LOCAL COHOMOLOGY AND PURE MORPHISMS

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Dedicated to Professor Phil Griffith

ABSTRACT. We study a question raised by Eisenbud, Mustaţă, and Stillman regarding the injectivity of natural maps from Ext modules to local cohomology modules. We obtain some positive answers to this question which extend earlier results of Lyubeznik. In the process, we also prove a vanishing theorem for local cohomology modules which connects theorems previously known in the case of positive characteristic and in the case of monomial ideals.

1. INTRODUCTION

Throughout this paper, the rings we consider are commutative, Noetherian, and contain an identity element. For an ideal $\mathfrak{a}$ of a ring R, the local cohomology modules $H^i_{\mathfrak{a}}(R)$ may be obtained as

$$H^i_{\mathfrak{a}}(R) = \varinjlim_t \operatorname{Ext}^i_R(R/\mathfrak{a}_t, R) \qquad \text{for } i \geqslant 0\,,$$

where $\{\mathfrak{a}_t\}_{t\geqslant 0}$ is a decreasing chain of ideals cofinal with the chain $\{\mathfrak{a}^t\}_{t\geqslant 0}$, and the maps in the directed system are those induced by the natural surjections

$$R/\mathfrak{a}_{t+1} \longrightarrow R/\mathfrak{a}_t\,.$$

Any chain of ideals which is cofinal with the chain $\{\mathfrak{a}^t\}_{t\geqslant 0}$ yields the same direct limit. In this context, Eisenbud, Mustaţă, and Stillman have raised the following questions:

Question 1.1. [EMS, Question 6.1] Let R be a polynomial ring over a field. For which ideals $\mathfrak{a}$ of R does there exist a chain of ideals $\{\mathfrak{a}_t\}_{t\geqslant 0}$ as above, such that for all $i \geqslant 0$ and all $t \geqslant 0$, the natural map

$$\operatorname{Ext}^i_R(R/\mathfrak{a}_t, R) \longrightarrow H^i_{\mathfrak{a}}(R)$$

is injective?

Question 1.2. [EMS, Question 6.2] Given a polynomial ring R over a field, for which ideals $\mathfrak{a}$ is the natural map $\operatorname{Ext}^i_R(R/\mathfrak{a}, R) \longrightarrow H^i_{\mathfrak{a}}(R)$ an inclusion?

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Question 1.1 is motivated by the fact that the R-modules $H^i_{\mathfrak{a}}(R)$ are typically not finitely generated, whereas modules of the form $\operatorname{Ext}^i_R(R/\mathfrak{b}, R)$ are finitely generated. Consequently a chain of ideals as in Question 1.1 yields a filtration of $H^i_{\mathfrak{a}}(R)$ by a natural family of finitely generated submodules.

Let R be a polynomial ring over a field. For an ideal $\mathfrak{a}$ generated by square-free monomials $m_1, \dots, m_r$, set $\mathfrak{a}^{[t]} = (m_1^t, \dots, m_r^t)$ for integers $t \geqslant 0$. Lyubeznik [Ly1, Theorem 1 (i)] proved that the natural maps

$$\operatorname{Ext}^i_R(R/\mathfrak{a}^{[t]}, R) \longrightarrow H^i_{\mathfrak{a}}(R)$$

are injective for all $i \geqslant 0$ and $t \geqslant 0$; see also Mustaţă [Mu, Theorem 1.1]. If R has positive characteristic, an ideal $\mathfrak{a}$ generated by square-free monomials has the property that $R/\mathfrak{a}$ is *F-pure*; see § 2. Our main result, Theorem 2.8, recovers Lyubeznik's result and also provides a positive answer to Question 1.1 for ideals defining F-pure rings, a case we single out for mention here:

Theorem 1.3. *Let R be a regular ring containing a field of characteristic $p > 0$, and $\mathfrak{a}$ an ideal such that $R/\mathfrak{a}$ is F-pure. Then the natural maps*

$$\operatorname{Ext}^i_R(R/\mathfrak{a}^{[p^t]}, R) \longrightarrow H^i_{\mathfrak{a}}(R)$$

are injective for all $i \geqslant 0$ and all $t \geqslant 0$.

Remark 1.4. If $d = \operatorname{depth}_R(\mathfrak{a}, R)$, then the natural map

$$\operatorname{Ext}^d_R(R/\mathfrak{a}, R) \longrightarrow H^d_{\mathfrak{a}}(R)$$

is injective. To see this, let $E^\bullet$ be a minimal injective resolution of R. Then $H^\bullet_{\mathfrak{a}}(R)$ is the cohomology of the complex $\Gamma_{\mathfrak{a}}(E^\bullet)$ and $\operatorname{Ext}^\bullet_R(R/\mathfrak{a}, R)$ is the cohomology of its subcomplex $\operatorname{Hom}_R(R/\mathfrak{a}, E^\bullet) = (0 :_{E^\bullet} \mathfrak{a})$. Since d is the least integer i such that $\Gamma_{\mathfrak{a}}(E^i)$ is nonzero, we are considering the cohomology of the rows of the diagram

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & 0 & \longrightarrow & \Gamma_{\mathfrak{a}}(E^d) & \longrightarrow & \Gamma_{\mathfrak{a}}(E^{d+1}) & \longrightarrow & \cdots \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
\cdots & \longrightarrow & 0 & \longrightarrow & (0 :_{E^d} \mathfrak{a}) & \longrightarrow & (0 :_{E^{d+1}})\mathfrak{a} & \longrightarrow & \cdots,
\end{array}$$

and the desired inclusion follows.

