J)\}$ with respect to ∂_v (cf. [Oa3, Theorem 5.7], [O-T1, Section 6]).

Methods for the algorithmic computation of 0-th integration worked out in [Oa3, Theorem 5.7] (i.e., the restriction to $v = 0$ of the Fourier transform of the module,) together with our theorem give the following algorithm. Let N denote $D_v/(\{\partial_v\} \cdot D_v + D_v \cdot \{1 - fv, \phi(J)\})$.

Algorithm 3. Input: $f \in R$; $\{P_1, \ldots, P_r\} \subseteq D$ generating J, D/J holonomic on $U = K^n \setminus \mathrm{V}(f)$.
Output: $k \in \mathbb{N}$; $\{Q_1, \ldots, Q_t\} \subseteq D$ such that $D/(Q_1, \ldots, Q_t)$ is isomorphic to $D/J \otimes_R R_f$ generated by $\overline{1} \otimes \frac{1}{f^{k+2}}$.
Begin.

1. For $P_i, i = 1, \ldots, r$ compute $\phi(P_i) \in D_v$ defined by replacing ∂_x by $\partial_x - v^2 f_x \partial_v$.
2. Compute the b-function $b(s)$ for integration of $D_v/D_v \cdot \{1 - fv, \phi(J)\}$ with respect to ∂_v. That is, find $K[v\partial_v] \cap \mathrm{in}_w(D_v \cdot \{1 - fv, \phi(J)\})$ where w is the weight assigning 1 to v, -1 to ∂_v and 0 to all other variables. Replace $v\partial_v$ by $-s - 1$.
3. Let k be the largest non-negative integer root of $b(s)$. If there is no such root, then output $I = D$, else continue. The integral N is generated by the coset of v^k in N.
4. Compute the annihilator I over D of v^k in N.
5. $D/J \otimes_R R_f$ is generated by $\overline{1} \otimes f^{-(k+2)}$ and isomorphic to D/I.
6. Return k and I.

End.

An algorithm to perform step 2 of the algorithm is given in [Oa2, Algorithm 4.5].

The steps 3 and 4 are nothing but an algorithm to get the 0-th integral of D-modules. Here is a more precise description of steps 3 and 4.

1. Let $G = \{g_1, \ldots, g_m\}$ be a Gröbner basis of $D_v \cdot \{1 - vf, \phi(J)\}$ with respect to w (see, e.g., [SST, section 1.1]).
2. Let G_k be the set
$$\{\,\mathrm{normalForm}(v^i g_j, \{\partial_v\} \cdot D_v) \mid j = 1, \ldots, m, 0 \le i \le k - \mathrm{ord}_w(g_j)\}.$$
3. Regard $D \cdot \{G_k\}$ as a left submodule of the free module
$$D\,v^0 + D\,v^1 + \cdots + D\,v^k = D^{k+1}$$
and find generators of $D \cdot \{G_k\} \cap D\,v^k$. The last intersection can be computed by using an order to eliminate $v^0, \ldots, v^{k-1}$, i.e, by using an order $\succ$ such that $a_i v^i \succ a_k v^k$ for all $i = 0, \ldots, k-1$ and $a_i, a_k \neq 0$ in D.

Here, we put

$$\mathrm{ord}_w(g) := \max\left\{w\cdot(\alpha,\beta) \mid g = \sum a_{\alpha\beta}v^\alpha\partial_v^\beta\right\}$$

following the notation of [SST]. normalForm$(g, \{\partial_v\}\cdot D_v)$ means taking the normal form of g with respect to the right ideal $\{\partial_v\}\cdot D_v$. For example,

$$\mathrm{normalForm}(v^2\partial_v^2, \{\partial_v\}\cdot D_v) = \mathrm{normalForm}(\partial_v^2 v^2 - 4\partial_v v + 2, \{\partial_v\}\cdot D_v) = 2.$$

