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Algorithmic stratification of $\mathbb{R}\mathscr{H}om_{\mathcal{D}}(\mathcal{M},\mathcal{N})$ for regular algebraic $\mathcal{D}$-modules on $\mathbb{C}^n$

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Abstract

Let M be a holonomic D-module on $\mathbb{C}^n$. We give an algorithm to stratify $\mathbb{C}^n$ such that on all strata X each restriction (derived inverse image) module $H^i(\rho_{X,\mathbb{C}^n}(M))$ is a connection. For regular holonomic modules this stratifies the solution complex $\mathbb{R}\mathscr{H}om_{\mathcal{D}_{\mathbb{C}^n,an}}(M_{an},\mathcal{O}_{an})$ of sheaves of (higher) holomorphic solutions of M. We also give an algorithm to compute the dimensions of the cohomology modules of the solution complex over all strata.

1. Introductions

The purpose of this paper is to give an algorithm as indicated in the abstract. The algorithm is based on an algorithm by T. Oaku and N. Takayama [12] for computing restrictions of a $\mathcal{D}$-module to a hyperplane. We then present an algorithm that finds the rank of the connection on each stratum and we close with a discussion of stratifying $\mathbb{R}\mathscr{H}om_{D_{\mathbb{C}^n,an}}(M_{an},N_{an})$ when M,N are both regular holonomic.

1.1. Notation

Let $K\subset\mathbb{C}$ be a computable field. We require that there exists an algorithm for finding associated primes in $R_n = K[x_1,\ldots,x_n]$ (e.g. [4] for $\mathbb{Q}$). Our modules are given via generators and relations with coefficients K; they are left modules unless specified otherwise. For $f\in R_n$, U_f stands for $\mathrm{Spec}(R_n[f^{-1}])$. If X is a $\mathbb{C}$-manifold and Y a locally closed submanifold then $d_{X/Y}$ is the relative $\mathbb{C}$-dimension $d_X - d_Y$. We denote by $T^*_Y(X)$ the conormal bundle given as the elements (y,λ) of the cotangent spaces $(T^*X)_y$ of $y\in Y$ which evaluate to zero on the tangent space of Y at y under the natural pairing $(T^*X)_y\times(TX)_y\to\mathbb{C}$.

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1.2. $\mathcal{D}$-basics

Let X be an algebraic manifold over $\mathbb{C}$, X^{an} the associated analytic space. We refer to [2, 3] for details of $\mathcal{D}$-module theory. The sheaf of $\mathbb{C}$-linear differential operators $\mathcal{D}_X$ acts on the sheaf of regular functions $\mathcal{O}_X$. Abusing notation, $\mathcal{O}_X \subseteq \mathcal{O}_{X^{an}}$ and $\mathcal{D}_{X^{an}} = \mathcal{O}_{X^{an}} \otimes_{\mathcal{O}_X} \mathcal{D}_X$. Let D_n be the Weyl algebra $R_n[\partial_1, \cdots, \partial_n]$. Let $\mathcal{D}_X$ -mods be the category of bounded complexes of finitely generated left $\mathcal{D}_X$-modules containing the category of left modules over $\mathcal{D}_X$ as subcategory in cohomological degree zero. We will write objects of $\mathcal{D}_X$ as $\mathcal{M}^\bullet$ and if $\mathcal{M}^\bullet$ happens to be a module in cohomological degree zero we write just $\mathcal{M}$. If $X = \mathbb{C}^n$ then all $\mathcal{D}_X$-modules are generated by their global sections; we use Roman letters for the "K-algebraic" sheaves induced by D_n-modules.