Remark 1.5. It is easy to see that Question 1.1 has a positive answer if $\mathfrak{a}$ is a set-theoretic complete intersection: if $f_1, \dots, f_n$ is a regular sequence generating $\mathfrak{a}$ up to radical, then the ideals $\mathfrak{a}_t = (f_1^t, \dots, f_n^t)$ form a descending chain with $\operatorname{Ext}^i_R(R/\mathfrak{a}_t, R) \hookrightarrow H^i_{\mathfrak{a}}(R)$ for all $i \geqslant 0$ and $t \geqslant 0$; for $i = n$, this follows from Remark 1.4, whereas if $i \neq n$, then $\operatorname{Ext}^i_R(R/\mathfrak{a}_t, R) = 0 = H^i_{\mathfrak{a}}(R)$.

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2. Pure homomorphisms and F-pure rings

Definition 2.1. A ring homomorphism $\varphi\colon R \longrightarrow S$ is *pure* if the map

$$\varphi \otimes 1\colon R \otimes_R M \longrightarrow S \otimes_R M$$

is injective for each R-module M. If R contains a field of characteristic $p > 0$, then R is *F-pure* if the Frobenius homomorphism $r \mapsto r^p$ is pure.

Evidently, pure homomorphisms are injective. Let R be a subring of S. If the inclusion $R \hookrightarrow S$ splits as a maps of R-modules, then it is pure. The converse is also true for module-finite extensions, see [HR2, Corollary 5.3].

Example 2.2. Let $R = \mathbb{K}[x_1, \dots, x_d]$ be a polynomial ring over a field $\mathbb{K}$, and let t be a positive integer. Then there is a $\mathbb{K}$-linear endomorphism φ of R with $\varphi(x_i) = x_i^t$ for $1 \leqslant i \leqslant d$. The inclusion $\varphi(R) \subseteq R$ splits since R is a free module over $\varphi(R)$ with basis $x_1^{e_1} \cdots x_d^{e_d}$ where $0 \leqslant e_i \leqslant t-1$. It follows that $\varphi\colon R \longrightarrow R$ is pure.

Let $\mathfrak{a}$ be an ideal of R generated by square-free monomials. Then $\varphi(\mathfrak{a}) \subseteq \mathfrak{a}$, so φ induces an endomorphism $\overline{\varphi}$ of $R/\mathfrak{a}$. The image of $\overline{\varphi}$ is spanned, as a $\mathbb{K}$-vector space, by those monomials in $x_1^t, \dots, x_d^t$ which are not in $\mathfrak{a}$. Using the map which is identity on these monomials, and kills the rest, we obtain a splitting of $\overline{\varphi}$. It follows that the endomorphism $\overline{\varphi}\colon R/\mathfrak{a} \longrightarrow R/\mathfrak{a}$ is pure.

Remark 2.3. The notion of F-pure rings was introduced by Hochster and Roberts in the course of their study of rings of invariants [HR1, HR2]. Examples of F-pure rings include regular rings, determinantal rings, Plücker embeddings of Grassmannians, polynomial rings modulo square-free monomial ideals, normal affine semigroup rings, homogeneous coordinate rings of ordinary elliptic curves, and, more generally, homogeneous coordinate rings of ordinary Abelian varieties. Moreover, pure subrings of F-pure rings are F-pure, and if R and S are F-pure algebras over a perfect field $\mathbb{K}$, then their tensor product $R \otimes_{\mathbb{K}} S$ is also F-pure.

Remark 2.4. Let $\varphi\colon R \longrightarrow S$ be a ring homomorphism. If $f \in R$, then φ localizes to give a map $R_f \longrightarrow S_{\varphi(f)}$. Similarly, if $\boldsymbol{f}$ is a sequence of elements of R, then φ induces a map of Čech complexes

$$\check{C}^\bullet_{\boldsymbol{f}}(R) \longrightarrow \check{C}^\bullet_{\varphi(\boldsymbol{f})}(S)\,.$$

Setting $\mathfrak{a} = (\boldsymbol{f})$, we have an induced map of local cohomology groups

$$\varphi_*\colon H^i_{\mathfrak{a}}(R) \longrightarrow H^i_{\mathfrak{a}S}(S) \qquad \text{for all } i \geqslant 0\,.$$

Note that for $r \in R$ and $\eta \in H^i_{\mathfrak{a}}(R)$, we have $\varphi(r)\varphi_*(\eta) = \varphi_*(r\eta)$.

Now suppose φ is an endomorphism of R with $\operatorname{rad}\mathfrak{a} = \operatorname{rad}\varphi(\mathfrak{a})R$. Then one obtains an induced *action*

$$\varphi_*\colon H^i_{\mathfrak{a}}(R) \longrightarrow H^i_{\varphi(\mathfrak{a})R}(R) = H^i_{\mathfrak{a}}(R)\,,$$

which is an endomorphism of the underlying Abelian group.

The archetypal example is the one where φ is the Frobenius endomorphism of a ring R of prime characteristic; in this case, for all ideals $\mathfrak{a}$ of R and integers $i \geqslant 0$, there is an induced action φ_* on $H^i_{\mathfrak{a}}(R)$ known as the *Frobenius action.*

If $\varphi\colon R \longrightarrow S$ is pure, then for all ideals $\mathfrak{a}$ of R and all integers $i \geqslant 0$, the induced map $\varphi_*\colon H^i_{\mathfrak{a}}(R) \longrightarrow H^i_{\mathfrak{a}S}(S)$ is injective, see [HR1, Corollary 6.8] or [HR2, Lemma 2.1]. In another direction, we have the following lemma, which will be a key ingredient in the proof of Theorem 2.8.