2. Proof of correctness of the algorithm

Let us first provide the

Proof of the Theorem: Given an operator $P \in D_v$ we shall define its *normal form* in D_v. The goal is a presentation of P where $\partial_i - v^2 f_i\partial_v$ take the position of ∂_i. In other words, we are aiming for a sum of the form $P = \sum \partial_v^a v^b p_{ab}(x)q_{ab}(\partial_x - v^2 f_x\partial_v)$ where p_{ab} and q_{ab} are polynomials in n variables over K. To this end, write first P as $\sum\partial_v^a v^b p_{ab}(x)q_{ab}(\partial_x)$. The operator $P^1 := \sum\partial_v^a v^b p_{ab}(x)q_{ab}(\partial_x - v^2 f_x\partial_v)$ has the property that $P - P^1$ will have lower degree in ∂_x than P. If $P - P^1 = 0$, quit. Otherwise write $P - P^1$ as $\sum\partial_v^a v^b p^1_{ab}(x)q^1_{ab}(\partial_x)$ and set $P^2 := \partial_v^a v^b p^1_{ab}(x)q^1_{ab}(\partial_x - v^2 f_x\partial_v)$. Repeat this procedure until we arrive at $P^l = 0$. Then $P = P^1 + \cdots + P^l$ and this sum is the desired normal form in D_v. We shall write $\tilde{P}$ for the normal form of P in D_v.

The normal form in D_v of an operator induces a normal form in $D_v/\{\partial_v\}\cdot D_v$ by removing all terms in $\{\partial_v\}\cdot D_v$ from the normal from of P in D_v. We denote the normal form of $P \in D_v$ in $D_v/\{\partial_v\}\cdot D_v$ by $\overline{\tilde{P}}$. Of course, $P+\{\partial_v\}\cdot D_v = \tilde{P}+\{\partial_v\}\cdot D_v$ as cosets. We notice that both normal forms are unique. As an example, consider the normal form of ∂_i in $D_v/(\partial_v)$: $P = \partial_i$, $P^1 = \partial_i - v^2 f_i\partial_v$, thus $P^2 = v^2 f_i\partial_v = \partial_v v^2 f_i - 2f_i v$, $P^3 = 0$ and hence $\overline{\tilde{P}} = (\partial_i - v^2 f_i\partial_v) - 2f_i v + \{\partial_v\}\cdot D_v$.

As a D-module, the quotient $N = D_v/(\{\partial_v\}\cdot D_v + D_v\cdot\{1 - vf, \phi(J)\})$ is spanned by the powers of v, which are in normal form. Let us try to understand the relations among these powers. To this end note that $(1 - fv)(\partial_i - v^2 f_i\partial_v) = (\partial_i - v^2 f_i\partial_v + f_i v)(1 - fv)$ which implies that $\overline{(1 - fv)\tilde{P}} = 0$ in N for all $P \in D_v$. This proves that $\overline{f(v\tilde{P})} = \overline{\tilde{P}} = \overline{P}$ in N for all $P \in D_v$ and we conclude that every element in N is divisible by f.

Moreover, the division by f is unique since if $f\overline{\tilde{Q}} = f\overline{\tilde{Q}'}$ in N then $\overline{vf(\tilde{Q} - \tilde{Q}')} = 0$ in N and that implies $\overline{\tilde{Q}} = \overline{\tilde{Q}'}$ in N. Conversely, for all $\overline{\tilde{P}} \in N$ there is a unique $\overline{\tilde{Q}} \in N$ for which $\overline{v\tilde{Q}} = \overline{\tilde{P}}$ because then necessarily $\overline{\tilde{Q}} = f\overline{\tilde{P}}$. So f is an invertible operator on N and the inverse is given by v acting on the normal form representative of a coset.

We shall now make use of the action we defined in the previous section.

The map $D_v \to D/J\otimes_R(R_f[s]\otimes f^{-s})$ defined via the action $\bullet$ is surjective if we send 1 to $\overline{1}\otimes 1\otimes f^{-s}$. Consider the combined map $D_v \to D/J\otimes_R (R_f[s]\otimes f^{-s}) \to (D/J\otimes_R(R_f[s]\otimes f^{-s}))/(s-2)(D/J\otimes_R(R_f[s]\otimes f^{-s}))$.

This is a surjective map of left D-modules and the kernel contains $\{\partial_v\} \cdot D_v + D_v \cdot \{1 - vf, \phi(J)\}$. If an operator in D_v is in the kernel, we can assume it to be in normal form in $D_v / \{\partial_v\} \cdot D_v$ since $\{\partial_v\} \cdot D_v$ is in the kernel. Any such operator may be written as $v^k P(\partial_x - v^2 f_x \partial_v)$ where $P(\partial_x) \in D$. We conclude that P must already be in the kernel since $v^k P(\partial_x - v^2 f_x \partial_v) \bullet (\overline{1} \otimes 1 \otimes f^{-s}) = v^k \bullet (\overline{P(\partial_x)} \otimes 1 \otimes f^{-s} = \overline{P(\partial_x)} \otimes f^{-k} \otimes f^{-s}$. Thus $P(\partial_x - v^2 f_x \partial_v) \bullet (\overline{1} \otimes 1 \otimes f^{-s}) \in (s-2) D/J \otimes_R (R_f[s] \otimes f^{-s})$ and therefore $f^j P(\partial_x) \in D[s] \cdot (J, (s-2))$ for some $j \in \mathbb{N}$. Since neither P nor J does contain s, setting $s = 2$ we see that $P(\partial_x)$ is in the f-saturation of J, $f^j P(\partial_i) \in J$.

