IRREGULARITY OF HYPERGEOMETRIC SYSTEMS VIA SLOPES ALONG COORDINATE SUBSPACES

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ABSTRACT. We study the irregularity sheaves attached to the A-hypergeometric D-module $M_A(\beta)$ introduced by I.M. Gel'fand et al. [GGZ87, GZK89], where $A \in \mathbb{Z}^{d\times n}$ is pointed of full rank and $\beta \in \mathbb{C}^d$. More precisely, we investigate the slopes of this module along coordinate subspaces.

In the process we describe the associated graded ring to a positive semigroup ring for a filtration defined by an arbitrary weight vector L on torus-equivariant generators. To this end we introduce the (A, L)-umbrella, a cell complex determined by A and L, and identify its facets with the components of the associated graded ring.

We then establish a correspondence between the full (A, L)-umbrella and the components of the L-characteristic variety of $M_A(\beta)$. We compute in combinatorial terms the multiplicities of these components in the L-characteristic cycle of the associated Euler–Koszul complex, identifying them with certain intersection multiplicities.

We deduce from this that slopes of $M_A(\beta)$ are combinatorial, independent of β, and in one-to-one correspondence with jumps of the (A, L)-umbrella. This confirms a conjecture of Sturmfels and gives a converse of a theorem of R. Hotta [Hot98, Ch. II, §6.2, Thm.]: $M_A(\beta)$ is regular if and only if A defines a projective variety.

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Contents

1. Introduction and overview

The solutions f of an ordinary linear differential equation $P \bullet f(x) = 0$ where

$$P = p_m(x)\partial_x^m + p_{m-1}(x)\partial_x^{m-1} + \cdots + p_1(x)\partial_x + p_0(x)$$

form a $\mathbb{C}$-vector bundle of dimension m away from the zero locus of p_m. At $p_m(x_0) = 0$, two types of singularities may occur: at a regular singular point x_0, the (multivalued) solutions have polynomial growth for $x \to x_0$ while in all other cases x_0 is called irregular. By Fuchs' Theorem, P is regular at the origin if and only if the Newton polygon $\mathrm{N}(P)$ of P is a quadrant (see [Inc44]). The slopes (or critical indices) of P are just the slopes of $\mathrm{N}(P)$; they represent a refined notion of irregularity (see [Lau85]) and indicate the growth of solutions near the critical point. Regularity at x_0 is equivalent to equality of formal and convergent solutions at x_0.

The concept of regularity in higher dimension is considerably more involved. Denote by $\mathcal{O}_X$ the structure sheaf of the complex manifold X and by $\mathcal{O}_{\widehat{X|Y}}$ the completion of $\mathcal{O}_X$ along the submanifold Y. For any coherent $\mathcal{D}_X$-module $\mathcal{M}$, Z. Mebkhout introduced the irregularity complex $\mathcal{I}rr_Y(M) = \mathbf{R}\mathcal{H}\mathrm{om}_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_{\widehat{X|Y}}/\mathcal{O}_{X|Y})$, $\mathcal{O}_{X|Y}$ being the restriction of $\mathcal{O}_X$ to Y [Meb89]. This intrinsically analytic notion is inspired by earlier work of B. Malgrange on regularity in the univariate case [Mal74], but quite difficult to use.

On the other hand, Y. Laurent [Lau87] gave a generalization for the concept of a slope to the multivariate case based on more algebraic methods. With X, Y as above, let V be the Kashiwara–Malgrange filtration along Y, let F be the order filtration, and put $L = pF + qV$. Then the rational number $p/q > 0$ is a slope of $\mathcal{M}$ along Y if the L-characteristic variety $\mathrm{Ch}^L(\mathcal{M}) = \mathrm{supp}(\mathrm{gr}^L(\mathcal{M})) \subseteq T^*X$ jumps (is not locally constant) at p/q.

The theorem of Fuchs generalizes to the multivariate case: by the analytic-algebraic comparison theorem for slopes [LM99, Thm. 2.4.2], $\mathcal{M}$ has no slopes along Y precisely if $\mathcal{I}rr_Y(\mathcal{M})$ is exact. In fact, the slopes agree with the jumps of the Gevrey filtration on the irregularity complex which provide a measure of growth for the solutions of $\mathcal{M}$ near Y.

All slopes of $\mathcal{M}$ along Y are rational and there are only a finite number of them, (see [Lau87]). A. Assi, F. Castro, and M. Granger [ACJG96] developed a Gröbner basis algorithm to compute slopes of algebraic D-modules using the algebraic counterparts of F, V, and L on the Weyl algebra D. In the process they proved a comparison theorem: slopes of modules over the Weyl algebra can be computed without leaving the algebraic category, where rationality and finiteness follow from the existence of the Gröbner fan (see [ACJG00]). Explicit formulæ for slopes of D-modules are very rare. The purpose of this article is to describe the slopes of A-hypergeometric D-modules.

I.M. Gel'fand, M.I. Graev, M.M. Kapranov and A.V. Zelevinskiĭ [GGZ87, GZK89] defined a class of D-modules that includes as particular cases the differential systems satisfied by the classical hypergeometric functions of Gauß, Appell, and others. These A-hypergeometric, or GKZ (after Gel'fand, Kapranov, and Zelevinskiĭ), systems are special cases of the equivariant D-modules of R. Hotta and M. Kashiwara [HK84, Hot98]. A $d \times n$ integer matrix A defines an action of a d-torus $\mathbb{T} := (\mathbb{C}^*)^d$ on $T_0^*\mathbb{C}^n = \mathbb{C}^n$. Our general hypothesis is that $\mathbb{N}A$ is a positive semigroup with $\mathbb{Z}A = \mathbb{Z}^d$. The closure of the orbit through $(1, \ldots, 1)$ is defined by the toric ideal $I_A \subseteq \mathbb{C}[\partial] =: R$ where $\partial := \partial_1, \ldots, \partial_n$ and $S_A := R/I_A = \mathbb{C}[\mathbb{N}A]$ is the associated semigroup ring. The Euler vector fields $E = (E_1, \ldots, E_d)$ are the pushforwards to this orbit of the Lie algebra generators $t_1\partial_1, \ldots, t_d\partial_d$ of $\mathbb{T}$. The A-hypergeometric system $M_A(\beta)$, depending on the Lie algebra character $\beta \in \mathbb{C}^d$, is the D-module defined by I_A and the Euler operators $E - \beta$. It arises in various situations in algebraic geometry such as in the theory of toric residues (see [CDS01]), the study of hyperplane arrangements (see [OT01]), and in the Picard–Fuchs equations governing the variation of Hodge structures for Calabi–Yau toric hypersurfaces (see [CK99]).

By a theorem of R. Hotta [Hot98, Ch. II, §6.2, Thm.], homogeneous A-hypergeometric systems are regular and hence have no slopes. In dimension one and in codimension one, slopes of $M_A(\beta)$ were studied by F. Castro and N. Takayama [CJT03] and M. Hartillo [HH03, HH05]. Cohen–Macaulayness of the toric rings in question makes these cases comparatively tractable. In our general situation, a key tool is a (generalization of a) computation by A. Adolphson [Ado94] identifying candidate components of the F-characteristic variety with the set of faces not containing 0 of the convex hull Δ_A^F of 0 and the columns $\mathbf{a}_1, \ldots, \mathbf{a}_n$ of A. In Section 2, L is the filtration on R defined by an arbitrary weight vector on $\partial_1, \ldots, \partial_n$. We introduce in Definition 2.7 the (A, L)-polyhedron Δ_A^L as the convex hull in projective space of 0 and all $\mathbf{a}_i^L := \mathbf{a}_i / \deg^L(\partial_i)$ for $i = 1, \ldots, n$. Its faces not containing zero form the (A, L)-umbrella Φ_A^L, a combinatorial object independent of β that encodes all information regarding the L-characteristic variety of $M_A(\beta)$.

FIGURE 1. Some (A, L)-umbrellas. (Shaded Δ_A^L with fat boundary Φ_A^L.)

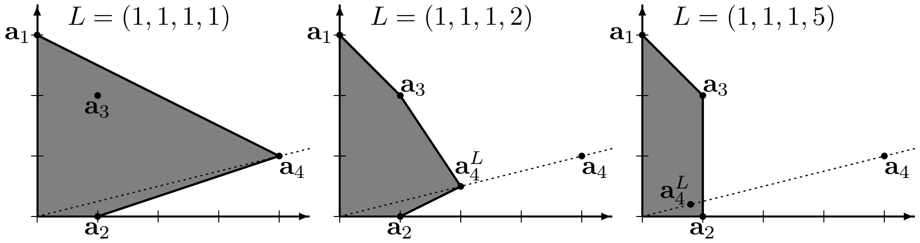

Figure 1 shows the (A, L)-umbrella in the example

$$A = \begin{pmatrix} 0 & 1 & 1 & 4 \\ 3 & 0 & 2 & 1 \end{pmatrix}$$

for the family of filtrations $L = (1, 1, 1, t)$ for three parameters resulting in combinatorially different umbrellas. In Theorem 2.14, we identify Φ_A^L with both the $\mathbb{Z}^d$-graded spectrum of the L-graded toric ring $S_A^L := \mathrm{gr}^L(S_A)$ and the collection of torus orbits in $\mathrm{Spec}(S_A^L)$. In Theorem 2.19, we characterize Cohen–Macaulayness of S_A^L by the corresponding property of its subrings generated by facet cones.

In Section 3, we extend L to the Weyl algebra $D = \mathbb{C}[x]\langle\partial\rangle$ in the variables $x := x_1, \dots, x_n$ in such a way that E is L-homogeneous and $W := \mathrm{gr}^L(D)$ is a polynomial ring. The family of filtrations in Figure 1 may be viewed as the restriction of $L = F + qV$ to R for the V-filtration along x_4 for $q = 0, 1, 4$. In Proposition 3.8 we show that all components of the L-characteristic variety of $M_A(\beta)$ correspond to faces $\tau \in \Phi_A^L$: each is the closure $\bar{C}_A^\tau$ of the conormal space C_A^τ to a torus orbit in $\mathrm{Spec}(S_A^L)$. The facet components correspond to orbits in the smooth, and hence Cohen–Macaulay, locus of S_A^L on which the L-symbols of the Euler vector fields form a regular sequence. From this we conclude in Proposition 3.10 that the facet components actually occur and their multiplicity is given by an index formula independent of the parameter β. This shows in particular that p/q is a slope of $M_A(\beta)$ at the origin whenever Φ_A^{pF+qV} jumps at p/q. We thus obtain a converse to Hotta's theorem in Corollary 3.16: regular A-hypergeometric systems are homogeneous.

Since orbits to nonfacets may be outside the Cohen–Macaulay locus of S_A^L, we consider in Section 4 the full Euler–Koszul complex $K_{A,\bullet}(S_A; \beta)$ from [MMW05]. In order to apply methods of homological algebra, we discuss the basics of good L-filtrations on R- and D-modules. Using the spectral sequence for the L-filtration on $K_{A,\bullet}(S_A; \beta)$, we identify its L-characteristic cycle with the intersection cycle between S_A^L and the L-graded Euler ideal $\mathrm{gr}^L(\langle E\rangle)$. In Theorem 4.11, we use Serre's Intersection Theorem [Ser65, Ch. V, §C.1, Thm. 1] to show that the L-characteristic variety of $K_{A,\bullet}(S_A; \beta)$ contains all candidate components. To show that this also holds for $M_A(\beta) = H_0(K_{A,\bullet}(S_A; \beta))$ we use in Theorem 4.16 results from [MMW05] and an induction argument on toric modules. In particular (see Corollary 4.17), the components of the L-characteristic variety of $M_A(\beta)$ are in one-to-one correspondence to the faces in the (A, L)-umbrella Φ_A^L. It follows (see

Corollary 4.18) that the slopes of $M_A(\beta)$ along coordinate subspaces at the origin correspond exactly to jump parameters p/q of Φ_A^{pF+qV}, confirming and extending a conjecture by B. Sturmfels.

The holonomic rank $\mathrm{rk}(M) := \dim_{\mathbb{C}(x)}(M \otimes_{\mathbb{C}[x]} \mathbb{C}(x))$ of a D-module M is the dimension of the $\mathbb{C}$-vector space of its solution space near a regular point. It is a classical result (see [GZK89, Ado94]) that for generic β the rank of $M_A(\beta)$ equals the volume of the convex hull Δ_A^F of 0 and $\mathbf{a}_1, \ldots, \mathbf{a}_n$ where the volume of the unit simplex is normalized to 1. The exceptional set $\mathcal{E}(A)$ of A is the set of rank-jumping parameters: $\beta \in \mathcal{E}(A)$ precisely if the rank of $M_A(\beta)$ exceeds the volume. By [MMW05], $\mathcal{E}(A)$ is a finite subspace arrangement. In Theorem 4.21 we give a general index/volume formula for the multiplicity $\mu_A^{L,\tau}$ of $\bar{C}_A^\tau$, $\tau \in \Phi_A^L$, in the L-characteristic cycle of $K_{A,\bullet}(S_A;\beta)$. We show then that for non-rank-jumping parameter β, or if τ is a facet, the number $\mu_A^{L,\tau}$ equals the multiplicity $\mu_{A,0}^{L,\tau}(\beta)$ of $\bar{C}_A^\tau$ in the L-characteristic cycle of $M_A(\beta)$. From [MMW05] it is known that the rank $\mu_{A,0}^{F,\emptyset}(\beta)$ of $M_A(\beta)$ is upper semicontinuous in the parameter β. Theorem 4.28 generalizes a weaker statement to all L and $\tau \in \Phi_A^L$: the multiplicity $\mu_{A,0}^{L,\tau}(\beta)$, $\tau \in \Phi_A^L$, is always minimal at generic β. We conjecture that $\mu_{A,0}^{L,\tau}(\beta)$ is upper semicontinuous in β.

Remark. In [SW06] we introduce a natural extension of $M_A(\beta)$ to a $\mathcal{D}$-module $\mathcal{M}_A(\beta)$ on the product $\prod_{j=1}^n \mathbb{P}^1_{\mathbb{C}}$ by noting that the generators E and I_A of the hypergeometric ideal extend to global differential operators. On each standard affine patch one obtains a D-module that may be viewed as a twisted A-hypergeometric system.