Lemma 2.5. *Let $(R, \mathfrak{m})$ be a local ring with a pure endomorphism φ such that $\varphi(\mathfrak{m})R$ is $\mathfrak{m}$-primary. Then, for all $i \geqslant 0$, the induced action*

$$\varphi_*\colon H^i_{\mathfrak{m}}(R) \longrightarrow H^i_{\mathfrak{m}}(R)$$

is surjective up to R-span, i.e., $\varphi_*(H^i_{\mathfrak{m}}(R))$ generates $H^i_{\mathfrak{m}}(R)$ as an R-module.

Proof. Consider an element $\eta \in H^i_{\mathfrak{m}}(R)$; we need to show that it belongs to the R-module spanned by $\varphi_*(H^i_{\mathfrak{m}}(R))$. The descending chain of R-modules

$$\langle \eta, \varphi_*(\eta), \varphi_*^2(\eta), \dots \rangle \ \supseteq\ \langle \varphi_*(\eta), \varphi_*^2(\eta), \dots \rangle \ \supseteq\ \langle \varphi_*^2(\eta), \varphi_*^3(\eta), \dots \rangle$$

stabilizes since $H^i_{\mathfrak{m}}(R)$ is Artinian. Hence there exists $e \geqslant 0$ such that

$$\varphi_*^e(\eta) \in \langle \varphi_*^{e+1}(\eta), \varphi_*^{e+2}(\eta), \dots \rangle\,. \tag{2.5.1}$$

Let e be the least such integer. If $e = 0$ we are done, whereas if $e \geqslant 1$ then the R-module

$$M = \frac{\langle \varphi_*^{e-1}(\eta), \varphi_*^{e}(\eta), \varphi_*^{e+1}(\eta), \dots \rangle}{\langle \varphi_*^{e}(\eta), \varphi_*^{e+1}(\eta), \dots \rangle}$$

is nonzero. But then, by the purity of φ, so is its image under

$$\varphi \otimes 1\colon R \otimes_R M \longrightarrow R \otimes_R M\,,$$

which contradicts (2.5.1). ∎

Remark 2.6. Let R be a regular ring with a flat endomorphism φ. We use R^φ to denote the R-bimodule which has R as its underlying Abelian group, the usual action of R on the left, and the right R-action with $r'r = \varphi(r)r'$ for $r \in R$ and $r' \in R^\varphi$. Let Φ be the functor on the category of R-modules with

$$\Phi(M) = R^\varphi \otimes_R M\,,$$

where $\Phi(M)$ is viewed as an R-module via the left R-module structure of R^φ. The iteration Φ^t is the functor with

$$\Phi^t(M) = R^\varphi \otimes_R \Phi^{t-1}(M) \qquad \text{for } t \geqslant 1\,,$$

where Φ^0 is interpreted as the identity functor. It is easily seen that

$$\Phi^t(M) = R^{\varphi^t} \otimes_R M\,.$$

(1) There is an isomorphism $\Phi(R) \cong R$ given by $r' \otimes r \mapsto r'\varphi(r)$. It follows that if M is a free R-module, then $\Phi(M) \cong M$. For a map α of free modules given by a matrix (α_{ij}), the map $\Phi(\alpha)$ is given by the matrix $(\varphi(\alpha_{ij}))$. Since φ is flat, Φ is an exact functor, and so it takes finite free resolutions to finite free resolutions. If M and N are R-modules, then there are natural isomorphisms

$$\Phi(\operatorname{Ext}^i_R(M,N)) \cong \operatorname{Ext}^i_R(\Phi(M),\Phi(N)) \qquad \text{for all } i \geqslant 0\,. \tag{2.6.1}$$

In particular, if $\mathfrak{a}$ is an ideal of R, then (2.6.1) implies that

$$\Phi(\operatorname{Ext}^i_R(R/\mathfrak{a},R)) \cong \operatorname{Ext}^i_R(R/\varphi(\mathfrak{a})R,R)\,.$$

(2) Suppose that the ideals $\{\varphi^t(\mathfrak{a})R\}_{t\geqslant 0}$ form a descending chain cofinal with the chain $\{\mathfrak{a}^t\}_{t\geqslant 0}$. Then, for each $i \geqslant 0$, the above isomorphism and its iterations fit into a commutative diagram

$$\begin{array}{ccccccc}
\cdots \longrightarrow & \operatorname{Ext}^i_R(R/\varphi^t(\mathfrak{a})R,R) & \longrightarrow & \operatorname{Ext}^i_R(R/\varphi^{t+1}(\mathfrak{a})R,R) & \longrightarrow \cdots \\
& \downarrow & & \downarrow & \\
\cdots \longrightarrow & \Phi^t(\operatorname{Ext}^i_R(R/\mathfrak{a},R)) & \longrightarrow & \Phi^{t+1}(\operatorname{Ext}^i_R(R/\mathfrak{a},R)) & \longrightarrow \cdots
\end{array}$$

where the maps in the top row are those induced by the natural surjections $R/\varphi^{t+1}(\mathfrak{a})R \longrightarrow R/\varphi^t(\mathfrak{a})R$, and the vertical maps are isomorphisms. Hence the bottom row has direct limit $H^i_{\mathfrak{a}}(R)$. It follows that $H^i_{\mathfrak{m}}(R) \cong \Phi(H^i_{\mathfrak{m}}(R))$.