$f^j P(\partial_x) \in J$ implies that $f^j P(\partial_x - v^2 f_x \partial_v) \in \phi(J)$. Modulo $D_v \cdot \{1 - vf\}$ this is equivalent to $P(\partial_x - v^2 f_x \partial_v) \in \phi(J)$ since $f^j P(\partial_x - v^2 f_x \partial_v)$ is in normal form. This simply means that P is in the kernel of $D_v \to D/J \otimes_R (R_f[s] \otimes f^{-s}) \to (D/J \otimes_R (R_f[s] \otimes f^{-s})) / (s-2) (D/J \otimes_R (R_f[s] \otimes f^{-s}))$ if and only if it is in $\{\partial_v\} \cdot D_v + D_v \cdot \{1 - vf, \phi(J)\}$.

Finally we observe that

$$\left(D/J \otimes_R (R_f[s] \otimes f^{-s})\right) / (s-2) \left(D/J \otimes_R (R_f[s] \otimes f^{-s})\right)$$

is isomorphic to $D/J \otimes_R R_f$ by means of the map $\overline{P} \otimes \frac{g(x,s)}{f^m} \otimes f^{-s} \to \overline{P} \otimes \frac{g(x,2)}{f^{m+2}}$. ☐

Remark 4. The theorem and its proof generalize nearly verbatim to the situation where M is any finitely generated D-module.

Now let us consider the situation in which $M = D/J$ is holonomic on U. Then $D_v / (\{\partial_v\} \cdot D_v + D_v \cdot \{1 - fv, \phi(J)\})$ is a finite D-module as it is isomorphic to the module $M \otimes_R R_f$ which is holonomic by theorem 1.3 of [K2]. We can find a generator by computing the b-function for integration along ∂_v for $D_v / D_v \cdot \{1 - vf, \phi(J)\}$ as in step 2 of algorithm 3. If k is the largest root of the b-function then v^k is a generator for $D_v / (\{\partial_v\} \cdot D_v + D_v \cdot \{1 - fv, \phi(J)\}) = M \otimes R_f$ (compare [O-T2], algorithm 5.4). Thus in order to represent the localization $M \otimes_R R_f$ as a quotient of D all one needs to do is to find the annihilator of v^k over D. This shows the correctness of our algorithm.

Remark 5. Again, the algorithm generalizes to the non-cyclic situation. Let $M = D^m / J$, $D^m = \oplus_1^m D e_j$. The modifications are as follows. Compute m separate b-functions $b_j(s)$ to the integration of $D_v(e_j + \phi(J)) / (D_v \cdot \{\phi(J), (1 - vf) e_j\})$ along ∂_v. $M \otimes_R R_f$ is generated by the cosets of $v^{k_j} e_j$ in N where k_j is the largest integer root of $b_j(s)$.

3. Categorical explanation of the algorithm

Let us now give a more categorical explanation of the validity of our algorithm.

Decompose $j : K^n \setminus V(f) \to K^n$ as $j = p \circ \iota \circ \phi$;

$$\begin{array}{ccc} W & \xrightarrow{\iota} & K^{n+1} \\ \uparrow{\scriptstyle\phi} & & \downarrow{\scriptstyle p} \\ K^n \setminus V(f) & \xrightarrow{j} & K^n \end{array}$$

where $\phi : K^n \setminus V(f) \to W = \{(v, x) \mid v \cdot f(x) = 1\}$ is defined by $x \mapsto (1/f(x), x)$; $\iota : W \hookrightarrow K^{n+1}$ is the closed embedding, and $p : K^{n+1} \to K^n$ is the natural projection. Then we have

$$M \otimes_R R_f = \int_j j^{-1}M = \int_p \int_\iota \int_\phi j^{-1}M$$

by the chain rule of integration functors (see, e.g., [B, p.251, 6.4 Proposition], [HT, 1.5.1]). Note that the chain rule holds in the derived category in general, but, in this case, $\int_j$ is an exact functor since $\mathcal{D}_{X \xleftarrow{j} U} = \mathcal{D}_X[1/f]$ is flat over $\mathcal{D}_U$ and j is an affine morphism. $\int_\iota \int_\phi$ is also an exact functor. Hence, we have only to compute 0-th integrals in each step of computing the integral functors. $\int_\phi j^{-1}M$ is obtained by the coordinate transformation represented by our ring map ϕ; consider the map