Based on results in this article we describe in [SW06] the L-characteristic variety of $\mathcal{M}_A(\beta)$ in terms of a combinatorial gadget generalizing the (A, L)-umbrella, and we determine the multiplicities in the L-characteristic Euler–Koszul cycle for generic β. In order to discuss the slopes of $\mathcal{M}_A(\beta)$ (i.e., the slopes of $M_A(\beta)$ at infinity) one then needs to understand for each point of $\prod_{j=1}^n \mathbb{P}^1_{\mathbb{C}}$ precisely which components of the L-characteristic variety pass through its cotangent space; we state a conjecture to this account in [SW06, Conj. 5.18]. The core of this issue seems to be related to the question whether one can relax the basic condition on A in this article: that it be pointed. Without pointedness, thus placing the origin in T_0^*X outside the corresponding toric variety, three issues come up: a) the definition of Δ_A^L must be changed; b) Lemma 3.14 fails; c) the arguments in Section 4 need adjustment. While we have some ideas for a) and our conjecture stated in [SW06] addresses b), c) is perhaps more involved, but should have interesting answers.

2. Filtrations on the toric ring

2.1. **Torus action and toric ring.**

Notation 2.1. By $\mathbb{Q}_+$ we mean the nonnegative rational numbers and we include 0 in $\mathbb{N}$.

Let $A = (a_{i,j}) \in \mathbb{Z}^{d\times n}$ be an integer matrix of rank d whose columns $\mathbf{a}_1, \ldots, \mathbf{a}_n \in \mathbb{Z}^d$ are nonzero. We assume that $\mathbb{N}A$ is a *positive* (see [BH93, §6.1]) (or *pointed*, see [MMW05, §1]) semigroup with $\mathbb{Z}A = \mathbb{Z}^d$. We write $\tau \subseteq A$ if τ is a subset of the column set of A and consider it both as a submatrix of A and a subset of the set of column indices $\{1, \ldots, n\}$. Then the *dimension* $\dim(\tau)$ is $\dim_{\mathbb{Q}}(\mathbb{Q}\tau) - 1$. For a vector or collection with index set $\{1,\ldots,n\}$, a lower index τ denotes the subvector or subcollection with indices in τ. We abbreviate $\overline{\tau} := \{1,\ldots,n\}\smallsetminus\tau$. For any set τ, its cardinality is denoted $|\tau|$.

We shall frequently denote by $\bar{C}$ the Zariski closure of a set C.

For $\mathbf{u}\in\mathbb{Z}^n$ define $\mathbf{u}_+$ by $(\mathbf{u}_+)_j = \max(0,\mathbf{u}_j)$ and put $\mathbf{u}_- = \mathbf{u}_+ - \mathbf{u}$. For $\mathbf{u},\mathbf{v}\in\mathbb{N}^n$ write $\min(\mathbf{u},\mathbf{v})$ for the vector whose j-th entry is $\min(\mathbf{u}_j,\mathbf{v}_j)$. For any vector $\mathbf{u}$ we mean by $\mathbf{u}>0$ that $\mathbf{u}$ is componentwise positive: $u_i>0$ for all i.

The base space in this note is $X := \operatorname{Spec}(\mathbb{C}[x]) = \mathbb{C}^n$ where $x := x_1,\ldots,x_n$. Let $R := \mathbb{C}[\partial]$ be the polynomial ring in n variables $\partial := \partial_1,\ldots,\partial_n$. Identifying ∂_i with the partial derivation $\partial/\partial x_i$, $\operatorname{Spec}(R)$ becomes the conormal space T_0^*X of X at 0. The d-torus $\mathbb{T} := (\mathbb{C}^*)^d = \operatorname{Spec}(\mathbb{C}[t_1^{\pm1},\ldots,t_d^{\pm1}])$ with coordinates $t := t_1,\ldots,t_d$ acts on $\operatorname{Spec}(R)$

$$(t,\xi)\longmapsto t\cdot\xi := (t^{\mathbf{a}_1}\xi_1,\ldots,t^{\mathbf{a}_n}\xi_n)\ . \tag{2.1.1}$$

This induces a $\mathbb{Z}^d$-grading on R by $\deg(\partial_i)=\mathbf{a}_i$. For a $\mathbb{Z}^d$-graded R-module N, we denote by $\deg(N)$ the set of its $\mathbb{Z}^d$-degrees.

Definition 2.2. We denote the orbit $\mathbb{T}\cdot\xi$ through $\xi\in T_0^*X$ by $\operatorname{Orb}(\xi)$. Let τ be a subset of columns of A. We define $\mathbf{1}_A^\tau\in\{0,1\}^n$ by

$$(\mathbf{1}_A^\tau)_j := \begin{cases}1 & \text{if} \quad \mathbf{a}_j\in\tau,\\ 0 & \text{if} \quad \mathbf{a}_j\notin\tau,\end{cases}$$

and denote by O_A^τ the orbit of $\mathbf{1}_A^\tau$.

The *toric ideal* $I_\tau\subseteq R_\tau := \mathbb{C}[\partial_\tau]$ *of* τ is the $\mathbb{Z}^d$-graded prime ideal generated by all $\Box_\mathbf{u} = \partial^{\mathbf{u}_+}-\partial^{\mathbf{u}_-}$ where $\mathbf{u}\in\mathbb{Z}^{|\tau|}$ such that $\tau\cdot\mathbf{u}=0$. We set $I_A^\tau := RI_\tau + J_\tau$ where J_τ is the R-ideal generated by $\{\partial_i \mid i\notin\tau\}$. Based on the following lemma, the *semigroup ring of* τ is

$$S_\tau := \mathbb{C}[\mathbb{N}\tau] = \bigoplus_{\mathbf{a}\in\mathbb{N}\tau}\mathbb{C}\cdot t^{\mathbf{a}} = \sum_{\mathbf{u}\in\mathbb{N}^{|\tau|}}\mathbb{C}\cdot(\partial_\tau^{\mathbf{u}} \mod I_\tau)\subseteq S_A,$$

The normalization of S_A is the Cohen–Macaulay ring $\tilde{S}_A = \bigoplus_{\mathbf{a}\in(\mathbb{Q}_+A\cap\mathbb{Z}^d)}\mathbb{C}\cdot t^{\mathbf{a}}$.

Lemma 2.3. *For $\tau\subseteq A$, $I_A^\tau = \{f\in R\mid f(O_A^\tau)=0\}$ and $S_\tau = R_\tau/I_\tau = R/I_A^\tau$.* $\square$

2.2. ***L*-filtration on the toric ring.** Let $L=(L_{\partial_1},\ldots,L_{\partial_n})\in\mathbb{Q}^n$ be a *weight vector*. It induces an increasing filtration L of $\mathbb{C}$-vector spaces on R via $[\partial^\mathbf{u}\in L_kR]\Leftrightarrow[L\cdot\mathbf{u}\le k]$. Note that L has a rational discrete index set. If $f\in L_kR\smallsetminus L_{<k}R$ then $k =: \deg^L(f)$ is the L-*degree* of f. Let $\sigma^L: R\to\operatorname{gr}^L(R)$ be the L-*symbol map* defined by $\sigma^L(f) = f \mod L_{<k}R$ if $\deg^L(f)=k$. An element $f\in R$ is L-*homogeneous* if $f=\sum_{L\cdot\mathbf{u}=k}f_\mathbf{u}\partial^\mathbf{u}$ for some $k\in\mathbb{Q}$ where $f_\mathbf{u}\in\mathbb{C}$. By abuse of notation, we identify R and $\operatorname{gr}^L(R)$ via the $\mathbb{C}$-linear isomorphism induced by $f\mapsto\sigma^L(f)$ for L-homogeneous f in right-normal form.

For any $\tau\subseteq A$, L induces a filtration on S_τ by

$$L_kS_\tau := \sum_{\deg^L(\partial_\tau^\mathbf{u})\le k}\mathbb{C}\cdot(\partial_\tau^\mathbf{u} \mod I_\tau).$$

With I_τ^L denoting the $\mathbb{Z}^d$-graded ideal $\operatorname{gr}^L(I_\tau)$, we abbreviate

$$S_\tau^L := \operatorname{gr}^L(S_\tau)\cong R/I_\tau^L.$$

The following is a mild extension of [Stu96, Cor. 4.4] (see also [Ado94, Lem. 4.11]).

Lemma 2.4. *One has the identity* $I_A^L = R\langle \sigma^L(\square_{\mathbf{u}}) \mid \mathbf{u} \in \mathbb{Z}^n, A \cdot \mathbf{u} = 0\rangle$.

Proof. Form a matrix B by adding the row L on top, and then a column $(1, 0, \ldots, 0)$ on the left of A. Then I_B is the L-homogenization of I_A relative to the new variable ∂_0. By [Stu96, Cor. 4.4], for any term order that eliminates ∂_0, I_B has a Gröbner basis that consists of L-homogenizations of binomials $\partial^{\mathbf{u}_+} - \partial^{\mathbf{u}_-}$ with $A \cdot \mathbf{u} = 0$. Dehomogenization leads to an L-Gröbner basis for I_A and the claim follows. $\square$

In this section we study the geometry of S_A^L in terms of A and L. Since the Gröbner fan of any R-ideal is defined over $\mathbb{Q}$, the study of real weights can be reduced to the case $L \in \mathbb{Q}^n$ (see [MS05, §7.4] for a discussion in the case where $L > 0$ defines a term order).

2.3. **The** (A, L)**-umbrella.** We consider the embedding of the affine space $\mathbb{Q}^d \supseteq A$ into the rational projective d-space

$$\mathbb{P}^d_{\mathbb{Q}} = \mathbb{P}_{\mathbb{Q}}(\mathbb{Q}^d \times \mathbb{Q})$$

via the map $q \mapsto (q : 1)$. Denote $\infty := \mathbb{P}^d_{\mathbb{Q}} \smallsetminus \mathbb{Q}^d$ the hyperplane at infinity.

In $\mathbb{P}^d_{\mathbb{Q}}$, any two distinct points $\mathbf{a}, \mathbf{b} \in \mathbb{P}^d_{\mathbb{Q}}$ are joined by two line segments. If H is a hyperplane in $\mathbb{P}^d_{\mathbb{Q}}$ containing neither $\mathbf{a}$ nor $\mathbf{b}$ then there is a unique line segment joining the points and not meeting H. This is exactly the convex hull of $\mathbf{a}$ and $\mathbf{b}$ in the affine space $\mathbb{P}^d_{\mathbb{Q}} \smallsetminus H$.

Definition 2.5. Let $H \subseteq \mathbb{P}^d_{\mathbb{Q}}$ be a hyperplane and let $U_H := \mathbb{P}^d_{\mathbb{Q}} \smallsetminus H$ be its complement. For $B \subseteq U_H$, the *convex hull* of B relative to H is the set $\mathrm{conv}_H(B)$ defined as the convex hull of B in the affine space U_H.

Note that, for varying H, $\mathrm{conv}_H(B)$ changes exactly when H is moved through a point of B. Within $\mathbb{Q}^d$, elements of convex hulls are linear combinations with nonnegative coefficients that add to unity. Convex hulls relative to $H \neq \infty$, with coordinates from $\mathbb{Q}^d \subseteq \mathbb{P}^d_{\mathbb{Q}}$, obey slightly different rules. Let $h \in \mathrm{Hom}_{\mathbb{Q}}(\mathbb{Q}^d, \mathbb{Q})$ be a linear form such that H is the closure of $h^{-1}(0)$ in $\mathbb{P}^d_{\mathbb{Q}}$. The line through 0 and $\mathbf{a} \in \mathbb{Q}^d \smallsetminus \{0\}$ meets ∞ in a point that we denote $\mathbf{a}/0$. For $q \in \mathbb{Q}$ let $\mathrm{sign}(q)$ be the usual signum function:

$$\mathrm{sign}(q) = \begin{cases} -1 & \text{if } q < 0; \\ 0 & \text{if } q = 0; \\ 1 & \text{if } q > 0. \end{cases}$$

Lemma 2.6. *Let* $B = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\} \subseteq U_H$ *with* $B \cap \infty = \{\mathbf{b}_{m+1}, \ldots, \mathbf{b}_n\}$ *and pick* $\{\mathbf{b}'_{m+1}, \ldots, \mathbf{b}'_n\} \subseteq \mathbb{Q}^d \cap U_H$ *such that* $\mathbf{b}_j = \mathbf{b}'_j/0$ *for* $m < j \leq n$. *If* $\mathbf{b} \in \mathrm{conv}_H(B) \cap \mathbb{Q}^d$ *then in coordinates of* $\mathbb{Q}^d$ *there is an equation* $\mathbf{b} = \sum_{j=1}^m \varepsilon_j \mathbf{b}_j + \sum_{j=m+1}^n \varepsilon'_j \mathbf{b}'_j$ *where* $\sum_{j=1}^m \varepsilon_j = 1$, $\mathrm{sign}(\varepsilon_j h(\mathbf{b}_j)) = \mathrm{sign}(h(\mathbf{b}))$ *for all* j *with* $\varepsilon_j \neq 0$, *and* $\mathrm{sign}(\varepsilon'_j h(\mathbf{b}'_j)) = \mathrm{sign}(h(\mathbf{b}))$ *for all* j *with* $\varepsilon'_j \neq 0$.

Proof. Let B_+ (resp. B_-) be the subsets of $B \smallsetminus \infty$ on which h evaluates positively (resp. negatively), and put $B_\infty = B \cap \infty$, $B' = \{\mathbf{b}'_{m+1}, \ldots, \mathbf{b}'_n\}$. A general element $\mathbf{b} \in \mathrm{conv}_H(B)$ is the convex combination of three points: $\mathbf{b}_+ \in \mathrm{conv}_H(B_+)$, $\mathbf{b}_- \in \mathrm{conv}_H(B_-)$, and $\mathbf{b}_\infty \in \mathrm{conv}_H(B_\infty)$.

Clearly, $\mathbf{b}_+ = \sum_{\mathbf{b}_j \in B_+} \varepsilon_j \mathbf{b}_j$ where $\varepsilon_j \geq 0$ and $\sum_{\mathbf{b}_j \in B_+} \varepsilon_j = 1$. A similar statement holds for $\mathbf{b}_-$. Now $\mathrm{conv}_H(\mathbf{b}_+, \mathbf{b}_-) \cap \mathbb{Q}^d$ is the union of rays $\{\lambda_+ \mathbf{b}_+ + \lambda_- \mathbf{b}_- \mid$

$\lambda_+ + \lambda_- = 1,\ \lambda_+\lambda_- \le 0\}$. Thus, if $\mathbf{b}_0 \in \mathrm{conv}_H(\mathbf{b}_+, \mathbf{b}_-)$ then $\mathrm{sign}(\lambda_+ h(\mathbf{b}_+)) = \mathrm{sign}(\mathbf{b}_0) = \mathrm{sign}(\lambda_- h(\mathbf{b}_-))$.