(3) Assume in addition that $(R,\mathfrak{m})$ is a regular local ring of dimension d, and that φ is a flat local endomorphism. In this case, the dimension formula

$$\dim R + \dim R/\varphi(\mathfrak{m})R = \dim R$$

implies that $\varphi(\mathfrak{m})R$ is $\mathfrak{m}$-primary. Let E denote the injective hull of $R/\mathfrak{m}$ as an R-module, and set $(-)^\vee = \operatorname{Hom}_R(-,E)$. Since R is Gorenstein, we have

$$E \cong H^d_{\mathfrak{m}}(R) \cong \Phi(H^d_{\mathfrak{m}}(R)) \cong \Phi(E)\,.$$

Hence (2.6.1) implies that $\Phi(M^\vee) \cong (\Phi(M))^\vee$ for each R-module M. Setting $M = \operatorname{Ext}^i_R(R/\mathfrak{a},R)$ and using local duality, we get

$$(\Phi(\operatorname{Ext}^i_R(R/\mathfrak{a},R)))^\vee \cong \Phi(H^{d-i}_{\mathfrak{m}}(R/\mathfrak{a}))\,.$$

Since $\Phi^t(-) = R^{\varphi^t} \otimes_R (-)$, we immediately obtain the isomorphisms

$$(\Phi^t(\operatorname{Ext}^i_R(R/\mathfrak{a},R)))^\vee \cong \Phi^t(H^{d-i}_{\mathfrak{m}}(R/\mathfrak{a})) \qquad \text{for all } t \geqslant 0\,.$$

Applying $(-)^\vee$ to the diagram in (2), we get the commutative diagram

$$\begin{array}{ccccccc}
\cdots \longleftarrow & H^{d-i}_{\mathfrak{m}}(R/\varphi^t(\mathfrak{a})R) & \longleftarrow & H^{d-i}_{\mathfrak{m}}(R/\varphi^{t+1}(\mathfrak{a})R) & \longleftarrow \cdots \\
& \uparrow & & \uparrow & \\
\cdots \longleftarrow & \Phi^t(H^{d-i}_{\mathfrak{m}}(R/\mathfrak{a})) & \longleftarrow & \Phi^{t+1}(H^{d-i}_{\mathfrak{m}}(R/\mathfrak{a})) & \longleftarrow \cdots
\end{array}$$

where the vertical maps are isomorphisms, and the maps in the first row are those induced by the natural surjections $R/\varphi^{t+1}(\mathfrak{a})R \longrightarrow R/\varphi^t(\mathfrak{a})R$.

In the archetypal example, R is a regular ring containing a field of positive characteristic, and φ is the Frobenius endomorphism. In this case, φ is flat by Kunz's theorem [Ku, Theorem 2.1]. The functor Φ is the Peskine-Szpiro functor of [PS], and the commutative diagram in Remark 2.6 (2) is precisely that obtained by Lyubeznik in [Ly2, Lemma 2.1]. The following is a mild generalization of [Ly2, Lemma 2.2].

Lemma 2.7. *Let $(R, \mathfrak{m})$ be a regular local ring with a flat local endomorphism φ, and let $\mathfrak{a}$ be an ideal such that $\varphi(\mathfrak{a}) \subseteq \mathfrak{a}$. Then φ induces an endomorphism $\overline{\varphi}$ of $R/\mathfrak{a}$, and hence an action $\overline{\varphi}_*\colon H^i_\mathfrak{m}(R/\mathfrak{a}) \longrightarrow H^i_\mathfrak{m}(R/\mathfrak{a})$. The composition*

$$R^\varphi \otimes_R H^i_\mathfrak{m}(R/\mathfrak{a}) \xrightarrow{\ \cong\ } H^i_\mathfrak{m}(R/\varphi(\mathfrak{a})R) \xrightarrow{\ \pi\ } H^i_\mathfrak{m}(R/\mathfrak{a})$$

is the map with $r' \otimes \eta \longmapsto r' \cdot \overline{\varphi}_*(\eta)$, where π is the map induced by the natural surjection $R/\varphi(\mathfrak{a})R \longrightarrow R/\mathfrak{a}$.

Proof. Since $\varphi(\mathfrak{m})R$ is $\mathfrak{m}$-primary, if $\boldsymbol{x}$ is a system of parameters for R, then so is its image $\varphi(\boldsymbol{x})$. The displayed isomorphism is a consequence of the flatness of φ as we saw in Remark 2.6. To analyze this isomorphism, let $\widetilde{\eta}$ be a lift of $\eta \in H^i_\mathfrak{m}(R/\mathfrak{a})$ to the module $\check{C}^i_{\boldsymbol{x}}(R/\mathfrak{a})$ of the Čech complex $\check{C}^\bullet_{\boldsymbol{x}}(R/\mathfrak{a})$. Then

$$\varphi(\widetilde{\eta}) \in \check{C}^i_{\varphi(\boldsymbol{x})}(R/\varphi(\mathfrak{a})R)$$

and the image of $r' \otimes \eta$ under the isomorphism is the image of $r' \cdot \varphi(\widetilde{\eta})$ in $H^i_\mathfrak{m}(R/\varphi(\mathfrak{a})R)$. Lastly, π maps this to $r' \cdot \overline{\varphi}_*(\eta) \in H^i_\mathfrak{m}(R/\mathfrak{a})$. $\square$

We are now ready to prove the main result:

Theorem 2.8. *Let R be a regular ring and $\mathfrak{a}$ an ideal of R. Suppose R has a flat endomorphism φ such that $\{\varphi^t(\mathfrak{a})R\}_{t\geqslant 0}$ is a decreasing chain of ideals cofinal with $\{\mathfrak{a}^t\}_{t\geqslant 0}$, and the induced endomorphism $\overline{\varphi}\colon R/\mathfrak{a} \longrightarrow R/\mathfrak{a}$ is pure. Then, for all $i \geqslant 0$ and $t \geqslant 0$, the natural map*

$$\operatorname{Ext}^i_R(R/\varphi^t(\mathfrak{a})R, R) \longrightarrow \operatorname{Ext}^i_R(R/\varphi^{t+1}(\mathfrak{a})R, R)$$

is injective.