$$y_i = x_i, \ (i = 1, \dots, n), \ y_{n+1} = 1/f(x).$$

Then, $\partial_{x_i} = \partial_{y_i} - \frac{f_i}{f^2}\partial_{y_i} = \partial_{y_i} - f_i v^2 \partial_{y_i}$ modulo $1 - fv = 0$, which commute with each other and define our ring map ϕ. $\int_\iota$ is nothing but Kashiwara equivalence corresponding to $\phi(D) \hookrightarrow D_v$. It follows that $\int_\iota \int_\phi j^{-1}M = D_v/(D_v \cdot \{\phi(J), 1 - fv\})$, compare also Proposition A.1 (p. 596) of [Oa1]. Integration under the projection p corresponds then to our last step accomplished by forming the quotient $\left(\int_\iota \int_\phi j^{-1}M\right) / \{\partial_v\} \cdot \left(\int_\iota \int_\phi j^{-1}M\right)$.

Since $j^{-1}M$ is holonomic, so is $\int_\iota \int_\phi j^{-1}M$ and hence the b-function for integration is nonzero, thus guaranteeing termination of the search for generators of $\int_j j^{-1}M$.

4. Examples

Example 6. For our first example we take $n = 1, x = x_1, K = \mathbb{C}, J = x\partial_x + \lambda, \lambda \in \mathbb{C}, f(x) = x$.

In this scenario, we have to compute the integral of the module $D_v/(1 - xv, x(\partial_x - v^2\partial_v) + \lambda)$ along ∂_v. This agrees with the left ideal $D_v \cdot \{1 - xv, \partial_x x - \partial_v v + 1 + \lambda\}$ from which one sees that the b-function is $s(s+1+\lambda)$. Thus the largest integer root is either 0 or $-1-\lambda$, depending on whether λ is a negative integer or not. So $M \otimes_R R_x$ is generated by $\overline{1} \otimes x^{-(-1-\lambda)-2}$ in the former and $\overline{1} \otimes x^{-2}$ in the latter case.

If for example $\lambda = -7$ we compute a b-function of $s(s-6)$ indicating that $M \otimes_R R_x$ is generated by $\overline{1} \otimes x^{-8}$. Since in this case $M = R = D \cdot x^7$, we conclude that $M \otimes_R R_f$ is in fact generated by $1/x$, as it should.

If on the other hand $\lambda = 1/2$ then $b(s) = s(s+3/2)$ and hence the largest integer root is 0. Thus, $M \otimes_R R_x$ is generated by v^0 corresponding to $1/x^2$. This is simply stating that M is already isomorphic to $M \otimes_R R_x$.

Example 7. In this example we consider the left ideal J generated by

$$\partial_x(x^2 - y^3),$$
$$\partial_y(x^2 - y^3).$$

These are annihilators of the function $1/f$, where $f = x^2 - y^3$, but they do not generate the annihilating ideal (see, e.g., [Oa2]). The left ideal J is not holonomic since the characteristic variety of J is $V(x^3 - y^2) \cup V(\xi_x, \xi_y)$, of which first component has dimension 3 in $\mathbb{C}^4 = \{(x, y; \xi_x, \xi_y)\}$. As to an algorithmic method to get the characteristic variety, see [Oa0]. It follows the output of a computer session using the computer algebra system Kan/sm1 ([T]) interspersed with comments. We remark that in this case M restricted to U is an $\mathcal{O}_U$-coherent free module of rank one where $U = \mathbb{C}^2 \setminus \mathrm{V}(f)$.

```
/ff
  [ ((x^2-y^3)*(Dx - v^2*2*x*Dt) + 2*x)
    ((x^2-y^3)*(Dy + v^2*3*y^2*Dt) -3*y^2)
    (v*(x^2-y^3)-1)
  ]
def
```

(this is $\phi(J)$)

```
sm1>ff [(v)] intbfm ::
[    $216*s^4+1296*s^3+2586*s^2+1716*s$ ]
[[6,1],[s+2,1],[6*s+11,1],[6*s+13,1],[s,1]]
```

(these are the factors of the b-function)

```
sm1>ff [(v)] -2 0 1 intall_s ;
Completed.
```