It suffices to show the lemma if $h(B') > 0$. Then $\mathbf{b}_\infty$ is of the form $\mathbf{b}'/0$ with $\mathbf{b}' \in \mathrm{conv}_H(B')$, and points of $\mathrm{conv}_H(B')$ are of the form $\sum_{\mathbf{b}'_j \in B'} \varepsilon_j \mathbf{b}'_j$ with $\sum_{\mathbf{b}'_j \in B'} \varepsilon_j = 1$ and all $\varepsilon_j \ge 0$. We are thus reduced to considering $\mathrm{conv}_H(\{\mathbf{b}_0, \mathbf{b}_\infty\})$ with $\mathbf{b}_\infty = \mathbf{b}'/0$ and $h(\mathbf{b}') > 0$. If $h(\mathbf{b}_0) > 0$ then $\mathrm{conv}_H(\{\mathbf{b}_0, \mathbf{b}_\infty\})$ is the ray $\{\mathbf{b}_0 + \lambda'\mathbf{b}' \mid \lambda' \ge 0\}$, while if $h(\mathbf{b}_0) < 0$ then it is the ray $\{\mathbf{b}_0 + \lambda'\mathbf{b}' \mid \lambda \le 0\}$. The condition on λ' can, if $\lambda' \neq 0$, be packaged as $\mathrm{sign}(\lambda' h(\mathbf{b}')) = \mathrm{sign}(h(\mathbf{b}_0))$. The lemma follows. ☐

We view the columns $\mathbf{a}_1, \ldots, \mathbf{a}_n \in \mathbb{Z}^d$ of A as points in $\mathbb{Q}^d = \mathbb{P}^d_{\mathbb{Q}} \setminus \infty$. By assumption, $\mathbb{N}A$ is positive and hence $h \in \mathrm{Hom}_{\mathbb{Q}}(\mathbb{Q}^d, \mathbb{Q})$ can be chosen such that $h(\mathbf{a}_j) > 0$ for all j. For any $\lambda \in \mathbb{Q}$, set $H_\lambda := h^{-1}(-\lambda)$ and $U_\lambda := \mathbb{P}^d_{\mathbb{Q}} \smallsetminus H_\lambda$.

Definition 2.7. Choose $h \in \mathrm{Hom}_{\mathbb{Q}}(\mathbb{Q}^d, \mathbb{Q})$ such that $h(\mathbf{a}_j) > 0$ for all j, and let ε be such that $0 < \varepsilon < |h(\mathbf{a}_j)/L_{\partial_j}|$ whenever $L_{\partial_j} \neq 0$. We set $\mathbf{a}^L_j := \mathbf{a}_j / L_{\partial_j}$ and call $\Delta^L_A := \mathrm{conv}_{H_\varepsilon}(\{0, \mathbf{a}^L_1, \ldots, \mathbf{a}^L_n\}) \subseteq \mathbb{P}^d_{\mathbb{Q}}$ the (A, L)-*polyhedron*.

Let the (A, L)-*umbrella* be the set Φ^L_A of faces of Δ^L_A which do not contain 0. In particular, Φ^L_A contains the empty face. Whenever it suits us, we identify $\tau \in \Phi^L_A$ with $\{j \mid \mathbf{a}^L_j \in \tau\}$, or with $\{\mathbf{a}_j \mid \mathbf{a}^L_j \in \tau\}$, or with the corresponding submatrix of A. By $\Phi^{L,k}_A \subseteq \Phi^L_A$, we denote the subset of faces of dimension k.

By $\Gamma^L_A := \bigcup \Phi^L_A$ (resp. $\Gamma^{L,k}_A := \bigcup \Phi^{L,k}_A$), we denote the underlying point set of Φ^L_A (resp. $\Phi^{L,k}_A$). Note that Γ^L_A is a piecewise linear manifold with boundary, homeomorphic to the $(d-1)$-disk. If $\mathbf{a} \in \Delta^L_A \smallsetminus \{0\}$ then the line through 0 and $\mathbf{a}$ meets Γ^L_A in $\Gamma^L_A(\mathbf{a})$.

The matrix A is called L-*homogeneous* if all $\mathbf{a}^L_j$ lie on a common hyperplane of $\mathbb{P}^d_{\mathbb{Q}}$. Every A is $\mathbf{0}$-homogeneous and we call $\Phi^{\mathbf{0}}_A$ the A-*umbrella*. If A is L-homogeneous then $I^L_A = I_A$ under the identification of $\mathrm{gr}^L(R)$ with R. Note that $\Phi^{\mathbf{0}}_A$ can be identified with the lattice of nonempty faces of the polyhedral cone $\mathbb{Q}_+ A$ via the face lattice of a cross-section $\mathbb{Q}_+ A \cap h^{-1}(1)$.

FIGURE 2. More (A, L)-umbrellas. (Shaded Δ^L_A with fat boundary Φ^L_A.)

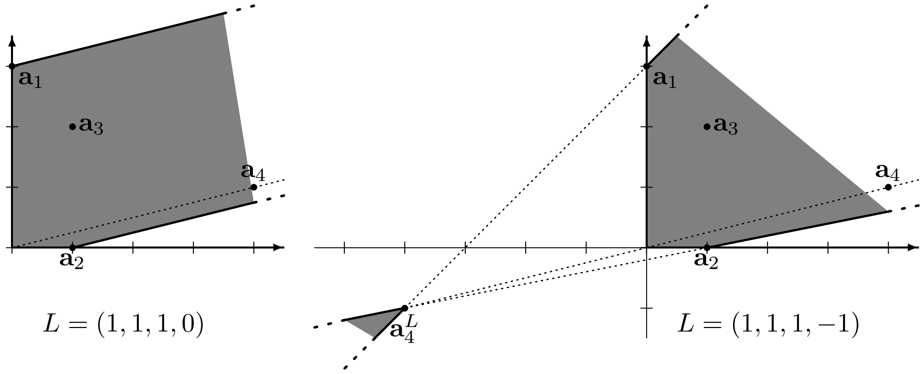

Figure 2 shows the intersection with $\mathbb{Q}^d$ of the (A, L)-umbrella for $L = (1, 1, 1, t)$ and $t \le 0$ in the example from the introduction, $A = \begin{pmatrix} 0 & 1 & 1 & 4 \\ 3 & 0 & 2 & 1 \end{pmatrix}$.

2.4. **Monomials in the graded toric ideal.** The following result generalizes Lemma 3.1 and 3.2 in [Ado94].

Lemma 2.8. *If $\mathbf{a}^L_{k_1}, \dots, \mathbf{a}^L_{k_m} \in \Gamma^L_A$ do not lie in a common $\tau \in \Phi^{L,d-1}_A$ then $\partial_{k_1} \cdots \partial_{k_m} \in \sqrt{I^L_A}$.*

Proof. We abbreviate $\mathbf{v} := L_\partial$.

Assume first that $v_{k_1} \neq 0$ and $\Gamma^L_A(\mathbf{a}_{k_1}) = \mathbf{a}_{k_1}/0 \in \infty$. In particular, $v_{k_1} > 0$ by Definition 2.7. The polyhedron $\Delta^L_A \cap \infty$ is the convex hull of the points $\mathbf{a}^L_i = \mathbf{a}_i/0$ with $v_i = 0$, and of the intersection points $(\mathbf{a}^L_i - \mathbf{a}^L_j)/0$ with ∞ of line segments from $\mathbf{a}^L_i$ to $\mathbf{a}^L_j$ with $v_i > 0 > v_j$. Therefore

$$\mathbf{a}^L_{k_1} = \sum_{v_i=0} \eta_i \mathbf{a}_i + \sum_{v_i>0>v_j} \eta_{i,j} \left(\mathbf{a}^L_i - \mathbf{a}^L_j\right)$$

for some $0 \le \eta_i, \eta_{i,j} \in \mathbb{Q}$. This equality gives rise to an element

$$\square = \partial_{k_1}^{s/v_k} - \prod_{v_i=0} \partial_i^{s\eta_i} \prod_{v_i>0>v_j} \partial_i^{s\eta_{i,j}/v_i} \partial_j^{-s\eta_{i,j}/v_j} \in I_A$$

where $s \in \mathbb{N}$ is chosen to clear all denominators. The L-degree of the left monomial is positive while that of the right one is zero. Thus $\sigma^L(\square) = \partial_{k_1}^{s/v_k} \in \mathrm{gr}^L(I_A)$ and the claim follows in this case.

We now keep the assumption $v_{k_1} \neq 0$ but assume that $\Gamma^L_A(\mathbf{a}_{k_1}) \notin \infty$. By hypothesis there is $\mathbf{a} \in \mathrm{conv}(\mathbf{a}^L_{k_1}, \dots, \mathbf{a}^L_{k_m})$ in the interior of Δ^L_A. By continuity of the function $\mathbf{a} \mapsto \Gamma^L_A(\mathbf{a})$, choosing $\mathbf{a}$ sufficiently close to $\mathbf{a}^L_{k_1}$ implies that $\Gamma^L_A(\mathbf{a}) \notin \infty$. As $\mathbf{a}$ lies in the interior of Δ^L_A and since $\Gamma^L_A(\mathbf{a}) \notin \infty$, there is $0 \neq t \in \mathbb{Q}$ with $\Gamma^L_A(\mathbf{a}) = t\mathbf{a} \in \Gamma^L_A$ and hence either $\mathrm{sign}(th(\mathbf{a})) = 1$ and $t > 1$, or $\mathrm{sign}(th(\mathbf{a})) = -1$ and $0 < t < 1$. In either case, $t\,\mathrm{sign}(th(\mathbf{a})) > \mathrm{sign}(th(\mathbf{a}))$. Using Lemma 2.6, we can write

$$\mathbf{a} = \sum_{v_{k_j}\neq 0} \varepsilon_{k_j} \mathbf{a}^L_{k_j} + \sum_{v_{k_j}=0} \varepsilon_{k_j} \mathbf{a}_{k_j} \tag{2.4.1}$$

for some $\varepsilon_{k_j} \in \mathbb{Q}$ with $\sum_{v_{k_j}\neq 0} \varepsilon_{k_j} = 1$, $\mathrm{sign}(\varepsilon_{k_j}/v_{k_j}) = \mathrm{sign}(h(\mathbf{a}))$ if $\varepsilon_{k_j} v_{k_j} \neq 0$, and $\mathrm{sign}(\varepsilon_{k_j}) = \mathrm{sign}(h(\mathbf{a}))$ if $\varepsilon_{k_j} \neq 0 = v_{k_j}$. Again by Lemma 2.6 and Definition 2.7, we can write

$$t\mathbf{a} = \sum_{v_i\neq 0} \eta_i \mathbf{a}^L_i + \sum_{v_i=0} \eta_i \mathbf{a}_i \tag{2.4.2}$$

for some $\eta_i \in \mathbb{Q}$ where $\sum_{v_i\neq 0} \eta_i = 1$, $\mathrm{sign}(\eta_i/v_i) = \mathrm{sign}(th(\mathbf{a}))$ if $\eta_i v_i \neq 0$, and $\mathrm{sign}(\eta_i) = \mathrm{sign}(th(\mathbf{a}))$ if $\eta_i \neq 0 = v_i$. Combining equations (2.4.1) and (2.4.2) we find an element

$$\square = \prod_{v_{k_j}\neq 0} \partial_{k_j}^{s|t\varepsilon_{k_j}/v_{k_j}|} \prod_{v_{k_j}=0} \partial_{k_j}^{s|t\varepsilon_{k_j}|} - \prod_{v_i\neq 0} \partial_i^{s|\eta_i/v_i|} \prod_{v_i=0} \partial_i^{s|\eta_i|} \in I_A \tag{2.4.3}$$

where $s \in \mathbb{N}$ chosen to clear all denominators. From

$$\begin{aligned}
\text{(2.4.4)} \qquad \deg^L\Big(\prod_{v_{k_j}\neq 0} \partial_{k_j}^{s|t\varepsilon_{k_j}/v_{k_j}|} \prod_{v_{k_j}=0} \partial_{k_j}^{s|t\varepsilon_{k_j}|}\Big) &= s|t| \sum_{v_{k_j}\neq 0} \varepsilon_{k_j} \operatorname{sign}(\varepsilon_{k_j}/v_{k_j}) \\
&= st \sum_{v_{k_j}\neq 0} \varepsilon_{k_j} \operatorname{sign}(th(\mathbf{a})) = st \operatorname{sign}(th(\mathbf{a})) > s \operatorname{sign}(th(\mathbf{a})) \\
= s \sum_{v_i \neq 0} \operatorname{sign}(\eta_i/v_i)\eta_i = s \sum_{v_i\neq 0} |\eta_i/v_i| v_i &= \deg^L\Big(\prod_{v_i\neq 0} \partial_i^{s|\eta_i/v_i|} \prod_{v_i=0} \partial_i^{s|\eta_i|}\Big),
\end{aligned}$$

we conclude that the L-leading term of $\Box$ is the left of the two monomials in (2.4.3). The claim follows in the case where at least one v_{k_j} is nonzero.

Suppose finally that $v_{k_j} = 0$ and hence $\mathbf{a}^L_{k_j} \in \infty$ for all $j = 1, \dots, m$. By assumption and Definition 2.7 we can pick an element $\mathbf{a}' \in \operatorname{conv}(\mathbf{a}^L_{k_1}, \dots, \mathbf{a}^L_{k_m}) \smallsetminus \partial\Delta^L_A \subseteq \operatorname{conv}(\mathbf{a}^L_{k_1}, \dots, \mathbf{a}^L_{k_m}) \smallsetminus \Gamma^L_A$. Then $\mathbf{a}' = \mathbf{a}/0$ with $\mathbf{a} \in \operatorname{conv}(\mathbf{a}_{k_1}, \dots, \mathbf{a}_{k_m})$. It follows that there is an equation of type (2.4.1) with conditions as indicated there. By construction, $\Gamma^L_A(\mathbf{a}') \neq \mathbf{a}' \in \infty$ and so $\Gamma^L_A(\mathbf{a}') \notin \infty$. Hence, there is a positive $t \in \mathbb{Q}$ with $\Gamma^L_A(\mathbf{a}') = t\mathbf{a}$ and so there is an equation of type (2.4.2) with conditions as indicated there. As $t \operatorname{sign}(th(\mathbf{a})) = -t > 0 > -1 = \operatorname{sign}(th(\mathbf{a}))$ we get an equation of type (2.4.4), and the claim follows as in the previous case. $\Box$

2.5. **Homogeneity in the graded toric ideal.**

Definition 2.9. Let τ be a set of columns of A. For $\mathbf{u} \in \mathbb{Z}^n$, we write $\operatorname{supp}(\mathbf{u}) \subseteq \tau$ if $\mathbf{u}_i \neq 0$ implies $\mathbf{a}_i \in \tau$. For $f \in R$, we write $\operatorname{supp}(f) \subseteq \tau$ if $\operatorname{supp}(\mathbf{u}) \subseteq \tau$ for all monomials $\partial^{\mathbf{u}}$ of f.