Proof. It suffices to verify the injectivity after localizing at maximal ideals, so we assume that $(R, \mathfrak{m})$ is a regular local ring. Let $d = \dim R$, and let E be the injective hull of $R/\mathfrak{m}$ as an R-module. Using $(-)^\vee = \operatorname{Hom}_R(-, E)$, local duality gives an isomorphism

$$\operatorname{Ext}^i_R(R/\varphi^t(\mathfrak{a})R, R)^\vee \cong H^{d-i}_\mathfrak{m}(R/\varphi^t(\mathfrak{a})R)\,,$$

and it suffices to show that the map

$$H^{d-i}_\mathfrak{m}(R/\varphi^{t+1}(\mathfrak{a})R) \longrightarrow H^{d-i}_\mathfrak{m}(R/\varphi^t(\mathfrak{a})R) \tag{2.8.1}$$

induced by the natural surjection

$$R/\varphi^{t+1}(\mathfrak{a})R \longrightarrow R/\varphi^t(\mathfrak{a})R$$

is surjective for each $t \geqslant 0$. In view of the isomorphisms

$$R^{\varphi} \otimes_R H^{d-i}_{\mathfrak{m}}(R/\varphi^t(\mathfrak{a})R) \cong H^{d-i}_{\mathfrak{m}}(R/\varphi^{t+1}(\mathfrak{a})R)$$

and the right exactness of tensor, it suffices to verify the surjectivity of (2.8.1) in the case $t = 0$. By Lemma 2.7, this reduces to checking that the $\overline{\varphi}$-action

$$\overline{\varphi}_*\colon H^{d-i}_{\mathfrak{m}}(R/\mathfrak{a}) \longrightarrow H^{d-i}_{\mathfrak{m}}(R/\mathfrak{a})$$

is surjective up to taking the R-span. This follows from Lemma 2.5. $\square$

Theorem 1.3 follows immediately from Theorem 2.8 by taking φ to be the Frobenius endomorphism. To recover the result for square-free monomial ideals [Ly1, Theorem 1 (i)], take φ as in Example 2.2.

3. Examples

We first construct an example of a module M over a regular local ring $(R, \mathfrak{m})$ such that $H^i_{\mathfrak{m}}(M) = 0$ but $\operatorname{Ext}^i_R(R/\mathfrak{a}, M)$ is nonzero for every $\mathfrak{m}$-primary ideal $\mathfrak{a}$ of R. It then follows that $H^i_{\mathfrak{m}}(M)$ cannot be realized as a union of appropriate Ext-modules. We use the following lemma:

Lemma 3.1. *Let $(R, \mathfrak{m})$ be a regular local ring of dimension d, and let $\mathfrak{a}$ be an $\mathfrak{m}$-primary ideal. Then, for each R-module M, there is an isomorphism*

$$\operatorname{Ext}^i_R(R/\mathfrak{a}, M) \cong \operatorname{Tor}^R_{d-i}(\operatorname{Ext}^d_R(R/\mathfrak{a}, R), M) \qquad \textit{for all } 0 \leqslant i \leqslant d\,.$$

Proof. Let $P_\bullet$ be a minimal free resolution of $R/\mathfrak{a}$. The complex $\operatorname{Hom}_R(P_\bullet, R)$ has homology $\operatorname{Ext}^\bullet_R(R/\mathfrak{a}, R)$. Since $\mathfrak{a}$ is an $\mathfrak{m}$-primary ideal of a regular ring R, we have $\operatorname{depth}_{\mathfrak{a}} R = d$, and so $\operatorname{Ext}^j_R(R/\mathfrak{a}, R)$ is nonzero only for $j = d$. It follows that, with a change of index, $\operatorname{Hom}_R(P_\bullet, R)$ is an acyclic complex of free modules resolving the module $\operatorname{Ext}^d_R(R/\mathfrak{a}, R)$. Hence

$$\begin{aligned}\operatorname{Ext}^i_R(R/\mathfrak{a}, M) = H^i(\operatorname{Hom}(P_\bullet, M)) &\cong H^i(\operatorname{Hom}(P_\bullet, R) \otimes_R M)\\ &\cong \operatorname{Tor}^R_{d-i}(\operatorname{Ext}^d_R(R/\mathfrak{a}, R), M)\,. \quad \square\end{aligned}$$

Example 3.2. Let $(R, \mathfrak{m})$ be a regular local ring of dimension $d > 0$, and x a nonzero element of $\mathfrak{m}$. Then $R/(x)$ has dimension $d-1$, so $H^d_{\mathfrak{m}}(R/(x)) = 0$. However, if $\mathfrak{a}$ is an $\mathfrak{m}$-primary ideal, then Lemma 3.1 implies that

$$\operatorname{Ext}^d_R(R/\mathfrak{a}, R/(x)) \cong \operatorname{Ext}^d_R(R/\mathfrak{a}, R) \otimes_R R/(x)\,,$$

which is nonzero. In particular, if $\{\mathfrak{a}_t\}_{t \geqslant 0}$ is a decreasing family of ideals cofinal with $\{\mathfrak{m}^t\}_{t \geqslant 0}$, then the modules $\operatorname{Ext}^d_R(R/\mathfrak{a}_t, R/(x))$ are nonzero for each t, and so the maps $\operatorname{Ext}^d_R(R/\mathfrak{a}_t, R/(x)) \longrightarrow H^d_{\mathfrak{m}}(R/(x))$ are not injective.