The order filtration on $\mathcal{D}_{X^{an}}$ induces the graded object $\mathrm{gr}_{(0,1)}(\mathcal{M})$ annihilated by the *characteristic ideal* $\mathrm{chI}(\mathcal{M})$ [11, 13]. The *characteristic cycle* $\mathrm{chC}(\mathcal{M})$ is the formal sum $\sum a_i V_i$ of irreducible components of the *characteristic variety* $\mathrm{chV}(\mathcal{M}) = \mathrm{Var}(\mathrm{chI}(\mathcal{M})) \subseteq T^*(X)$ where $a_i \in \mathbb{N}$ is the multiplicity of $\mathrm{gr}_{(0,1)}(\mathcal{M})$ along V_i. If $U \subseteq X$ is open, $\mathrm{chV}(\mathcal{M}|_U) = \mathrm{chV}(\mathcal{M}) \cap T^*(U)$, and $\mathrm{chC}(\mathcal{M}|_U) = \sum a_i(V_i \cap T^*(U))$. If X and $\mathcal{M}$ are algebraic, then so are $T^*(X)$, $\mathrm{gr}_{(0,1)}(\mathcal{M})$ and $\mathrm{chV}(\mathcal{M})$. The *singular locus* $\mathrm{sing}(\mathcal{M})$ is the projection of $\mathrm{chV}(\mathcal{M})$ into $X \subset T^*(X)$ and it is always a closed set since $\mathrm{chV}(\mathcal{M})$ is a conical variety (i.e., its defining ideal is homogeneous in the derivation variables).

1.3. Holonomic modules and the $\mathscr{S}ol$-functor

The *holonomic* modules $\mathrm{Hol}(X) = \{\mathcal{M}^\bullet \in \mathcal{D}_X\text{-mods} : \dim \mathrm{chV}(H^i(\mathcal{M}^\bullet)) = \dim(X)\,\forall\, i\}$ form a full subcategory of $\mathcal{D}_X$ -mods, closed under subquotients and extensions. Near $p \notin \mathrm{sing}(\mathcal{M})$, $\mathcal{M} \in \mathrm{Hol}(X)$ is a connection and its *rank* is the finite number $\mathrm{rk}(\mathcal{M}) = \dim_{\mathbb{C}}(\mathrm{Hom}_{\mathcal{D}_{X,p}}(\mathcal{M}_p, \mathcal{O}_p))$.

Let $\mathcal{M}^\bullet, \mathcal{L}^\bullet \in \mathrm{Hol}(X)$. Then by [8] the derived functors of the sheaf homomorphisms from $\mathcal{M}^\bullet_{an}$ to $\mathcal{L}^\bullet_{an}$ satisfy $\dim_{\mathbb{C}}(\mathscr{E}xt^i_{\mathcal{D}_{X^{an}}}(\mathcal{M}^\bullet_{an}, \mathcal{L}^\bullet_{an}))_p < \infty \forall p \in X$ and the sheaves $\mathscr{E}xt^i_{\mathcal{D}_{X^{an}}}(\mathcal{M}^\bullet_{an}, \mathcal{L}^\bullet_{an})$ are constructible. In [8, 9] the following question was raised.

Problem 1 (Kashiwara): Let $X = \mathbb{C}^n$. Suppose $M^\bullet, L^\bullet \in \mathrm{Hol}(X)$ are induced by complexes of by D_n-modules. Compute a stratification $\mathfrak{S}$ of X such that $\mathbb{R}\mathscr{H}om_{\mathcal{D}_{X^{an}}}(M^\bullet_{an}, L^\bullet_{an})$ is locally constant on each $S \in \mathfrak{S}$, and find an algorithm to compute $\dim_{\mathbb{C}} \mathscr{E}xt^i_{\mathcal{D}_{X^{an}}}(M^\bullet_{an}, L^\bullet_{an})_p\ \forall p \in X \in \mathfrak{S}, \forall i.$

In the next section we produce such an algorithm for $L^\bullet = R_n$ and $M^\bullet$ regular algebraic: $\mathcal{M}^\bullet \in \mathrm{Hol}(X)$ is called *regular at* $p \in X$ if and only if

$$\mathbb{R}\mathscr{H}om_{\mathcal{D}_{X^{an}}}(\mathcal{M}^\bullet_{an}, \mathcal{O}_{X^{an}})_p \to \mathbb{R}\mathrm{Hom}_{\mathcal{D}_{X^{an}}}(\mathcal{M}^\bullet_{an,p}, \hat{\mathcal{O}}_{X^{an},p}) \cong \mathbb{R}\mathrm{Hom}_{\mathbb{C}}(\rho_{p,X}(\mathcal{M}^\bullet), \mathbb{C}) \tag{1.1}$$

is a quasi-isomorphism. Here we denote by $\rho_{p,X}(-)$ the derived inverse image functor $(\mathcal{D}_X/\mathcal{I}(p)\cdot\mathcal{D}_X)\otimes^L_{\mathcal{D}_X}(-)_{an}$ where $\mathcal{I}(p)$ is the $\mathcal{O}_X$-ideal defining p. Regularity at p means that near p the (derived) formal solutions are in fact convergent. One calls $\mathcal{M}$ *regular*, $\mathcal{M}\in\mathrm{Reg}(X)$, if for all $i : X^{an}\hookrightarrow\overline{X^{an}}$ with compact $\overline{X^{an}}$ the direct image $i_+(\mathcal{M})$ is regular at all $p\in\overline{X^{an}}$.