Lemma 2.10. *Let $\tau \in \Phi^L_A$ and pick $\mathbf{u} \in \mathbb{Z}^n$ such that $A \cdot \mathbf{u} = 0$.*

(1) If $\operatorname{supp}(\Box_{\mathbf{u}}) \subseteq \tau$ then $\sigma^L(\Box_{\mathbf{u}}) = \Box_{\mathbf{u}}$. In particular, the toric ideal I_τ is L-homogeneous.

(2) If $\operatorname{supp}(\partial^{\mathbf{u}_\pm}) \subseteq \tau$ and $\operatorname{supp}(\partial^{\mathbf{u}_\mp}) \not\subseteq \tau$ then $\sigma^L(\Box_{\mathbf{u}}) = \mp\partial^{\mathbf{u}_\mp}$.

Proof. We abbreviate $\mathbf{v} := L_\partial$.

Consider first the case where the facet $\tau \in \Phi^{L,d-1}_A$ lies entirely in ∞. Then $\mathbf{a}^L_i \in \tau$ implies $v_i = 0$ and hence $\operatorname{supp}(\Box_{\mathbf{u}}) \subseteq \tau$ implies that $\Box_{\mathbf{u}}$ is L-homogeneous of degree zero. Suppose $\operatorname{supp}(\mathbf{u}_+) \subseteq \tau$ but $\operatorname{supp}(\mathbf{u}_-) \not\subseteq \tau$. As $\tau \subseteq \infty$ is a facet of Δ^L_A and by Definition 2.7, the interior of Δ^L_A meets neither ∞ nor H_ε. Hence Δ^L_A is completely contained in the $\mathbb{P}^d_{\mathbb{Q}}$-closure of one of the regions $\{t \in \mathbb{Q}^d \mid h(t) \geq -\varepsilon\}$ and $\{t \in \mathbb{Q}^d \mid h(t) \leq -\varepsilon\}$. Since $0 \in \Delta^L_A$, it must be the former. Moreover, by definition of ε, $h(\mathbf{a}^L_j) \notin [-\varepsilon, 0]$ for all $v_j \neq 0$, whence $v_j \geq 0$ in all cases. Thus $\deg^L(\partial^{\mathbf{u}_+}) = 0$ and $\deg^L(\partial^{\mathbf{u}_-}) > 0$, hence $\sigma^L(\Box_{\mathbf{u}}) = -\partial^{\mathbf{u}_-}$. The case $\operatorname{supp}(\mathbf{u}_-) \subseteq \tau \not\supseteq \operatorname{supp}(\mathbf{u}_+)$ is similar.

Now let $\tau \in \Phi^L_A$ be not contained in ∞, or let $\tau \in \Phi^L_A \smallsetminus \Phi^{L,d-1}_A$ be a nonfacet face contained in ∞. Then there is a linear form $h_\tau \in \operatorname{Hom}_{\mathbb{Q}}(\mathbb{Q}^d, \mathbb{Q})$ such that the closure of $h_\tau^{-1}(1)$ in $\mathbb{P}^d_{\mathbb{Q}}$ meets Δ^L_A precisely in τ, and for which $\operatorname{sign}(v_i)h_\tau(\mathbf{a}^L_i) \leq \operatorname{sign}(v_i)$ whenever $v_i \neq 0$.

For all $\mathbf{a}^L_i \in \tau$ with $v_i = 0$ the line through 0 and $\mathbf{a}_i$ meets τ in $\mathbf{a}^L_i \in \infty$. This means that $\mathbf{a}_i$ is parallel to $h_\tau^{-1}(1)$ and so $h_\tau(\mathbf{a}_i) = 0$. Thus, $A \cdot \mathbf{u} = 0$ implies that

$$0 = h_\tau(A \cdot \mathbf{u}) = h_\tau\left(\sum_{i=1}^n u_i \mathbf{a}_i\right) = \sum_{u_i > 0 \neq v_i} u_i v_i h_\tau(\mathbf{a}^L_i) - \sum_{u_i < 0 \neq v_i} (-u_i) v_i h_\tau(\mathbf{a}^L_i).$$

If $\operatorname{supp}(\Box_{\mathbf{u}}) \subseteq \tau$ and $u_i v_i \neq 0$ then $h_\tau(\mathbf{a}_i^L) = 1$ and hence $\deg^L(\partial^{\mathbf{u}_+}) = \deg^L(\partial^{\mathbf{u}_-})$.

Now suppose $\operatorname{supp}(\partial^{\mathbf{u}_+}) \subseteq \tau \not\supseteq \operatorname{supp}(\partial^{\mathbf{u}_-})$. Hence $h_\tau(\mathbf{a}_i^L) = 1$ if $u_i > 0 \neq v_i$, $\operatorname{sign}(v_i) h_\tau(\mathbf{a}_i^L) \leq \operatorname{sign}(v_i)$ if $u_i < 0 \neq v_i$, and $\operatorname{sign}(v_i) h_\tau(\mathbf{a}_i^L) < \operatorname{sign}(v_i)$ for at least one $u_i < 0$. Thus,

$$\begin{aligned}\deg^L(\partial^{\mathbf{u}_+}) = \sum_{u_i>0} v_i u_i &= \sum_{u_i>0\neq v_i} v_i u_i h_\tau(\mathbf{a}_i^L) \\ &= \sum_{u_i<0\neq v_i} v_i(-u_i) h_\tau(\mathbf{a}_i^L) < \sum_{u_i<0\neq v_i} v_i(-u_i) = \deg^L(\partial^{\mathbf{u}_-}).\end{aligned}$$

The case $\operatorname{supp}(\mathbf{u}_-) \subseteq \tau \not\supseteq \operatorname{supp}(\mathbf{u}_+)$ is similar. ☐

2.6. **Minimal associated prime ideals.** We now identify the components of the L-graded toric ring S_A^L.

Definition 2.11. For $\mathbf{u} \in \mathbb{Z}^n$ let

$$\tau_{\mathbf{u}}^L = \bigcap_{\operatorname{supp}(\mathbf{u}) \subseteq \tau \in \Phi_A^L} \tau$$

be the smallest element of Φ_A^L containing $\operatorname{supp}(\mathbf{u})$ and put $\tau_{\mathbf{u}}^L = A$ if there is none. Note that $[\partial^{\mathbf{u}} \in J_\tau] \Rightarrow [\tau_{\mathbf{u}}^L \not\subseteq \tau]$ for $\tau \subseteq A$ and $\mathbf{u} \in \mathbb{N}^n$.

Recall that I_A, I_A^τ, and J_τ are $\mathbb{Z}^d$-graded prime ideals.

Lemma 2.12. *Let $\tilde{I}_A^L \subseteq R$ be generated by all elements of the following types:*

(1) $\partial_{k_1} \cdots \partial_{k_m}$ where $\mathbf{a}_{k_1}^L, \ldots, \mathbf{a}_{k_m}^L$ do not lie in a common facet $\tau \in \Phi_A^{L,d-1}$;
(2) $\Box_{\mathbf{u}}$ where $\mathbf{u} \in \mathbb{Z}^n$, $A \cdot \mathbf{u} = 0$, and $\tau_{\mathbf{u}}^L \neq A$.

Then $\tilde{I}_A^L = \bigcap_{\tau \in \Phi_A^{L,d-1}} I_A^\tau$.

Proof. Let $I = \bigcap_{\tau \in \Phi_A^{L,d-1}} I_A^\tau$. We show first that $\tilde{I}_A^L \subseteq I$. Let I_0 denote the ideal of R generated by the elements from (1). Then clearly $I_0 \subseteq J_\tau \subseteq I_A^\tau$ for any $\tau \in \Phi_A^{L,d-1}$. Pick $\Box_{\mathbf{u}}$ as in (2). By part (1) of Lemma 2.10, $\sigma^L(\Box_{\mathbf{u}}) = \Box_{\mathbf{u}}$. Let $\tau \in \Phi_A^{L,d-1}$ and suppose $\Box_{\mathbf{u}} \notin J_\tau$. Then (without loss of generality) $\operatorname{supp}(\mathbf{u}_+) \subseteq \tau$. By part (2) of Lemma 2.10, since $\sigma^L(\Box_{\mathbf{u}}) = \Box_{\mathbf{u}}$, $\operatorname{supp}(\mathbf{u}_-) \subseteq \tau$ as well. Hence $\Box_{\mathbf{u}} \in I_\tau \subseteq I_A^\tau$ and so $\tilde{I}_A^L \subseteq I$.

For the converse inclusion suppose now $m \in I$. We write $m = m_0 + \sum_{\tau \in \Phi_A^L} m_\tau$ where $m_0 \in I_0$, and

$$m_\tau = \sum_{\tau_{\mathbf{u}}^L = \tau} c_{\mathbf{u}} \partial^{\mathbf{u}} \in \mathbb{C}[\partial_\tau]$$

collects the monomials in m minimally supported in τ. Since $I_0 \subseteq \tilde{I}_A^L \subseteq I$ we may assume that $m_0 = 0$. Now pick any $\tau \in \Phi_A^L$. Then $m = \hat{m}_\tau + m_{\overline{\tau}}$ where $\hat{m}_\tau = \sum_{\tau' \subseteq \tau} m_{\tau'}$ are the terms in m supported in τ. Since $m_{\overline{\tau}} \in J_\tau \subseteq I_A^\tau$ and $m \in I \subseteq I_A^\tau$, $\hat{m}_\tau = m - m_{\overline{\tau}} \in I_A^\tau \cap \mathbb{C}[\partial_\tau] = I_\tau \subseteq \tilde{I}_A^L$ in view of (2). Since every $\hat{m}_\tau$ is in $\tilde{I}_A^L$, so is every m_τ, and hence $m \in \tilde{I}_A^L$ as well. ☐

Lemma 2.13. *The radical ideal $\tilde{I}_A^L$ from Lemma 2.12 equals $\sqrt{I_A^L}$.*

Proof. By Lemma 2.8, the elements in 2.12.(1) are in $\sqrt{I_A^L}$. By part (1) of Lemma 2.10, all elements from 2.12.(2) are in I_A^L. Hence $\tilde{I}_A^L \subseteq \sqrt{I_A^L}$.

By Lemmas 2.4 and 2.12, it suffices to show conversely that, for any $\mathbf{u} \in \mathbb{Z}^n$ with $A \cdot \mathbf{u} = 0$, $\sigma^L(\Box_{\mathbf{u}}) \in I_A^\tau$ for all $\tau \in \Phi_A^{L,d-1}$. Pick such $\mathbf{u}$ and let $\tau \in \Phi_A^{L,d-1}$. If $\tau_{\mathbf{u}}^L \subseteq \tau$ then by part (1) of Lemma 2.10, $\sigma^L(\Box_{\mathbf{u}}) = \Box_{\mathbf{u}} \in I_\tau \subseteq I_A^\tau$. If $\tau_{\mathbf{u}_+}^L \subseteq \tau$ but $\tau_{\mathbf{u}_-}^L \not\subseteq \tau$ then by part (2) of Lemma 2.10, $\sigma^L(\Box_{\mathbf{u}}) = -\partial^{\mathbf{u}_-} \in J_\tau \subseteq I_A^\tau$. Similarly, $\sigma^L(\Box_{\mathbf{u}}) \in I_A^\tau$ if $\tau_{\mathbf{u}_-}^L \subseteq \tau$ but $\tau_{\mathbf{u}_+}^L \not\subseteq \tau$. Finally, if $\tau_{\mathbf{u}_+}^L \not\subseteq \tau \not\supseteq \tau_{\mathbf{u}_-}^L$ then $\sigma^L(\Box_{\mathbf{u}}) \in J_\tau \subseteq I_A^\tau$. ☐

The following consequence of Lemmas 2.3, 2.12, and 2.13 generalizes [Ado94, Lem. 3.2].

Theorem 2.14. *The set of $\mathbb{Z}^d$-graded prime ideals of R containing I_A^L equals $\{I_A^\tau \mid \tau \in \Phi_A^L\}$ and hence the (A,L)-umbrella encodes the geometry of S_A^L:*

$$\operatorname{Spec}(S_A^L) = \operatorname{Var}(I_A^L) = \bigcup_{\tau \in \Phi_A^{L,d-1}} \bar{O}_A^\tau = \bigsqcup_{\tau \in \Phi_A^L} O_A^\tau.$$

Adjacencies of orbit strata correspond to inclusions in the (A,L)-umbrella:

$$\left[O_A^{\tau'} \subseteq \bar{O}_A^\tau\right] \quad \Leftrightarrow \quad [\tau' \subseteq \tau]\,. \qquad \Box$$

In particular, Theorem 2.14 identifies the $\mathbb{Z}^d$-graded prime ideals containing $I_A = I_A^{\mathbf{0}}$ with the elements of $\Phi_A^{\mathbf{0}}$ from Definition 2.7.

2.7. **Index formula for multiplicities.** By Theorem 2.14, there is a composition chain of $\mathbb{Z}^d$-graded R-modules

$$0 = N_0 \subsetneq N_1 \subsetneq \cdots \subsetneq N_{l-1} \subsetneq N_l = S_A^L \tag{2.7.1}$$

with $N_i/N_{i-1} \cong S_{\tau_i}(-\mathbf{u}_i)$ for some $\tau_i \in \Phi_A^L$ and $\mathbf{u}_i \in \mathbb{Z}^d$. As $\mathbb{Z}^d$-graded vector spaces $S_A^L = S_A$ and the $\mathbb{Z}^d$-graded Hilbert function of both rings has values in $\{0,1\}$. Thus, the composition chain (2.7.1) induces a partition of $\mathbb{Z}^d$-degrees

$$\deg(S_A^L) = \deg(S_A) = \mathbb{N}A = \bigsqcup_{i=1}^{l} (\mathbf{u}_i + \mathbb{N}\tau_i) = \bigsqcup_{i=1}^{l} \deg(S_{\tau_i}(-\mathbf{u}_i)). \tag{2.7.2}$$

Definition 2.15. If $\tau \in \Phi_A^{L,d-1}$ is a facet then the number $\nu_A^{L,\tau}$ of indices i in the chain (2.7.1), and hence in the partition (2.7.2), with $\tau_i = \tau$ is the *multiplicity* of S_A^L along $\bar{O}_A^\tau$. Note that $\nu_A^{L,\tau}$ is the length of the localization of S_A^L at I_A^τ and hence independent of the particular composition chain.