Example 3.4 below is due to Eisenbud: given positive integers $a \leqslant b - 2$, there exists a polynomial ring R and a finitely generated graded R-module M, such that the natural map $\operatorname{Ext}^i_R(R/\mathfrak{a}, M) \longrightarrow H^i_{\mathfrak{m}}(M)$ is not injective for all $a < i < b$ and all $\mathfrak{m}$-primary ideals $\mathfrak{a}$. This is based on a construction of Evans and Griffith [EG, Theorem A].

Theorem 3.3 (Evans-Griffith). *Let $\mathbb{K}$ be an infinite field and take a sequence of positive integers, $n_0 < n_1 < \cdots < n_s$. Then there exists a polynomial ring R over $\mathbb{K}$, with a homogeneous prime ideal $\mathfrak{p}$, such that the local cohomology module $H^i_{\mathfrak{m}}(R/\mathfrak{p})$ is nonzero if and only if $i \in \{n_0, n_1, \dots, n_s\}$. Moreover, if $n_0 \geqslant 2$, then $R/\mathfrak{p}$ may be chosen to be a normal domain.* □

Example 3.4 (Eisenbud). Let $a \leqslant b-2$ be positive integers. By Theorem 3.3, there exists a polynomial ring R with a homogeneous prime $\mathfrak{p}$, such that $\operatorname{depth} R/\mathfrak{p} = a$, $\dim R/\mathfrak{p} = b$ and $H^j_{\mathfrak{m}}(R/\mathfrak{p}) = 0$ for all $a < j < b$. Let $\mathfrak{a}$ be an $\mathfrak{m}$-primary ideal. Then $\operatorname{Ext}^a_R(R/\mathfrak{a}, R/\mathfrak{p})$ is nonzero so, by Lemma 3.1,

$$\operatorname{Tor}^R_{d-a}(\operatorname{Ext}^d_R(R/\mathfrak{a}, R), R/\mathfrak{p}) \neq 0 \qquad \text{where } d = \dim R\,.$$

By the rigidity of Tor over regular local rings, [Li], it follows that

$$\operatorname{Tor}^R_j(\operatorname{Ext}^d_R(R/\mathfrak{a}, R), R/\mathfrak{p}) \neq 0 \qquad \text{for all } 0 \leqslant j \leqslant d-a\,.$$

By another application of Lemma 3.1, the module $\operatorname{Ext}^i_R(R/\mathfrak{a}, R/\mathfrak{p})$ is nonzero if $a \leqslant i \leqslant d$. Now if $\{\mathfrak{a}_t\}_{t\geqslant 0}$ is any decreasing family of ideals cofinal with $\{\mathfrak{m}^t\}_{t\geqslant 0}$, it follows that the maps

$$\operatorname{Ext}^i_R(R/\mathfrak{a}_t, R/\mathfrak{p}) \longrightarrow H^i_{\mathfrak{m}}(R/\mathfrak{p})$$

are not injective for each $a < i < b$ and each $t \geqslant 0$.

Example 3.5. Let $\mathbb{K}$ be a field and consider the $\mathbb{K}$-linear ring homomorphism

$$\alpha\colon R = \mathbb{K}[w, x, y, z] \longrightarrow \mathbb{K}[s^4, s^3t, st^3, t^4]$$

where α sends w, x, y, z to the elements s^4, s^3t, st^3, t^4 respectively. Let $\mathfrak{a}$ be the kernel of α. Using vanishing theorems such as [HL, Theorem 2.9], it may be verified that $H^i_{\mathfrak{a}}(R) = 0$ for $i \geqslant 3$.

If $\mathbb{K}$ has characteristic $p > 0$, Hartshorne [Ha] showed that $\mathfrak{a}$ is a set-theoretic complete intersection, i.e., that there exist elements f, g in R such that $\mathfrak{a} = \operatorname{rad}(f, g)$. In this case, the ideals $\mathfrak{a}_t = (f^t, g^t)$ form a descending chain cofinal with $\{\mathfrak{a}^t\}$ for which the maps $\operatorname{Ext}^i_R(R/\mathfrak{a}_t, R) \longrightarrow H^i_{\mathfrak{a}}(R)$ are injective for all $i \geqslant 0$ and $t \geqslant 0$; see Remark 1.5.

Next, suppose that $\mathbb{K}$ has characteristic 0. If $\mathfrak{b}$ is an ideal with $\operatorname{rad}\mathfrak{b} = \mathfrak{a}$ such that $\operatorname{Ext}^i_R(R/\mathfrak{b}, R) \longrightarrow H^i_{\mathfrak{a}}(R)$ is injective for all $i \geqslant 0$, then

$$\operatorname{Ext}^3_R(R/\mathfrak{b}, R) = 0 = \operatorname{Ext}^4_R(R/\mathfrak{b}, R)$$

and so $R/\mathfrak{b}$ is Cohen-Macaulay. This leads to the following question:

Question 3.6. Let $\mathbb{K}$ be a field of characteristic 0 and, as in Example 3.5, let $\mathfrak{a} \subset R = \mathbb{K}[w, x, y, z]$ be an ideal with $R/\mathfrak{a} \cong \mathbb{K}[s^4, s^3t, st^3, t^4]$. Is the ideal $\mathfrak{a}$ set-theoretically Cohen-Macaulay, i.e., does there exist an ideal $\mathfrak{b} \subset R$ with $\operatorname{rad}\mathfrak{b} = \mathfrak{a}$, such that the ring $R/\mathfrak{b}$ is Cohen-Macaulay?