The category $\mathrm{Reg}(X)$ contains $\mathcal{O}_X$ and is closed under the formation of subquotients and extensions. If $Y\subseteq X$ is smooth, the derived inverse image (restriction) functor $\rho_{Y,X}(-)$ sends connections to connections with cohomological shift $d_{X/Y}$. More generally, holonomicity and regularity of complexes are stable under $\rho_{Y,X}(-)$ and the direct image functor $f_+(-)$ for all analytic maps $f : Y\to X$. It remains an important open question to find an algorithm that determines regularity of a $\mathcal{D}$-module near a given point: so far there is a partial solution by Assi, Castro and Granger [1] in terms of an algorithm that can test for a real number to be a slope of a given module with respect to a hyperplane.

2. Stratification by connections and computation of rank

2.1. Kashiwara equivalence and connections

We let X be an algebraic manifold and suppose that $i : Y\hookrightarrow X$ is a closed algebraic subvariety. Let $\mathcal{M}\in\mathcal{O}_X$ -mods and set

$$\Gamma_Y(\mathcal{M})(U)=\{m\in\mathcal{M}(U) : {\mathcal{I}_Y}^k\cdot m=0\forall k\gg 0\}.$$

Its derived functor $\mathbb{R}\Gamma_Y(-) : \mathcal{O}_X\text{-mods}\to\mathcal{O}_X\text{-mods}$, the *local cohomology functor with supports in* Y, sends $\mathcal{D}_X$-modules to $\mathcal{D}_X$-modules and preserves $\mathrm{Hol}(X)$ and $\mathrm{Reg}(X)$. Algorithms to compute $\mathbb{R}\Gamma_Y(-)$ are given in [12] and can be executed for example with [7].

Suppose now that in addition to the above Y is smooth. A theorem of Kashiwara says that i_+ sets up an equivalence of categories from $\mathcal{D}_Y$-modules to $\mathcal{D}_X$-modules supported on Y, and $i_+\circ\rho_{Y,X}(\mathcal{M}^\bullet)=\mathbb{R}\Gamma_Y(\mathcal{M}^\bullet)$ for $\mathcal{M}\in\mathrm{Hol}(X)$, see [3], VI.7.13. In particular, if $M\in D_n$-mods and $\mathrm{supp}(M)=p\in\mathbb{C}^n$ then $M=(i_{\mathbb{C}^n,p})_+(\mathbb{C}^\nu)=(H^n_p(R_n))^\nu$ and ν is the multiplicity of $\mathrm{chC}(M)$ along $T^*_p(\mathbb{C}^n)$.

We can now state the following algorithm in which $I(S)$ denotes for any subset of $\mathbb{C}^n$ the ideal of polynomials vanishing entirely on S, and $\overline{(-)}$ denotes Zariski closure.

Algorithm 2.1 (Stratifying to connections):
Input: $M^\bullet\in\mathrm{Hol}(D_n)$.
Output: An algebraic stratification $\mathfrak{S}$ of $\mathbb{C}^n$ such that $\rho^i_{X,\mathbb{C}^n}(M^\bullet)$ is a connection along each stratum X for all $i\in\mathbb{N}$.

1. Let $\mathfrak{S}=\emptyset$, $\mathfrak{Q}=\{(\mathbb{C}^n,M^\bullet)\}$.
2. If $\mathfrak{Q}$ is empty, exit. Otherwise pick an item $(X,N^\bullet)$ from $\mathfrak{Q}$.
3. Let $Y\subset\overline{X}$ be the projection to $\mathbb{C}^n$ of $\overline{\mathrm{chV}(N^\bullet)\setminus T^*_X(\mathbb{C}^n)}$.