Proposition 2.16. *For all $\tau \in \Phi_A^{L,d-1}$, $\nu_A^{L,\tau} = [\mathbb{Z}^d : \mathbb{Z}\tau]$. In particular, the degree of I_A^L equals $\sum_{\tau \in \Phi_A^{L,d-1}} \nu_A^{L,\tau} = \sum_{\tau \in \Phi_A^{L,d-1}} [\mathbb{Z}^d : \mathbb{Z}\tau]$.*

Proof. Fix $\tau \in \Phi_A^{L,d-1}$. For disjoint $\mathbf{u}+\mathbb{N}\tau$ and $\mathbf{v}+\mathbb{N}\tau$, $\mathbf{u}-\mathbf{v} \notin \mathbb{Z}\tau$ since $\mathbb{N}\tau$ contains a shifted copy of its normalization $\mathbb{Q}_+\tau \cap \mathbb{Z}\tau$. This means that $\nu_A^{L,\tau} \le [\mathbb{Z}^d : \mathbb{Z}\tau]$ and it remains to show that $\bigsqcup_{\tau_i=\tau}(\mathbf{u}_i + \mathbb{N}\tau_i)$ meets every coset of $\mathbb{Z}^d/\mathbb{Z}\tau$. Pick $\mathbf{u} \in \mathbb{N}\tau$ outside $\mathbb{Q}_+\tau'$ for all $\tau' \in \Phi_A^L$ with $\tau' \subsetneq \tau$. Since $\mathbb{N}A$ contains a shifted copy of its normalization $\mathbb{Q}_+A \cap \mathbb{Z}^d$, for $k \gg 0$

$$k\mathbf{u} + (\mathbb{Q}_+\tau \cap \mathbb{Z}^d) \subseteq k\mathbf{u} + (\mathbb{Q}_+A \cap \mathbb{Z}^d) \subseteq \mathbb{N}A = \deg(S_A^L) = \bigsqcup_{i=1}^{l} \deg(S_{\tau_i}(-\mathbf{u}_i)).$$

By Lemma 2.17 below, for $k \gg 0$

$$k\mathbf{u} + (\mathbb{Q}_+\tau \cap \mathbb{Z}^d) \subseteq \bigsqcup_{\tau_i=\tau} (\mathbf{u}_i + \mathbb{N}\tau_i)$$

and the left hand side meets every coset of $\mathbb{Z}^d/\mathbb{Z}\tau$. This yields $\nu_A^{L,\tau} \geq [\mathbb{Z}^d : \mathbb{Z}\tau]$. $\square$

Lemma 2.17. *Let $\tau \in \Phi_A^{L,d-1}$ and suppose $\mathbf{u} \in \mathbb{N}\tau \smallsetminus \mathbb{Q}_+\tau'$ for all $\tau' \in \Phi_A^L$ with $\tau' \subsetneq \tau$. Fix $\mathbf{u}' \in \mathbb{Z}^d$ and $\tau' \in \Phi_A^L$ such that $\tau' \neq \tau$. Then for $k \gg 0$*

$$(k\mathbf{u} + \mathbb{Q}_+\tau) \cap (\mathbf{u}' + \mathbb{Q}_+\tau') = \emptyset.$$

Proof. Suppose that $k\mathbf{u} + \mathbf{v} = \mathbf{u}' + \mathbf{v}'$ with $\mathbf{v} \in \mathbb{Q}_+\tau$ and $\mathbf{v}' \in \mathbb{Q}_+\tau'$. Pick a linear form $0 \neq h_{\tau,\tau'} \in \operatorname{Hom}_{\mathbb{Q}}(\mathbb{Q}^d, \mathbb{Q})$ with $h_{\tau,\tau'}(\tau) \geq 0$, $h_{\tau,\tau'}(\tau') \leq 0$. Then, by hypothesis, $h_{\tau,\tau'}(\mathbf{u}) > 0$. But $kh_{\tau,\tau'}(\mathbf{u}) + h_{\tau,\tau'}(\mathbf{v}) = h(\mathbf{u}') + h_{\tau,\tau'}(\mathbf{v}')$ implies that $kh_{\tau,\tau'}(\mathbf{u}) \leq h(\mathbf{u}')$ which is impossible for $k \gg 0$. $\square$

2.8. **Newton filtration and Cohen–Macaulayness.** Until the end of this section, we fix a weight vector L with $L_{\partial_i} > 0$ for all i. Let $V = \bigoplus_{\mathbf{a}\in\mathbb{N}A} V_{\mathbf{a}}$ be a $\mathbb{Z}^d$-graded vector space. Then the *Newton filtration* $N = N_A^L$ on V with respect to Δ_A^L is defined by

$$N_iV = \bigoplus_{\mathbf{a}\in i\cdot\Delta_A^L} V_{\mathbf{a}}.$$

Note that $\operatorname{gr}^N(V) = V$ as $\mathbb{Z}^d$-graded vector spaces. For $\tau \in \Phi_A^L$ or $\tau = A$, we denote $V(\tau) = \bigoplus_{\mathbf{a}\in\mathbb{Q}_+\tau} V_{\mathbf{a}}$. Generalizing [Kou76, Prop. 2.6], there is a complex

$$0 \longrightarrow V_{d-1} \longrightarrow V_{d-2} \longrightarrow \cdots \longrightarrow V_0 \longrightarrow 0 \tag{2.8.1}$$

where the $\mathbb{Z}^d$-graded vector space V_i is the direct sum of all $V(\tau)$ for which $\tau \in \Phi_A^{L,i}$ is not contained in the boundary of Γ_A^L (see Definition 2.7). The cohomology of this complex is concentrated in homological degree $d-1$ and equal to V.

Now assume that V is a $\mathbb{Z}^d$-graded $\mathbb{C}$-algebra. Then $\operatorname{gr}^N(V)(\tau) = V(\tau)$ as $\mathbb{C}$-algebras for $\tau \in \Phi_A^L$. For $\tau, \tau' \in \Phi_A^L$ (or $\tau = A$) with $\tau \supseteq \tau'$, the maps

$$\gamma_{\tau,\tau'}\colon \operatorname{gr}^N(V)(\tau) \twoheadrightarrow \operatorname{gr}^N(V)(\tau')$$

in the complex (2.8.1) for $\operatorname{gr}^N(V)$ are natural projections of $\mathbb{C}$-algebras. In particular, (2.8.1) is a complex of $\operatorname{gr}^N(V)$-modules.

Lemma 2.18. *For all $\tau \in \Phi_A^L$ and all $k > 0$, $\mathbb{C}[\mathbb{N}\cdot k\cdot(\tau \cap \Phi_A^{L,0})] \subseteq S_A^L(\tau)$ is a module-finite ring extension for all $\tau \in \Phi_A^L$.*

Proof. One can construct a $\mathbb{Z}^d$-graded composition chain of S_A^L as in (2.7.1) such that the resulting partition (2.7.2) refines the partition $\mathbb{Q}_+A = \mathbb{Q}_+\tau \sqcup (\mathbb{Q}_+A \smallsetminus \mathbb{Q}_+\tau)$, and hence $\deg(S_{\tau_i}(-\mathbf{u}_i))$ meets $\mathbb{Q}_+\tau$ only if $\tau_i \subseteq \tau$. Namely, given any chain as in (2.7.1) we refine it at any index $i \in \{1,\ldots,l\}$ for which $\deg(S_{\tau_i}(-\mathbf{u}_i)) = \mathbf{u}_i + \mathbb{N}\tau_i$ meets both $\mathbb{Q}_+\tau$ and $\mathbb{Q}_+A \smallsetminus \mathbb{Q}_+\tau$. Let $\ell \in \operatorname{Hom}_{\mathbb{Q}}(\mathbb{Q}^d, \mathbb{Q})$ be a separating linear form, $\ell(\tau) \geq 0$ and $\ell(\tau_i) \leq 0$. If $\tau_i \subseteq \tau$ (and hence $\mathbf{u}_i \notin \mathbb{Q}_+\tau$), we chose $\mathbf{u}'_i \in (\mathbf{u}_i + \mathbb{N}\tau_i) \cap \mathbb{Q}_+\tau$. The submodule $S_{\tau_i}(-\mathbf{u}'_i)$ of $S_{\tau_i}(-\mathbf{u}_i)$ has its $\mathbb{Z}^d$-degrees entirely in $\mathbb{Q}_+\tau$, and the cokernel of the inclusion has smaller dimension. If conversely $\tau_i \not\subseteq \tau$, choose $\ell \in \operatorname{Hom}_{\mathbb{Q}}(\mathbb{Q}^d, \mathbb{Q})$ above such that $\ell(\tau_i) \neq 0$ and pick $\mathbf{u}'_i \in \mathbf{u}_i + \mathbb{N}\tau_i$ with $\ell(\mathbf{u}'_i) < 0$. Then $S_{\tau_i}(-\mathbf{u}'_i) \subseteq S_{\tau_i}(-\mathbf{u}_i)$ and has degrees completely outside of $\mathbb{Q}_+\tau$.

Iterating this procedure we arrive at a composition chain as claimed. Then, however, the statement of the lemma is obvious: for each $i \in \{1, \dots, l\}$ with $\deg(S_{\tau_i}(-\mathbf{u}_i)) \cap \mathbb{Q}_+\tau \neq \emptyset$, $\tau_i \subseteq \tau$, and $(S_{\tau_i}(-\mathbf{u}_i))(\tau) \cong S_{\tau_i}$ is even module-finite over the subring $\mathbb{C}[\mathbb{N} \cdot k \cdot (\tau_i \cap \Phi_A^{L,0})]$ of $\mathbb{C}[\mathbb{N} \cdot k \cdot (\tau \cap \Phi_A^{L,0})]$. □

Theorem 2.19. *If $S_A^L(\tau)$ is Cohen–Macaulay for all $\tau \in \Phi_A^L$ not contained in the topological boundary of Γ_A^L then S_A^L is Cohen–Macaulay.*

Proof. There is an integer $k > 0$ such that $k \cdot \Gamma_A^L$ has vertex set $k \cdot \Gamma_A^{L,0} \subseteq \mathbb{N}A$. Within the space of sequences $f_1, \dots, f_d$ in S_A^L for which each f_i is a sum of terms whose $\mathbb{Z}^d$-degrees are in $k \cdot \Gamma_A^{L,0}$, choose one sequence that is generic.

Each f_i is homogeneous with respect to $N = N_A^L$ and can hence be identified with its N-symbol in $\mathrm{gr}^N(S_A^L)$. By Lemma 2.18, the vector space spanned by $\gamma_{A,\tau}(f_1), \dots, \gamma_{A,\tau}(f_d) \in \mathbb{C}[\mathbb{N}\tau]$ contains a system of parameters on $S_A^L(\tau)$ for all $\tau \in \Phi_A^L$. This system of parameters is a regular sequence by the Cohen–Macaulay hypothesis. By the (spectral sequence) argument in [Kou76, §2.12], the Koszul complex induced by $f_1, \dots, f_d$ on the complex (2.8.1) with $V = S_A^L$ is a resolution. It follows that the Koszul complex induced by $f_1, \dots, f_d$ on $\mathrm{gr}^N(S_A^L)$, and hence on S_A^L, is a resolution as well. Therefore, $f_1, \dots, f_d$ is a regular sequence on S_A^L and the claim follows. □

3. Characteristic variety of the hypergeometric system

3.1. **Characteristic varieties and slopes.** The *Weyl algebra* $D = \mathbb{C}[x]\langle\partial\rangle$ in n variables $x = x_1, \dots, x_n$ is the ring of $\mathbb{C}$-linear differential operators on $X = \mathbb{C}^n$ and contains R as a commutative subring. The $\mathbb{Z}^d$-grading on R extends to D by setting $-\deg(x_i) = \mathbf{a}_i = \deg(\partial_i)$.

With $L_x = (L_{x_1}, \dots, L_{x_n})$ and $L_\partial = (L_{\partial_1}, \dots, L_{\partial_n})$, $L = (L_x, L_\partial) \in \mathbb{Q}^{2d}$ is a *weight vector on D* if $L_x + L_\partial \geq 0$. Fix any such weight vector L; it defines an increasing filtration L on D by $[x^{\mathbf{u}}\partial^{\mathbf{v}} \in L_kD] \Leftrightarrow [L \cdot (\mathbf{u}, \mathbf{v}) \leq k]$. Since the Gröbner fan of any D-ideal is defined over $\mathbb{Q}$ (see [ACJG00]), the study of real weights can be reduced to the present rational case. If $P \in L_kD \smallsetminus L_{<k}D$ then $k =: \deg^L(P)$ is the *L-degree* of P. The multiplicative, but not additive, *L-symbol map*

$$\sigma^L : D \longrightarrow \mathrm{gr}^L(D) =: W$$

is defined by $\sigma^L(P) = P \mod L_{<k}R$ if $\deg^L(P) = k$. An element of the form $P = \sum_{L\cdot(\mathbf{u},\mathbf{v})=k} P_{\mathbf{u},\mathbf{v}} x^{\mathbf{u}}\partial^{\mathbf{v}} \in D$ with $P_{\mathbf{u},\mathbf{v}} \in \mathbb{C}$ is *L-homogeneous.* By abuse of notation, we identify L-homogeneous elements in D with their image under σ^L in W. We restrict ourselves to the case $L_x + L_\partial > 0$ in which case $W \cong \mathbb{C}[x, \partial]$ can be considered as the ring of polynomial functions on the cotangent space T^*X

The definition of the characteristic variety of a D-module is based on the concept of good filtrations discussed in more detail in Section 4 and [Sch85, Ch. II, §§1.1–1.3]. Let F be a filtration on a ring T. Then G is called a *good F-filtration* on a T-module N (see [Sch85, Ch. II, Def. 1.1.1]), if there are generators $n_1, \dots, n_m$ of N and $\mathbf{u} \in \mathbb{Z}^m$ such that for all k one has

$$G_kN = \sum_{i=1}^{m} F_{k+u_i}T \cdot n_i.$$

Note that good F-filtrations on N exist if and only if N is T-finite. From the definition follows the fact that all good F-filtrations on N are equivalent in the sense that for all $k, l \in \mathbb{N}$ there are $k_l, l_k \in \mathbb{N}$ with $G_k N \subseteq G'_{l_k} N$ and $G'_l \subseteq G_{k_l}$.