While the requirement of F-purity in Theorem 1.3 is certainly a strong hypothesis, it appears to be a crucial ingredient. In the following example, we have regular rings $R_p = R/pR$ of prime characteristic p and ideals $\mathfrak{a}_p = \mathfrak{a}R_p$ such that the maps

$$\operatorname{Ext}^4_{R_p}(R_p/\mathfrak{a}_p^{[p^t]}, R_p) \longrightarrow H^4_{\mathfrak{a}_p}(R_p)$$

are injective if and only if $R_p/\mathfrak{a}_p$ is F-pure; the set of primes for which this is the case is infinite, as is its complement.

Example 3.7. Let $E \subset \mathbb{P}^2_{\mathbb{Q}}$ be an elliptic curve, and consider the Segre embedding of $E \times \mathbb{P}^1_{\mathbb{Q}}$ in $\mathbb{P}^5_{\mathbb{Q}}$. Clearing denominators in a set of generators for the defining ideal of the homogeneous coordinate ring, we obtain an ideal $\mathfrak{a}$ of $R = \mathbb{Z}[u, v, w, x, y, z]$ such that $R/\mathfrak{a} \otimes_{\mathbb{Z}} \mathbb{Q}$ is the coordinate ring of $E \times \mathbb{P}^1_{\mathbb{Q}}$. For prime integers p, let $R_p = R/pR$ and $\mathfrak{a}_p = \mathfrak{a}R_p$. For all but finitely many primes p, the reduction mod p of E is a smooth elliptic curve E_p and $R_p/\mathfrak{a}_p$ is a homogeneous coordinate ring for $E_p \times \mathbb{P}^1_{\mathbb{Z}/p}$. We restrict our attention to such primes. Since $\operatorname{depth} R_p/\mathfrak{a}_p = 2$, the Auslander-Buchsbaum formula implies that $\operatorname{pd}_{R_p} R_p/\mathfrak{a}_p = 4$. Using the flatness of Frobenius, we see that

$$\operatorname{pd}_{R_p} R_p/\mathfrak{a}_p^{[p^t]} = 4\,,$$

and hence that

$$\operatorname{Ext}^4_{R_p}(R_p/\mathfrak{a}_p^{[p^t]}, R_p) \neq 0 \qquad \text{for all } t \geqslant 0\,.$$

On the other hand, $H^4_{\mathfrak{a}_p}(R_p)$ is zero if E_p is supersingular and nonzero if E_p is ordinary, see [HS, Example 3, page 75] or [Ly2, page 219]. By well-know results on elliptic curves, there are infinitely primes p for which E_p is supersingular, and infinitely many for which it is ordinary. Consider the natural map

$$\operatorname{Ext}^i_{R_p}(R_p/\mathfrak{a}_p^{[p^t]}, R_p) \longrightarrow H^i_{\mathfrak{a}_p}(R_p)\,. \tag{3.7.1}$$

Ordinary primes. If E_p is ordinary then its coordinate ring is F-pure, and it follows that $R_p/\mathfrak{a}_p$ is F-pure as well. In this case, Theorem 1.3 implies that the map (3.7.1) is injective for all $i \geqslant 0$ and $t \geqslant 0$.
Supersingular primes. If p is a prime such that E_p is supersingular, then $H^4_{\mathfrak{a}_p}(R_p) = 0$ so the map (3.7.1) is not injective for $i = 4$. We do not know whether there exists an $\mathfrak{a}_p$-primary ideal $\mathfrak{b}$ for which the maps

$$\operatorname{Ext}^i_{R_p}(R_p/\mathfrak{b}, R_p) \longrightarrow H^i_{\mathfrak{a}_p}(R_p)$$

are injective for all $i \geqslant 0$. Since $H^i_{\mathfrak{a}_p}(R_p) = 0$ for $i \geqslant 4$ in the supersingular case, the existence of such an ideal would imply that $\mathfrak{a}_p$ is set-theoretically Cohen-Macaulay; see also [SW, § 3].

4. A vanishing criterion

The observations from § 2 yield the following vanishing theorem, which links Lyubeznik's positive characteristic result [Ly2, Theorem 1.1] to a theorem for monomial ideals recorded below as Corollary 4.2.

Theorem 4.1. *Let $(R, \mathfrak{m})$ be a regular local ring, $\mathfrak{a}$ an ideal, and φ a flat local endomorphism such that $\{\varphi^t(\mathfrak{a})R\}_{t \geqslant 0}$ is a decreasing chain of ideals cofinal with the chain $\{\mathfrak{a}^t\}_{t \geqslant 0}$. Then $H^i_{\mathfrak{a}}(R) = 0$ if and only if some iteration of the induced action*

$$\overline{\varphi}_* \colon H^{\dim R - i}_{\mathfrak{m}}(R/\mathfrak{a}) \longrightarrow H^{\dim R - i}_{\mathfrak{m}}(R/\mathfrak{a})$$

is zero.