4. If $X \cap Y = \emptyset$ add $\{X, N^\bullet\}$ to $\mathfrak{S}$, strike $(X, N^\bullet)$ from $\mathfrak{Q}$ and return to Step 2.
5. If $X \cap Y$ is not empty, let $g' \in I(Y) \setminus I(\overline{X})$.
6. Add $\{X \cap U_{g'}, N^\bullet[g'^{-1}]\}$ to $\mathfrak{S}$, strike $(X, N^\bullet)$ from $\mathfrak{Q}$.
7. Write $X \cap \mathrm{Var}(g') = \coprod X_j$ such that each X_j is irreducible (over K), smooth and affine: $X_j = U_j \cap C_j$, $U_j = U_{f_j}$, C_j closed in $\mathbb{C}^n$.
8. Add to $\mathfrak{Q}$ the pairs $(V_j, \mathbb{R}\Gamma_{\overline{X}_j}(N^\bullet[f_j^{-1}]))$ and return to Step 2.

Return $\mathfrak{S}$.

Proof (of correctness): We note first that $\bigcup_{X\in\mathfrak{S}} X$ is a disjoint union equal to $\mathbb{C}^n$. It is in Step 7 that we use the hypothesis on the existence of an algorithm for primary decomposition of ideals in R_n.

Since $\mathbb{R}\Gamma_{\overline{X}}(-)\circ\mathbb{R}\Gamma_{\overline{X'}}(-) = \mathbb{R}\Gamma_{\overline{X}\cap\overline{X'}}(-)$, for all pairs $(X, N^\bullet)$ that ever enter Q one always has $N^\bullet = \mathbb{R}\Gamma_{\overline{X}}(M^\bullet[f^{-1}])$ and $X = \overline{X} \setminus \mathrm{Var}(f)$ is smooth, affine and irreducible over K. Hence, in Step 5, $Y \subsetneq X$ since every cohomology module of $N^\bullet$ is supported within X and therefore is a connection on a set that is Zariski-dense in X.

Let $X = C \cap U_f$ be in $\mathfrak{S}$. Then $\mathrm{chC}(H^i_{\overline{X}}(M^\bullet[f^{-1}])) = a_i \cdot T^*_X(\mathbb{C}^n)$ plus some components not meeting X. By Kashiwara equivalence $\mathrm{chC}(\rho^i_{X,\mathbb{C}^n}(M^\bullet)) = a_i \cdot X$. Hence each $\rho^i_{X,\mathbb{C}^n}(M^\bullet)$ has empty singular locus and so is a connection on X.

Termination is a consequence of Noetherian induction. $\square$

2.2. The rank of a connection on a locally closed subset

Let $M^\bullet \in \mathrm{Hol}(\mathbb{C}^n)$ be algebraic and let $X \subseteq \mathbb{C}^n$ be locally closed and algebraic. Assume that $M^\bullet \cong \mathbb{R}\Gamma_{\overline{X}}(M^\bullet)$ and that $\rho^i_{X,\mathbb{C}^n}(M^\bullet)$ is a connection for all i. We want to know the ranks of these connections. If one can identify $p \in X$ explicitly, then it suffices to compute $\rho^{i+\dim(X)}_{p,\mathbb{C}^n}(M^\bullet)$. However, it may be difficult or even impossible to explicitly give the coordinates of a single point on the stratum.

On the other hand, if F is smooth, closed and equidimensional in X then $\rho^i_{F,\mathbb{C}^n}(M^\bullet)$ is a connection of rank $\mathrm{rk}\left(\rho^{i-d_{X/F}}_{X,\mathbb{C}^n}(M^\bullet)\right)$. The idea is therefore to reduce to zero-dimensional F. Let V be an affine subspace of $\mathbb{C}^n$. Call V *generic for* X if $d_V + d_X = n$ and $\deg(\overline{X}) = \#(X \cap V)$. Within the appropriate Grassmannian over K a Zariski-open set of subspaces V is generic for X; indeed, V is generic for X if and only if $I(V) + I(X)$ is radical of degree $\deg(\overline{X})$ and of dimension zero.