Definition 3.1. The *L-characteristic variety* $\mathrm{Ch}^L(M)$ of a finite D-module M on X is the support of $\mathrm{gr}^L(M)$ in T^*X for some good L-filtration on M. A finite D-module M is *L-holonomic* if $\dim \mathrm{Ch}^L(M) = n$.

The independence of $\mathrm{Ch}^L(M)$ of the choice of the good L-filtration on M follows from [Sch85, Ch. II, Prop. 1.3.1.a]. The following algebraic statement is a special case of a result by G.G. Smith [Smi01, Thm. 1.1].

Theorem 3.2. *The dimension of any component of* $\mathrm{Ch}^L(M)$ *is at least* n. □

Important special cases of filtrations on D are the the *order filtration* $F = (F_x, F_\partial) = (\mathbf{0}, \mathbf{1})$ and the *V-filtration* $V = (V_x, V_\partial)$ along the coordinate variety

(3.1.1) $$Y := \mathrm{Var}(x_{\mathfrak{V}}) \subseteq X, \quad \mathfrak{V} \subseteq \{1, \ldots, n\},$$

defined by $-V_{x_i} = 1 = V_{\partial_i}$ for $i \in \mathfrak{V}$ and $V_{x_i} = 0 = V_{\partial_i}$ for $i \notin \mathfrak{V}$. The notion of slopes along Y (see Definition 3.4) involves the family of intermediate filtrations L between F and V defined by the linear combination of weight vectors

(3.1.2) $$L = pF + qV, \quad p/q \in \mathbb{Q}_{>0} \cup \{\infty\}.$$

Note that $L_x + L_\partial > 0$ since $p > 0$. If $p'/q' = p/q$ then the filtrations L and L' are identical, up to a dilation in the index: $L_{p'k} = L'_{pk}$. By abuse of notation, we shall frequently identify the filtration L with the number p/q. For $Y \subseteq X$ closed and reduced, let $T^*_Y X$ be (the closure of) the conormal bundle of (the smooth points of) Y in X. With notation as in (3.1.1) and (3.1.2), the ring $W = \mathrm{gr}^L(D)$ can be considered as the ring of polynomial functions on the cotangent space $T^*T^*_Y X$ of $T^*_Y X$. For $i \in \mathfrak{V}$, $-x_i$ can be interpreted as the partial derivative with respect to the variable ∂_i. This sets up an explicit isomorphism between $T^*T^*_Y X$ and T^*X .

Remark 3.3. For $L = pF + qV$ as in (3.1.2), $\mathrm{Ch}^L(M)$ is the global algebraic version of Y. Laurent's *microcharacteristic variety* $\Sigma^r_\Lambda(\mathcal{M})$ *of type* $r = p/q + 1$ in $T^*\Lambda$ with $\Lambda = T^*_Y X$ (see [Lau87, §3.2]). Our algebraic L-filtration corresponds to the filtration on the sheaf of analytic differential operators on X along Y induced by Y. Laurent's microlocal filtration $F_{\Lambda,r}$ along Λ (see [Lau87, Def. 3.2.1]). By a flatness argument, A. Assi et al. [ACJG96, Lem. 1.1.2] show that the analytification functor commutes with the grading by these two corresponding filtrations. Thus, the components of our $\mathrm{Ch}^L(M)$ which meet the preimage of Y in T^*X correspond to the components of $\Sigma^r_\Lambda(\mathcal{M})$.

The preceding remark motivated the following algebraic version of Y. Laurent's *critical indices* (see [Lau87, §3.4]) which are also called slopes.

Definition 3.4. For $L = pF + qV$ as in (3.1.2), we mean by *$f(L')$ jumps at $L' = L$* that the set-valued function

$$\mathbb{Q} \ni p'/q' \longmapsto L' = p'F + q'V \mapsto f(L')$$

is not locally constant at p/q. A *slope* of a finite D-module M *at $y \in Y$ along* Y is a value $L = p/q$ such that the set of components of $\mathrm{Ch}^{L'}(M)$ which meet $T^*_y X$ jumps at $L' = L$.

It follows from the existence of the Gröbner fan (see [ACJG00]) that a fixed D-module has only finitely many and only rational slopes along all coordinate varieties. Y. Laurent proved this finiteness and rationality along general varieties in the microlocal setting in [Lau87, Thm. 3.4.1]. He also showed that $\Sigma^r_\Lambda(\mathcal{M})$ is involutive (see [Lau87, Prop. 3.5.2]) and Lagrangian for holonomic $\mathcal{M}$ (see [Lau87, Cor. 4.1.2.(ii)]). In view of Remark 3.3, this implies for holonomic M that $\mathrm{Ch}^L(M)$ is purely n-dimensional for $L = pF + qV$ as in (3.1.2).

3.2. **Hypergeometric system and candidate components.** We now define our main object of interest, the hypergeometric D-module $M_A(\beta)$, introduced in [GGZ87, GZK89].

Definition 3.5. The *Euler vector fields* $E = (E_1, \dots, E_d)$ of A are defined by $E_i := \sum_j a_{i,j} x_j \partial_j$ for $i = 1, \dots, d$. The *A-hypergeometric* (or *GKZ*) *system* defined by A and a complex parameter vector $\beta \in \mathbb{C}^d$ is the D-module

$$M_A(\beta) := D/D\langle I_A, E - \beta\rangle$$

on X defined by the toric ideal I_A and the *Euler operators* $E - \beta$.

The A-hypergeometric system is always holonomic (see [Ado94]). Our goal is to understand its L-characteristic varieties and slopes under the assumption that

$$L_x + L_\partial = (c, \dots, c) =: \mathbf{c} \text{ for some rational } c > 0. \tag{3.2.1}$$

This guarantees that W is a polynomial ring and E is L-homogeneous of positive L-degree. Note that $M_A(\beta)$ is L-homogeneous if A is L-homogeneous as defined in Definition 2.7. The following statement is a consequence of Buchberger's algorithm.

Lemma 3.6. *One has the identity* $\mathrm{gr}^L(DI_A) = WI_A^L$. □

The vector fields $t_1\partial_{t_1}, \dots, t_d\partial_{t_d}$ span the tangent space at any point of the torus $\mathbb{T} = (\mathbb{C}^*)^d$. Hence, for any $\xi \in T_0^*X$, the tangent space of the orbit $\mathrm{Orb}(\xi)$ is spanned by the pushforwards $E_i^T := \sum_j a_{i,j}\partial_j(-x_j)$ of $t_i\partial_{t_i}$ under the map $\mathbb{T} \to T_0^*X$, $t \mapsto t \cdot \xi$ from (2.1.1). Since E_i is L-homogeneous of positive degree with $\sigma^L(E_i) = -\sigma^L(E_i^T)$, the equations $\sigma^L(E - \beta) = 0$ impose the conormal condition to the orbit $\mathrm{Orb}(\xi) \subseteq T_0^*X$ in $T^*T_0^*X = T^*X$. In what follows we abuse notation by writing $\sigma^L(E) = E$.

Definition 3.7. For a subset τ of $\{1, \dots, n\}$, we denote by $C_A^\tau \subseteq T^*X$ the conormal space to the orbit $O_A^\tau \subseteq T_0^*X$ from Definition 2.2. We denote by $P_C \subseteq W$ the defining ideal of any irreducible variety $C \subseteq T^*X$ and abbreviate $P_{C_A^\tau}$ by P_τ.

Proposition 3.8. *The L-characteristic variety of* $M_A(\beta)$ *is*

$$\mathrm{Ch}^L(M_A(\beta)) = \bigcup_{\tau \in \phi_A^L(\beta)} \bar{C}_A^\tau = \bigsqcup_{\tau \in \phi_A^L(\beta)} C_A^\tau$$

for some subset $\phi_A^L(\beta) \subseteq \Phi_A^L$. *In particular, the hypergeometric system* $M_A(\beta)$ *is L-holonomic for any L.*

Proof. By definition, $\mathrm{Ch}^L(M_A(\beta)) \subseteq \mathrm{Var}(W\langle I_A^L, E\rangle)$. By Theorem 2.14 and the preceding arguments,

$$\mathrm{Var}(W\langle I_A^L, E\rangle) = \bigsqcup_{\tau \in \Phi_A^L} C_A^\tau = \bigcup_{\tau \in \Phi_A^L} \bar{C}_A^\tau. \tag{3.2.2}$$

Then Theorem 3.2 assures that $\mathrm{Ch}^L(M_A(\beta))$ is purely n-dimensional and hence a union of closures of conormals C_A^τ for certain $\tau \in \Phi_A^L$. $\square$

3.3. **Existence of facet components.** In this subsection, we use an elementary localization argument to give an index formula for the multiplicity of components in the characteristic cycle $\mathrm{CC}^L(M_A(\beta))$ (see Definition 4.4) corresponding to facets in the (A,L)-umbrella. It shows in particular that these multiplicities are independent of β and positive, and that all facet components occur in $\mathrm{Ch}^L(M_A(\beta))$. We deduce from this a converse to R. Hotta's Theorem: regular A-hypergeometric systems are homogeneous in the usual sense.

Definition 3.9. We denote by $\mu_{A,0}^{L,C}(\beta)$ the multiplicity of $\mathrm{gr}^L(M_A(\beta))$ along C. (The reason for the appearance of the subscript "0" will become apparent in Section 4.) This is the length of the W_{P_C}-module $\mathrm{gr}^L(M_A(\beta)) \otimes_W W_{P_C}$. We write $\mu_{A,0}^{L,\tau}(\beta)$ if $C = \bar{C}_A^\tau$.

Theorem 3.10. *For all $\beta \in \mathbb{C}^d$ and all $\tau \in \Phi_A^{L,d-1}$,*

$$\mu_{A,0}^{L,\tau}(\beta) = \nu_A^{L,\tau} = [\mathbb{Z}^d : \mathbb{Z}\tau].$$

In particular, $\Phi_A^{L,d-1} \subseteq \phi_A^L(\beta)$ for all $\beta \in \mathbb{C}^d$.

Proof. Let $\tau \in \Phi^{L,d-1}$ and relabel columns such that $\mathbf{a}_1^L, \ldots, \mathbf{a}_d^L \in \tau$ and such that $\mathbf{a}_1, \ldots, \mathbf{a}_d$ are linearly independent. We have to show that $C_A^\tau \subseteq \mathrm{Ch}^L(M_A(\beta))$. First, we verify that E is a regular sequence on $W[\partial_\tau^{-1}]/W[\partial_\tau^{-1}]I_A^L$. After Gauss reduction on E and multiplying E_i by ∂_i^{-1}, $E_i \equiv x_i$ modulo terms independent of x_j for all $j \le d$. After a change of the coordinates $x_1, \ldots, x_d$ in $W[\partial_\tau^{-1}]$, leaving I_A^L invariant, $E_i = x_i$. As I_A^L does not involve the variables $x_1, \ldots, x_d$, E is a regular sequence modulo $W[\partial_\tau^{-1}]I_A^L$ as claimed.

Since $W[\partial_\tau^{-1}]I_A^L = \mathrm{gr}^L(D[\partial_\tau^{-1}]I_A)$ by Lemma 3.11 below, $\mathrm{gr}^L(D[\partial_\tau^{-1}]\langle I_A, E-\beta\rangle) = \mathrm{gr}^L(D[\partial_\tau^{-1}]I_A) + \mathrm{gr}^L(D[\partial_\tau^{-1}]\langle E\rangle)$ by the argument in [SST00, Thm. 4.3.5]. Again by Lemma 3.11,

$$\mathrm{gr}^L(M_A(\beta))[\partial_\tau^{-1}] = W[\partial_\tau^{-1}]/W[\partial_\tau^{-1}]\langle I_A^L, E\rangle.$$

Since $\partial_j \notin P_\tau$ for all $j \in \tau$, this yields

$$\mathrm{gr}^L(M_A(\beta))_{P_\tau} = W_{P_\tau}/W_{P_\tau}\langle I_A^L, E\rangle \cong (R_{I_A^\tau}/I_A^L)(x_{d+1}, \ldots, x_n)$$

with the ring isomorphism defined by the above coordinate change, and hence

$$\mu_{A,0}^{L,\tau}(\beta) = \ell(\mathrm{gr}^L(M_A(\beta))_{P_\tau}) = \ell(R_{I_A^\tau}/I_A^L) = \nu_A^{L,\tau}.$$

Then Proposition 2.16 finishes the proof. $\square$

Lemma 3.11. *For $I \subseteq D$ and $\tau \subseteq \{1, \ldots, n\}$, $\mathrm{gr}^L(D[\partial_\tau^{-1}]I) = \mathrm{gr}^L(D[\partial_\tau^{-1}])\,\mathrm{gr}^L(I)$. In particular, $\mathrm{gr}^L((D/I)[\partial_\tau^{-1}]) = (\mathrm{gr}^L(D/I))[\partial_\tau^{-1}]$.*

Proof. The inclusion $\mathrm{gr}^L(D[\partial_\tau^{-1}]I) \supseteq \mathrm{gr}^L(D[\partial_\tau^{-1}])\,\mathrm{gr}^L(I)$ holds trivially. For $Q \in D[\partial_\tau^{-1}]$, $\partial_\tau^{\mathbf{u}}Q \in D$ for some $\mathbf{u} \in \mathbb{N}^{|\tau|}$. If $\{P_i\}$ is a finite set of generators for I then the opposite inclusion follows from

$$\sigma^L\big(\sum_i Q_i P_i\big) = \partial_\tau^{-\mathbf{u}}\sigma^L\big(\sum_i \partial_\tau^{\mathbf{u}} Q_i P_i\big) \in \mathrm{gr}^L(D[\partial_\tau^{-1}])\,\mathrm{gr}^L(I)$$

with a common $\mathbf{u}$ for the finitely many $Q_i \in D[\partial_\tau^{-1}]$. In particular,

$$\begin{aligned}\operatorname{gr}^L((D/I)[\partial_\tau^{-1}]) &= \operatorname{gr}^L(D[\partial_\tau^{-1}]/D[\partial_\tau^{-1}]I) = \operatorname{gr}^L(D[\partial_\tau^{-1}])/\operatorname{gr}^L(D[\partial_\tau^{-1}])\operatorname{gr}^L(I)\\ &= W[\partial_\tau^{-1}]/W[\partial_\tau^{-1}]\operatorname{gr}^L(I) = (W/\operatorname{gr}^L(I))[\partial_\tau^{-1}] = (\operatorname{gr}^L(D/I))[\partial_\tau^{-1}]. \qquad \square\end{aligned}$$

Example 3.12. Let A be as in Figure 1; we consider $L = F + tV$ where V is the V-filtration along $\operatorname{Var}(x_4)$, induced by the weight vector $(0,0,0,-1,0,0,0,1)$. We consider specifically $t \in \{0,1,4\}$. As these weights are generic, for all $\tau \subseteq \{1,\dots,4\}$ the conormal closures $\bar{C}_A^\tau$ are coordinate subspaces $\operatorname{Var}(x_\tau, \partial_{\bar\tau})$. The facets in Φ_A^L for these values of t are shown in Figure 1.