(computing the integration)

```
 0-th cohomology:  [    1 ,
   [    -3*x*Dx-2*y*Dy-18 ,
        3*y^2*Dx+2*x*Dy ,
        -2*y^3*Dy+2*x^2*Dy-18*y^2 ]  ]
-1-th cohomology:  [    0 , [    ]  ]
```

The integration $\int_j j^{-1}M$ is not $\mathcal{O}_X$-coherent, although of course it is still coherent over the sheaf of differential operators $\mathcal{D}_X$ on $X = \mathbb{C}^2$. Since localization is an exact functor, the first cohomology group (corresponding to the first higher order integration) was known to be zero. The 0-th cohomology above coincides with the annihilating ideal of the function f^{-3}.

Example 8. Let $n = 3$, and consider the ideal J generated by the system

$$\begin{array}{lcl}(x^3 - y^2z^2)\partial_x & + & 3x^2, \\ (x^3 - y^2z^2)\partial_y & - & 2yz^2, \\ (x^3 - y^2z^2)\partial_z & - & 2yz^2.\end{array}$$

These operators are annihilators of the exponential function $e^{1/f}$ where $f(x, y, z) = x^3 - y^2z^2$. The characteristic variety of $M = D/J$ has six components, defined by the prime ideals (y, x), (z, x), (ξ_x, ξ_y, ξ_z), (ξ_y, z, x), (ξ_z, y, x) and $(y\xi_y - z\xi_z, 2x\xi_x + 3z\xi_z, 8z\xi_x^3 + 27\xi_y^2\xi_z, 8y\xi_x^3 + 27\xi_y\xi_z^2, -4z^2\xi_x^2 + 9x\xi_y^2, -4yz\xi_x^2 + 9x\xi_y\xi_z, -4y^2\xi_x^2 + 9x\xi_z^2, 2z^3\xi_x\xi_z + 3x^2\xi_y^2, 2yz^2\xi_x + 3x^2\xi_y, 2y^2z\xi_x + 3x^2\xi_z, -yz^3\xi_z + x^3\xi_y, -y^2z^2 + x^3, x^3\xi_y^2 - z^4\xi_z^2)$. All but the first two are of dimension three. This implies that the non-holonomic locus of M is contained in the hypersurface $x = 0$. Here, we used [Asir] to obtain the primary ideal decomposition.

Hence we may apply our algorithm to compute $M \otimes_R R_x$. We remark that contrary to the previous example in this case $j^{-1}M$ is holonomic but not coherent as $\mathcal{O}_U$-module on $U = X \setminus \mathrm{V}(x)$. Using Kan/sm1 again one obtains the eight operators

$$\begin{array}{l}-3y\partial_y + 3z\partial_z,\ -2xyz^2\partial_x - 3x^3\partial_y - 4yz^2,\ -2xy^2z\partial_x - 3x^3\partial_z - 4y^2z, \\ 6xz^3\partial_x\partial_z + 9x^3\partial_y^2 + 6xz^2\partial_x + 6yz^2\partial_y + 6z^3\partial_z + 12z^2, \\ -6y^2z^3\partial_z + 4x^4\partial_x + 12x^3z\partial_z + 8x^3 + 12, \\ 6yz^4\partial_z^2 - 4x^4\partial_x\partial_y - 12x^3z\partial_y\partial_z + 18yz^3\partial_z - 8x^3\partial_y - 12\partial_y, \\ 8x^5\partial_x^2 + 24x^4z\partial_x\partial_z + 18x^3z^2\partial_z^2 + 64x^4\partial_x + 102x^3z\partial_z + 80x^3 + 24x\partial_x + 48, \\ -6z^5\partial_z^3 + 4x^4\partial_x\partial_y^2 + 12x^3z\partial_y^2\partial_z - 36z^4\partial_z^2 + 8x^3\partial_y^2 - 36z^3\partial_z + 12\partial_y^2\end{array}$$

which annihilate the function $x^{-2}e^{1/f}$. The characteristic variety of this holonomic left ideal of D has the same last four components as the characteristic variety of M while the first two components (of dimension 4) are replaced by (ξ_z, ξ_y, x) and (z, y, x) (of dimension 3).

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[1]Yokohama City University
E-mail address: `oaku@math.yokohama-cu.ac.jp`

[2]Kobe University and MSRI
E-mail address: `takayama@math.kobe-u.ac.jp`

[3]University of Minnesota and MSRI
E-mail address: `walther@math.umn.edu`