Let V be X-generic. Then the cohomology of $\rho_{V,\mathbb{C}^n}(M^\bullet) \in \mathrm{Hol}(V)$ is supported in $F := V \cap X = V \cap \overline{X}$. By Kashiwara equivalence the rank of $\rho^i_{X,\mathbb{C}}(M^\bullet)$ in any point of F is the rank of $\rho^{i+\dim(X)}_{F,\mathbb{C}^n}(M^\bullet)$. Let $m' = \deg \mathrm{gr}_{(0,1)}(H^{\dim(X)+i}_V(M^\bullet))$ and $m = \deg(I(F))$.

Lemma 2.1: *With the above notation,* $\mathrm{rk}(\rho^i_{X,\mathbb{C}^n}(M^\bullet)) = m'/m$.

Proof: Suppose F is just one point. So $\rho^{i+\dim(X)}_{F,\mathbb{C}^n}(M^\bullet)$ is a vector space of some dimension $\nu = \operatorname{rk}(\rho^i_{X,\mathbb{C}^n}(M^\bullet))$, and its pushforward to $\mathbb{C}^n$ is isomorphic to $H^{i+\dim(X)}_F(M^\bullet) = \bigoplus_{t=1}^{\nu} D_n/D_n \cdot I(F)$. Since cohomology of $M^\bullet$ is supported in X and $F = X \cap V$, $H^{i+\dim(X)}_F(M^\bullet) = H^{i+\dim(X)}_V(M^\bullet)$. On the other hand, $T^*_F(\mathbb{C}^n)$ is the only n-dimensional component of $\operatorname{gr}_{(0,1)}(H^{\dim(X)+i}_V(M^\bullet))$, so $\nu = \deg \operatorname{gr}_{(0,1)}(H^{\dim(X)+i}_V(M^\bullet))$. Since $m = 1$, the formula holds in this case.

If F consists of several points, the degree of $\operatorname{gr}_{(0,1)}(H^{\dim(X)+i}_V(M^\bullet))$ is the sum of all local degrees. Since the local degree is constant along F (equal to the rank of $\rho^i_{X,\mathbb{C}^n}(M^\bullet)$ along X), the claimed formula holds always. □

We remark that if a D_n-module M is cyclic, then $\deg \operatorname{chI}(M) = \deg(\operatorname{gr}_{(0,1)}(M))$ while in general the module structure is more delicate than the annihilator.

Together with the previous algorithm we obtain

Algorithm 2.2 (Stratification and rank):
Input: $M^\bullet \in \operatorname{Hol}(\mathbb{C}^n)$.
Output: A stratification $\{(X_i, C_i, U_i, f_i, r_{i,j})\}$ such that

- $U_i = U_{f_i}$ is open in $\mathbb{C}^n$,
- $C_i = \operatorname{Var}(g_{i,1}, \dots, g_{i,c_i})$ is closed in $\mathbb{C}^n$,
- $X_i = C_i \cap U_i$ is affine smooth and $\rho^j_{X_i,\mathbb{C}^n}(M^\bullet)$ is a connection of rank $r_{i,j}$,
- $(i_{\mathbb{C}^n, X_i})_+ \circ (\rho^j_{X_i,\mathbb{C}^n}(M^\bullet)) = H^{j+\dim(X_i)}_{C_i}(M^\bullet[f_i^{-1}])$.

1. Use Algorithm 2.1 to find a stratification $\{(X_i, C_i, U_i, f_i)\}$ of $\mathbb{C}^n$ such that the cohomology sheaves of $\mathbb{R}\Gamma_{C_i}(M^\bullet[f_i^{-1}])$ induce connections along X_i.
2. For all i, pick an X_i-generic space V_i, put $F_i = V_i \cap X_i$.
3. Then $r_{i,j} = \deg \operatorname{gr}_{(0,1)}(H^{\dim(X_i)+j}_{V_i \cap C_i}(M^\bullet[f_i^{-1}]))/\deg(I(F_i))$.

Remark 2.3: Of course, if $p \in X_i$ is known, instead of Steps 2, 3 one just computes $\rho_{p,\mathbb{C}^n}(M^\bullet)$ with [12].

If $M \in \operatorname{Reg}(\mathbb{C}^n)$, Algorithm 2.2 solves Problem 1 for $L^\bullet = R_n$ because of the isomorphism (1.1).