Proof. Let $d = \dim R$. The direct limit

$$H^i_{\mathfrak{a}}(R) = \varinjlim_t \operatorname{Ext}^i_R(R/\varphi^t(\mathfrak{a})R, R)$$

vanishes if and only if for each $t \in \mathbb{N}$, there exists $k \in \mathbb{N}$ such that the map

$$\operatorname{Ext}^i_R(R/\varphi^t(\mathfrak{a})R, R) \longrightarrow \operatorname{Ext}^i_R(R/\varphi^{t+k}(\mathfrak{a})R, R) \tag{4.1.1}$$

induced by the surjection $R/\varphi^{t+k}(\mathfrak{a})R \longrightarrow R/\varphi^t(\mathfrak{a})R$ is zero. By local duality, the map (4.1.1) is zero if and only if

$$H^{d-i}_{\mathfrak{m}}(R/\varphi^{t+k}(\mathfrak{a})R) \longrightarrow H^{d-i}_{\mathfrak{m}}(R/\varphi^t(\mathfrak{a})R)$$

is the zero map. By Remark 2.6 (3) and the flatness of $R^\varphi \otimes_R -$, this is equivalent to the map

$$H^{d-i}_{\mathfrak{m}}(R/\varphi^k(\mathfrak{a})R) \longrightarrow H^{d-i}_{\mathfrak{m}}(R/\mathfrak{a})$$

being zero. By Lemma 2.7, this last condition is equivalent to the k-th iterate of the action $\overline{\varphi}_* \colon H^{d-i}_{\mathfrak{m}}(R/\mathfrak{a}) \longrightarrow H^{d-i}_{\mathfrak{m}}(R/\mathfrak{a})$ being zero. □

Using Theorem 4.1 we next recover a vanishing theorem for monomial ideals, [Ly1, Theorem 1 (iii)]. In [Mi, Corollary 6.7] Miller proves a stronger statement connecting $H^i_{\mathfrak{a}}(S)$ and $H^{\dim S - i}_{\mathfrak{m}}(S/\mathfrak{a})$ via Alexander duality.

Corollary 4.2. *Let S be a polynomial ring over a field, and let $\mathfrak{a}$ be an ideal generated by square-free monomials. Then $H^i_{\mathfrak{a}}(S) = 0$ if and only if $H^{\dim S - i}_{\mathfrak{m}}(S/\mathfrak{a}) = 0$.*

Proof. Let $S = \mathbb{K}[x_1, \dots, x_d]$, and let φ be the $\mathbb{K}$-linear endomorphism with $\varphi(x_i) = x_i^2$ for $1 \leqslant i \leqslant d$. Then φ is flat, and induces a pure endomorphism of $S/\mathfrak{a}$, see Example 2.2.

Each of the modules in question is graded, so the issue of vanishing is unchanged under localization at the homogeneous maximal ideal of S. We can therefore work over the regular local ring $(R, \mathfrak{m})$, where we need to show that $H^i_{\mathfrak{a}}(R) = 0$ if and only if $H^{d-i}_{\mathfrak{m}}(R/\mathfrak{a}R) = 0$. The endomorphism φ localizes to give a flat endomorphism of R. Moreover, since purity localizes,

φ induces a pure endomorphism $\overline{\varphi}$ of $R/\mathfrak{a}R$. By Theorem 4.1, $H^i_{\mathfrak{a}}(R) = 0$ if and only if some iterate of the action

$$\overline{\varphi}_*\colon H^{d-i}_{\mathfrak{m}}(R/\mathfrak{a}R) \longrightarrow H^{d-i}_{\mathfrak{m}}(R/\mathfrak{a}R)$$

is zero. But $\overline{\varphi}_*$ is injective since $\overline{\varphi}$ is pure, so an iterate of $\overline{\varphi}_*$ is zero precisely if $H^{d-i}_{\mathfrak{m}}(R/\mathfrak{a}R) = 0$. □

References

[EMS] D. Eisenbud, M. Mustaţă, and M. Stillman, *Cohomology on toric varieties and local cohomology with monomial supports*, J. Symbolic Comput. **29** (2000), 583–600.

[EG] E. G. Evans Jr. and P. A. Griffith, *Local cohomology modules for normal domains*, J. London Math. Soc. (2) **19** (1979), 277–284.

[Ha] R. Hartshorne, *Complete intersections in characteristic* $p > 0$, Amer. J. Math. **101** (1979), 380–383.

[HS] R. Hartshorne and R. Speiser, *Local cohomological dimension in characteristic* p, Ann. of Math. (2) **105** (1977), 45–79.

[HR1] M. Hochster and J. Roberts, *Rings of invariants of reductive groups acting on regular rings are Cohen-Macaulay*, Adv. Math. **13** (1974), 115–175.

[HR2] M. Hochster and J. Roberts, *The purity of the Frobenius and local cohomology*, Adv. Math. **21** (1976), 117–172.

[HL] C. Huneke and G. Lyubeznik, *On the vanishing of local cohomology modules*, Invent. Math. **102** (1990), 73–93.

[Ku] E. Kunz, *Characterizations of regular local rings for characteristic* p, Amer. J. Math. **91** (1969), 772–784.

[Li] S. Lichtenbaum, *On the vanishing of* Tor *in regular local rings*, Illinois J. Math. **10** (1966), 220–226.

[Ly1] G. Lyubeznik, *On the local cohomology modules* $H^i_{\mathfrak{a}}(R)$ *for ideals* $\mathfrak{a}$ *generated by monomials in an* R*-sequence*, in: Complete Intersections, Lecture Notes in Math. **1092**, Springer, 1984, pp. 214–220.

[Ly2] G. Lyubeznik, *On the vanishing of local cohomology in characteristic* $p > 0$, Compos. Math. **142** (2006), 207–221.

[Mi] E. Miller, *The Alexander duality functors and local duality with monomial support*, J. Algebra **231** (2000), 180–234.

[Mu] M. Mustaţă, *Local cohomology at monomial ideals*, J. Symbolic Comput. **29** (2000), 709–720.

[PS] C. Peskine and L. Szpiro, *Dimension projective finie et cohomologie locale*, Inst. Hautes Études Sci. Publ. Math. **42** (1973), 47–119.

[SW] A. K. Singh and U. Walther, *On the arithmetic rank of certain Segre products*, Contemp. Math. **390** (2005), 147–155.

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