For $t = 0$ one finds $\mu_{A,0}^{L,\{1,4\}}(\beta) = 12$ and $\mu_{A,0}^{L,\{2,4\}}(\beta) = 1$ with sum $13 = \operatorname{vol}_{\mathbb{Z}^d}(\Delta_A^F)$ (for relevant notation and more information, see Definition 4.19 and the continuation in Example 4.24). For $t = 1$ one finds three facets, with $\mu_{A,0}^{L,\{1,3\}}(\beta) = 3$, $\mu_{A,0}^{L,\{3,4\}}(\beta) = 7$ and $\mu_{A,0}^{L,\{2,4\}}(\beta) = 1$. On the other hand, $t = 4$ yields two facets only, with $\mu_{A,0}^{L,\{1,3\}}(\beta) = 3$ and $\mu_{A,0}^{L,\{2,3\}}(\beta) = 2$. The respective sums, $3+7+1$ and $3+2$, are the degrees of the L-graded toric ideals (see Proposition 2.16).

Lemma 3.13. *The map $L' \mapsto \Phi_A^{L',d-1}$ jumps at $L' = L$ if and only if the map $L' \mapsto \Phi_A^{L'}$ jumps at $L' = L$.*

Proof. The "only if" part in the statement follows trivially from $\Phi_A^{L',d-1} \subseteq \Phi_A^{L'}$. Assume that $\Phi_A^{L',d-1}$ is locally constant at L and let $\tau \in \Phi_A^L$. Then there is a $\tau' \in \Phi_A^{L,d-1}$ such that τ is a face of τ'. By assumption, $\tau' \in \Phi_A^{L',d-1}$ for L' close to L. Since τ' is not contained in a hyperplane through origin and since the $\mathbf{a}_j^{L'}$ depend on L' only by scaling, τ remains a face of τ' for L' close to L. In particular, $\tau \in \Phi_A^{L'}$ for L' close to L and hence $\Phi_A^{L'}$ is locally constant at L. $\square$

Theorem 3.10 and Lemma 3.13 yield the existence of slopes of $M_A(\beta)$ corresponding to jumps of the (A,L)-umbrella with L as in (3.1.2). The following result on the candidate components $\bar{C}_A^\tau$ of $\operatorname{Ch}^L(M_A(\beta))$ allows us to show that all these slopes occur at the origin.

Lemma 3.14. *For all $\tau \in \Phi_A^L$, the candidate component $\bar{C}_A^\tau$ meets T_0^*X.*

Proof. Since the ideal $W\langle I_A^L, E\rangle$ is homogeneous in the x-variables, this holds also for its associated prime ideals P_τ where $\tau \in \Phi_A^L$. Thus $\bar{C}_A^\tau = \operatorname{Var}(P_\tau)$ meets T_0^*X as claimed. $\square$

Corollary 3.15. *For $L = pF + qV$ as in* (3.1.2), *if Φ_A^L jumps at $L = L'$ then $L' = p'/q'$ is a slope of $M_A(\beta)$ along Y at $0 \in Y$.* $\square$

In particular, we obtain the following converse to a Theorem by R. Hotta [Hot98, Ch. II, §6.2, Thm.].

Corollary 3.16. *Regular A-hypergeometric systems are homogeneous with respect to the order filtration F.*

Proof. The polytope $\operatorname{conv}(A)$ is an intersection of closed half-spaces and $0 \notin \operatorname{conv}(A)$ by positivity of $\mathbb{N}A$. Thus there is a facet τ of $\operatorname{conv}(A)$ and a linear form $\ell \in \operatorname{Hom}_{\mathbb{Q}}(\mathbb{Q}^d, \mathbb{Q})$ such that $\ell(\mathbf{a}_i) \geq 1$ with equality equivalent to $i \in \tau$. If $\tau \in \Phi_A^{F,d-1}$ then τ is a facet of $\Delta_A^F = \operatorname{conv}(\{0, \mathbf{a}_1, \dots, \mathbf{a}_n\})$. But then $0 \in \Delta_A^F$ and $\ell(0) = 0 < 1$

implies that $\ell(\mathbf{a}_i) \leq 1$ and hence $\ell(\mathbf{a}_i) = 1$ for $i = 1, \ldots, n$. So in this case A is F-homogeneous. Assume now that $\tau \notin \Phi_A^{F,d-1}$ and let $L = F + \frac{q}{p}V$ where $\mathfrak{V} = \overline{\tau}$ in (3.1.1). For $i \in \tau$, $\mathbf{a}_i^L = \mathbf{a}_i$ and hence $\ell(\mathbf{a}_i^L) = 1$. But for $i \notin \tau$ and $p/q \to 0$, $\mathbf{a}_i^L \to 0$ and hence eventually $\ell(\mathbf{a}_i^L) < 1$. Thus, $\tau \in \Phi_A^{L,d-1}$ for $p/q \to 0$ while $\tau \notin \Phi_A^{F,d-1}$ and hence $L \mapsto \Phi_A^L$ can not be constant. Then, by Corollary 3.15, $M_A(\beta)$ has slopes. By [LM99, Thm. 2.4.2], $M_A(\beta)$ is hence irregular. ☐

We will see later in Corollary 4.18 that Corollary 3.15 actually gives a complete list of all slopes along coordinate varieties at the origin.

Remark 3.17. For generic β, the equivalence of the regularity of $M_A(\beta)$ with the homogeneity of A was already obtained in [SST00, Thm. 2.4.11].

4. Characteristic cycle of the Euler–Koszul complex

In this section we assume that $L_x + L_\partial = \mathbf{c}$ is a constant positive vector as in (3.2.1). By way of a dilation we further may, and do, assume that the index set of the L-filtration is $\mathbb{Z}$.

By Proposition 3.8, all components of the L-characteristic variety of $M_A(\beta)$ are of the form $\bar{C}_A^\tau$ where $\tau \in \Phi_A^L$. In this section we prepare the way for Corollary 4.17 which ascertains the presence of every such candidate component. The approach is to consider $M_A(\beta)$ as the 0-th homology of a Koszul type complex by operators $E-\beta$ on the D-module $D \otimes_R N$ for the $\mathbb{Z}^d$-graded R-module $N = S_A$. For modules N having a composition series with quotients of type S_θ, $\theta \in \Phi_A^{\mathbf{0}}$, we can apply results in Sections 2 and 3 combined with homological algebra. For basic results on filtered rings and modules we refer to [Sch85, Ch. II, §§1.1-1.3].

4.1. **Good filtrations and toric modules.** In order to combine homological methods with good L-filtrations on D-modules, we need the L-filtration on D to be Noetherian, which means the following.

The *Rees ring* of a ring T with a $\mathbb{Z}$-indexed filtration F is the graded ring

$$\mathrm{Rees}^F(T) = \bigoplus_{i \in \mathbb{Z}} F_i(T)t^i \subseteq T[t, t^{-1}]$$

and F is called *Noetherian* if $\mathrm{Rees}^F(T)$ is Noetherian.

Lemma 4.1. *The filtrations L on D and L_∂ on R are Noetherian.*

Proof. Let us write $(\mathbf{u}, \mathbf{v}) := (L_x, L_\partial)$, then $\mathbf{u} + \mathbf{v} = \mathbf{c} > 0$ by hypothesis. The elements of $\mathrm{Rees}^L(D)$ are of the form $\sum_{k \in \mathbb{Z}} \sum_{\mathbf{u} \cdot \mathbf{u}' + \mathbf{v} \cdot \mathbf{v}' \leq k} c_{\mathbf{u}', \mathbf{v}', k} x^{\mathbf{u}'} \partial^{\mathbf{v}'} t^k$. Apply the change of variables $x_i \mapsto x_i t^{-u_i}$, $\partial_i \mapsto \partial_i t^{u_i}$ and set $m = k + \mathbf{u} \cdot (\mathbf{v}' - \mathbf{u}')$. Then the elements are transformed to $\sum_{m \in \mathbb{Z}} \sum_{(\mathbf{v}+\mathbf{u}) \cdot \mathbf{v}' \leq m} x^{\mathbf{u}'} \partial^{\mathbf{v}'} t^m$. Thus, $\mathrm{Rees}^L(D) \cong \mathrm{Rees}^{F'}(D)$ where $F' := (0, \mathbf{v} + \mathbf{u}) = (0, \mathbf{c})$, which is Noetherian by [Sch85, Ch. II, Prop. 1.1.8] since $F'_0 D = \mathbb{C}[x]$ is Noetherian. Similarly, a change of variables $\partial_i \mapsto \partial_i t^{1-v_i}$ shows that $\mathrm{Rees}^{L_\partial}(R) \cong \mathrm{Rees}^{F_\partial}(R)$ is Noetherian. ☐

The *Rees module* of a T-module N with an F-filtration G is the graded $\mathrm{Rees}^F(T)$-module

$$\mathrm{Rees}^G(N) = \bigoplus_{i \in \mathbb{Z}} G_i(N)t^i \subseteq N[t, t^{-1}].$$

For Noetherian F, G is a good F-filtration on N in the sense of Subsection 3.1 if and only if $\mathrm{Rees}^G(N)$ is Noetherian over $\mathrm{Rees}^F(T)$, (see [Sch85, Ch. II, Prop. 1.1.7]). The advantage of this new definition is the following. Consider a short exact sequence of T-modules

$$0 \longrightarrow N' \longrightarrow N \longrightarrow N'' \longrightarrow 0\,, \tag{4.1.1}$$

a good F-filtration G on N, and the induced F-filtrations G' on N' and G'' on N''. Then the associated Rees sequence

$$0 \longrightarrow \mathrm{Rees}^{G'}(N') \longrightarrow \mathrm{Rees}^{G}(N) \longrightarrow \mathrm{Rees}^{G''}(N'') \longrightarrow 0$$

is exact and hence G' and G'' are good according to the new definition. By definition of G' and G'', the maps in the sequence (4.1.1) are strict and hence

$$0 \longrightarrow \mathrm{gr}^{G'}(N') \longrightarrow \mathrm{gr}^{G}(N) \longrightarrow \mathrm{gr}^{G''}(N'') \longrightarrow 0$$

is exact.

For D-modules of the type $M = D \otimes_R N$ where N is an R-module, a good L_∂-filtration on N induces a good L-filtration on M by

$$L_k(D \otimes_R N) = \sum_{i+j=k} (L_x)_i \mathbb{C}[x] \otimes_{\mathbb{C}} (L_\partial)_j N. \tag{4.1.2}$$

Here, $(L_x)_i\mathbb{C}[x]$ denotes the level-i piece of the filtration L_x on $\mathbb{C}[x]$. For many purposes, we can replace a given good L-filtration on M by that in (4.1.2) and then Lemma 3.6 generalizes as follows.

Lemma 4.2. *For any R-module N, $\mathrm{gr}^L(D \otimes_R N) = W \otimes_R \mathrm{gr}^{L_\partial}(N)$ as W-modules.* □

The following definition (see [MMW05, Def. 4.5]) describes a class of R-modules that arise naturally in the study of A-hypergeometric systems. Recall that, by Theorem 2.14, the $\mathbb{Z}^d$-graded prime ideals containing I_A are of the form I_A^θ for $\theta \in \Phi_A^{\mathbf{0}}$.

Definition 4.3. A $\mathbb{Z}^d$-graded R-module N is *toric* if it has a *toric filtration*

$$0 = N_0 \subsetneq N_1 \subsetneq \cdots \subsetneq N_{l-1} \subsetneq N_l = N$$

which means that for all i, $N_i/N_{i-1} = S_{\theta_i}(\mathbf{u}_i)$ for some $\theta_i \in \Phi_A^{\mathbf{0}}$ and some $\mathbf{u}_i \in \mathbb{Z}^d$. The minimal such l is called the *toric length* of N. A *toric morphism* is a $\mathbb{Z}^d$-graded R-linear map between toric modules.

4.2. **Characteristic cycle and Euler–Koszul homology.** Recall that by Theorem 3.2 any component of the L-characteristic variety $\mathrm{Ch}^L(M)$ of a finite D-module M is at least n-dimensional.

Definition 4.4. The *L-characteristic cycle* of an L-holonomic D-module M is the formal sum of n-dimensional irreducible varieties in T^*X

$$\mathrm{CC}^L(M) := \sum_C \mu^{L,C}(M) \cdot C, \quad \mu^{L,C}(M) := \ell\big(\mathrm{gr}^L(M)_{P_C}\big),$$

for some good L-filtration on M.

A bounded complex of D-modules $K_\bullet$ is called *homologically L-holonomic* if all its homology modules $H_i(K_\bullet)$ are L-holonomic. The *L-characteristic cycle* of a homologically L-holonomic complex of finite D-modules $K_\bullet$ is

$$\mathrm{CC}^L(K_\bullet) := \sum_{i\in\mathbb{Z}}(-1)^i\,\mathrm{CC}^L(H_i(K_\bullet)) = \sum_C \mu^{L,C}(K_\bullet)\cdot C,$$

where

$$\mu^{L,C}(K_\bullet) := \sum_{i\in\mathbb{Z}}(-1)^i\mu_i^{L,C}(K_\bullet),\quad \mu_i^{L,C}(K_\bullet) := \ell\big(\mathrm{gr}^L(H_i(K_\bullet))_{P_C}\big).$$

For an L-holonomic D-module M considered as a complex with trivial differential concentrated in homological degree zero, the two definitions of $\mathrm{CC}^L(M)$ coincide.