3. $\mathbb{R}\mathscr{H}om$ between general regular holonomic modules

Let $X = \mathbb{C}^n$ and $M^\bullet, L^\bullet \in \operatorname{Reg}(X)$. We recall that regular holonomic modules are stable under local cohomology, restriction, and the external tensor product $\mathcal{M} \boxtimes \mathcal{L} = p_1^*(\mathcal{M}) \otimes_{\mathcal{O}_X \times \mathcal{O}_X} p_2^*(\mathcal{L})$ where $X \xleftarrow{p_1} X \times X \xrightarrow{p_2}$ are the canonical projections. Stratifying $\mathbb{R}\mathscr{H}om_{\mathcal{D}_{X^{an}}}(M^\bullet_{an}, L^\bullet_{an})$ is a consequence of the algorithms of the previous section as follows. There is a global duality functor $\mathbb{D}_X(\mathcal{M}^\bullet) = \mathbb{R}\mathscr{H}om_{\mathcal{D}_X}(\mathcal{M}^\bullet, \mathcal{D}_X \otimes_{\mathcal{O}_X} \omega_X^{-1})[\dim X]$. Over D_n, the dual of M is τ applied to the n-th cohomology group of $\operatorname{Hom}_{D_n}(F_\bullet, D_n)$ for any D_n-free resolution $F_\bullet$ of M, where τ is the anti-equivalence $x^\alpha\partial^\beta \to (-\partial)^\beta x^\alpha$.

For notational convenience, let χ be the analytic manifold $\mathbb{C}^n$. Put $\mathscr{S}ol(M^\bullet) = \mathbb{R}\mathscr{H}om_{\mathcal{D}_\chi}(M^\bullet_{an}, \mathcal{O}_\chi)$ and consider the diagonal embedding $\chi \to \chi\times\chi$. By [3] and regularity of $M^\bullet$ and $L^\bullet$,

$$\begin{aligned}\mathbb{R}\mathscr{H}om_{\mathcal{D}_\chi}(M^\bullet_{an}, L^\bullet_{an}) &= \omega_\chi \otimes^L_{\mathcal{D}_\chi} (\rho_{\chi,\chi\times\chi}(\mathbb{D}_\chi M^\bullet_{an} \boxtimes L^\bullet_{an})[d_X])[-d_X]\\ &= \omega_\chi \otimes^L_{\mathcal{D}_\chi} (\rho_{\chi,\chi\times\chi}(\mathbb{D}_\chi M^\bullet_{an} \boxtimes L^\bullet_{an}))\\ &= \mathscr{S}ol(\mathbb{D}_\chi(\rho_{\chi,\chi\times\chi}(\mathbb{D}_\chi M^\bullet_{an} \boxtimes L^\bullet_{an}))\\ &= \mathscr{S}ol(\rho_{\chi,\chi\times\chi}(M^\bullet_{an} \boxtimes \mathbb{D}_\chi L^\bullet_{an}))\end{aligned}$$

and by (1.1) its stalk at $p \in \chi$ is $\mathbb{C}$-dual to $\rho_{p,X} \circ \rho_{X,X\times X}(M^\bullet \boxtimes \mathbb{D}_X L^\bullet) = \rho_{p\times p,X\times X}(M^\bullet \boxtimes \mathbb{D}_X L^\bullet)$. So stratifying $\mathbb{R}\mathscr{H}om_{\mathcal{D}_\chi}(M^\bullet_{an}, L^\bullet_{an})$ amounts to stratifying $\mathscr{S}ol(M^\bullet \boxtimes \mathbb{D}_X L^\bullet)$ along the diagonal.

In an independent paper [14] we discuss applications of these methods of two types: we first consider the Malgrange module M_f associated to a singularity f and its endomorphism s with minimal Bernstein-Sato polynomial $b_f(s)$, see [10]. The second class of examples that we study are hypergeometric systems of the type introduced by Gelfand, Graev, Kapranov and Zelevinsky in [5, 6]. In both cases the interest derives from the change of the stratification (or certain multiplicities of the strata) under the variation of an external quantity. For the singularity examples, this external quantity is the choice of a factor of the Bernstein-Sato polynomial. In the GKZ-case we vary the parameter vector β. The explicit computations give rise to speculations and questions that are listed together with comments and progress known to the author.

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