A D-module is holonomic in the usual sense if it is L-holonomic for $L = F$. The independence of $\mathrm{CC}^L(M)$ of the particular choice of the good L-filtration follows from [Sch85, Ch. II, Prop. 1.3.1.a]. Essentially by definition, the L-characteristic variety $\mathrm{Ch}^L(M)$ of an L-holonomic M in Definition 3.1 is the union

$$\mathrm{Ch}^L(M) = \bigcup_{\mu^{L,C}(M)>0} C.$$

For $L = pF + qV$ as in (3.1.2), $\mathrm{CC}^L(M)$ is the global algebraic version of Y. Laurent's *microcharacteristic cycle* $\widetilde{\Sigma}^r_\Lambda(\mathcal{M})$ *of type* $r = p/q + 1$ in $T^*\Lambda$ with $\Lambda = T^*_Y X$ whose support is $\Sigma^r_\Lambda(\mathcal{M})$ (see Remark 3.3). Lemma 4.1 and [Sch85, Ch. II, Prop. 1.3.1.b] yield the following statement.

Lemma 4.5. *The L-characteristic cycle* CC^L *is additive.* □

Our main technical tool for the study of $\mathrm{CC}(M_A(\beta))$ is the Euler–Koszul functor from [MMW05]. For a $\mathbb{Z}^d$-graded left D-module M and a $\mathbb{Z}^d$-homogeneous $y \in M$,

$$(E_i - \beta_i)\circ y := (E_i - \beta_i + \deg_i(y))y \tag{4.2.1}$$

defines d commuting D-linear endomorphisms $E - \beta$ of M.

Definition 4.6. Let N be a $\mathbb{Z}^d$-graded R-module and $\beta \in \mathbb{C}^d$. The *Euler–Koszul complex* $K_{A,\bullet}(N;\beta)$ is the Koszul complex of the endomorphisms $E-\beta$ on the left D-module $D\otimes_R N$. Its i-th homology $H_{A,i}(N;\beta) := H_i(K_{A,\bullet}(N;\beta))$ is the i-th *Euler–Koszul homology* of N.

To a good L_∂-filtration on a $\mathbb{Z}^d$-graded R-module N we associate a good L-filtration on $K_{A,\bullet}(N;\beta)$ by setting for $i_1 < \cdots < i_k$ the L-degree of the $(i_1,\ldots,i_k)$-th unit vector in $K_{A,k}(R;\beta) = D^{\binom{n}{k}}$ equal to $\sum_{t=1}^k \deg^L(E_{i_t})$, and using equation (4.1.2). Then Lemma 4.2 implies that

$$\mathrm{gr}^L(K_{A,\bullet}(N;\beta)) = W\otimes_R K_\bullet(\mathrm{gr}^L(N);E), \tag{4.2.2}$$

by which, abusing notation, we mean the usual Koszul complex induced by the L-symbols of E on $W\otimes_R \mathrm{gr}^L(N)$.

By [MMW05, Thm. 5.1], $K_{A,\bullet}(N;\beta)$ is homologically F-holonomic for all toric modules N. Using Proposition 3.8 and Lemma 4.5, one shows by induction on the toric length of N combined with a spectral sequence argument as in the proof of Theorem 4.11 that $K_{A,\bullet}(N;\beta)$ is homologically L-holonomic for all toric modules N and for all L.

Definition 4.7. For a toric module N, we call

$$\mathrm{CC}^L(N) := \mathrm{CC}^L(K_{A,\bullet}(N;\beta)) = \sum_{i=0}^{d}(-1)^i\,\mathrm{CC}^L(H_{A,i}(N;\beta))$$

the *L-characteristic Euler–Koszul cycle* of N. Recall Definition 3.9. We denote the *i-th Euler–Koszul L-multiplicity* and the *Euler–Koszul L-characteristic* of N along C by

$$\mu_{A,i}^{L,C}(N;\beta) := \mu_i^{L,C}(K_{A,\bullet}(N;\beta)),\quad \mu_A^{L,C}(N) := \mu^{L,C}(K_{A,\bullet}(N;\beta)).$$

If $N = S_A$ we skip the argument N, while if $C = \bar{C}_A^\tau$ we write τ rather than $\bar{C}_A^\tau$ as upper index.

The independence of β suggested by the notation $\mu_A^{L,C}(N)$ will be established in Theorem 4.11 below. Lemma 4.5 and the long exact Euler–Koszul homology sequence imply the following statement.

Lemma 4.8. *The Euler–Koszul L-characteristic $\mu_A^{L,C}$ along C is additive.* □

4.3. **Euler–Koszul characteristic and intersection multiplicity.** We now interpret $\mu_A^{L,\tau}$ as an intersection multiplicity (see [Ser65, Ch. V, §B]). That will lead to an explicit formula in Theorem 4.21.

Definition 4.9. For two W-modules M and M' and a variety $C \subsetneq \mathrm{Spec}(W)$ not strictly contained in an irreducible component of $\mathrm{supp}(M \otimes_W M')$, the *intersection multiplicity of M and M' along C* is the alternating sum

$$\chi^C(M, M') := \sum_i (-1)^i \ell\left(\mathrm{Tor}_i^{W_{P_C}}(M_{P_C}, M'_{P_C})\right)$$

of lengths of W_{P_C}-modules. If $C = \bar{C}_A^\tau$ we write τ rather than $\bar{C}_A^\tau$ as upper index.

Lemmas 4.1 and 4.2 and [Sch85, Ch. II, Prop. 1.3.1.b] yield the following statements.

Lemma 4.10. *Fix a W-module M and a variety $C \subseteq \mathrm{Spec}(W)$. Whenever defined, the quantity $\chi^C(M, W \otimes_R \mathrm{gr}^L(N))$ is independent of the choice of a good L-filtration on the R-module N and additive in N.* □

Theorem 4.11. *For any toric module N and all $\beta \in \mathbb{C}^d$,*

$$\mu_A^{L,C}(N) = \chi^C(W/W\langle E\rangle, W \otimes_R \mathrm{gr}^L(N))$$

and is hence independent of β. For any $i \in \mathbb{N}$, $\mu_{A,i}^{L,C}(N;\beta) > 0$ only if $C = \bar{C}_A^\tau$ for some $\tau \in \Phi_A^L$, while $\mu_A^{L,C}(N) > 0$ if and only if $\dim(N) = d$ and $C = \bar{C}_A^\tau$ for some $\tau \in \Phi_A^L$.

Proof. We consider $K_\bullet = K_{A,\bullet}(N;\beta)$ and fix a variety C for which $\mu_A^{L,C}(N)$ is defined. By Corollary A.2 and equation (4.2.2), there is a sequence of homologically graded W_{P_C}-modules $K^1, \dots, K^r$ with W_{P_C}-linear endomorphisms $d^i \colon K^i \to K^i$ such that

$$K^1 = W_{P_C} \otimes_R K_\bullet(\mathrm{gr}^L(N); E),$$

$K^{i+1} = H(K^i, d^i)$ for $i = 1, \dots, r-1$, and $K^r = \mathrm{gr}^L(H_{A,\bullet}(N;\beta))_{P_C}$ as homologically graded modules. Thus for all $\beta \in \mathbb{C}^d$, $\mu_A^{L,C}(N) = \chi(K^r) = \cdots = \chi(K^1)$ is independent of β by additivity of $\chi(K) := \sum_i (-1)^i \ell(H_i(K,d))$. Since E is a regular sequence in W, Koszul homology against E computes Tor over both W and W_{P_C} and hence

$$\begin{aligned}\chi(K^1) &= \sum_{i=1}^{d}(-1)^i \ell\left(\operatorname{Tor}_i^{W_{P_C}}(W_{P_C}/W_{P_C}\langle E\rangle, W_{P_C}\otimes_R \operatorname{gr}^L(N))\right)\\ &= \chi^C(W/W\langle E\rangle, W\otimes_R \operatorname{gr}^L(N)).\end{aligned}$$

By Lemmas 4.8 and 4.10 we may assume now that $N = S_\theta$ for some $\theta \in \Phi_A^{\mathbf{0}}$ with the standard good L_∂-filtration induced by the single generator $1 \in S_\theta$. Then the complex K^1 above is the Koszul complex of E on $W_{P_C}\otimes_R S_\theta^L$. By [BH93, Prop. 1.6.5], its homology is annihilated by both $\operatorname{gr}^L(I_A^\theta) = R\cdot I_\theta^L + J_\theta \supseteq I_A^L$ and E. Then equation (3.2.2) shows that $\mu_{A,i}^{L,C}(S_\theta;\beta) > 0$ for some i implies that $C = \bar{C}_A^\tau$ is an irreducible component of $\operatorname{Spec}(W/W\langle E\rangle \otimes_R S_A^L)$ with $\theta \supseteq \tau \in \Phi_A^L$. By Theorem 2.14,

$$\operatorname{codim}(W/W\langle E\rangle) + \operatorname{codim}(W\otimes_R S_\theta^L) = d + n - \dim(S_\theta) \geq n = \operatorname{codim}(\bar{C}_A^\tau).$$

So Serre's Intersection Theorem [Ser65, Ch. V, §C.1, Thm. 1] shows that $\mu_A^{L,\tau}(S_\theta) > 0$ if and only if $\dim(S_\theta) = d$, which is to say $\theta = A$. □

In the special case $L = F$, Theorem 4.11 yields the holonomicity of Euler–Koszul homology of toric modules proved in [MMW05, Thm. 5.1].

In order to discuss $\operatorname{CC}^L(M_A(\beta))$ we need more information about $\mu_{A,0}^{L,\tau}(\beta)$: Theorem 4.11 does not state that $\mu_{A,0}^{L,\tau}(\beta) > 0$. We show next that for generic β this inequality does indeed hold. In Subsections 4.4 and 4.6 we will show that the genericity assumption is not necessary and in fact identify a global combinatorial lower bound for $\mu_{A,0}^{L,\tau}(\beta)$, similar to classical statements about the holonomic rank of $M_A(\beta)$ in [GZK89].

Corollary 4.12. *For generic (more precisely, not rank-jumping) $\beta \in \mathbb{C}^d$*

$$\mu_{A,0}^{L,\tau}(\beta) = \mu_A^{L,\tau} = \chi^\tau(W/W\langle E\rangle, W\otimes_R S_A^L) > 0$$

for all $\tau \in \Phi_A^L$. In particular, $\operatorname{Ch}^L(M_A(\beta)) = \bigcup_{\tau\in\Phi_A^L} \bar{C}_A^\tau$.

Proof. By assumption on β and [MMW05, Thm. 6.6], the Euler–Koszul complex $K_{A,\bullet}(S_A;\beta)$ is a resolution of $M_A(\beta)$. Hence $\operatorname{CC}^L(K_{A,\bullet}(S_A;\beta)) = \operatorname{CC}^L(M_A(\beta))$ and $\mu_{A,0}^{L,\tau}(\beta) = \mu_A^{L,\tau} > 0$ by Theorem 4.11. So the last claim follows from Proposition 3.8. □

Recall that $\tilde{S}_A$ is the normalization of S_A. In Section 4.4, we show that $\mu_{A,0}^{L,\tau}(\beta) > 0$ for all β. This is done by reducing to $\tilde{S}_A$ and applying the following statement.

Corollary 4.13. *For all $\tau \in \Phi_A^L$,*

$$\mu_{A,0}^{L,\tau}(\tilde{S}_A;\beta) = \mu_A^{L,\tau}(\tilde{S}_A) = \mu_A^{L,\tau} = \chi^\tau(W/W\langle E\rangle, W\otimes_R S_A^L) > 0.$$

Proof. Since $\tilde{S}_A$ is a Cohen–Macaulay module of full dimension over S_A, its Euler–Koszul complex is a resolution of $H_{A,0}(\tilde{S}_A;\beta)$ by [MMW05, Thm. 6.6]. Hence $\mu_{A,i}^{L,\tau}(\tilde{S}_A;\beta) = 0$ for all $i > 0$ which proves the first equality. There is a short exact sequence

$$0 \longrightarrow S_A \longrightarrow \tilde{S}_A \longrightarrow Q \longrightarrow 0 \tag{4.3.1}$$

where $\dim(Q) < d$ and the rest of the claim follows from Theorem 4.11 and Corollary 4.12. $\square$

4.4. **Rigidity of Euler–Koszul multiplicities.** In this section we show in Theorem 4.16 that the strict positivity of $\mu^{L,\tau}_{A,0}(\beta)$ in Corollary 4.12 holds without the genericity assumption. As a consequence, we obtain a complete description of the L-characteristic variety and the slopes of the A-hypergeometric system along coordinate varieties at the origin.

Definition 4.14. Let τ be a subset of the column set of A. We set $W_\tau = \mathbb{C}[x_\tau, \partial_\tau]$ so that $W/WI^\tau_A = \mathbb{C}[x_{\overline{\tau}}] \otimes_{\mathbb{C}} (W_\tau \otimes_{R_\tau} S_\tau)$.

Let $g \in \mathrm{Gl}(\mathbb{Z}, d)$ and put $A' = gA$ and $\beta' = g\beta$. Then note that $S_A = S_{A'}$ and $E' - \beta' = g(E - \beta)$. Hence, the Euler–Koszul complexes $K_{A,\bullet}(S_A; \beta)$ and $K_{A',\bullet}(S_{A'}; \beta')$ are homotopy equivalent. In particular, when investigating homotopy-invariant properties of $K_{A,\bullet}(S_A; \beta)$ one may replace A by gA.

Lemma 4.15. *Let $\theta \in \Phi^{\mathbf{0}}_A$ and assume that the top $\dim(\theta)$ rows A' of θ are a lattice basis for the $\mathbb{Z}$-row span of θ. Then, for all $\tau \in \Phi^L_A$,*

$$\mu^{L,\tau}_{A,i}(S_\theta; \beta) = \begin{cases} \mu^{L',\tau'}_{A',i}(S_{\theta'}; \beta') & \text{if } \tau \subseteq \theta; \\ 0 & \text{if } \tau \not\subseteq \theta; \end{cases}$$

where $L' = (L_{x_{A'}}, L_{\partial_{A'}})$, $\tau = \tau'$, $\theta' = A'$, and $\beta' := (\beta_1, \ldots, \beta_{\dim(\theta)})$.

Proof. By [MMW05, Lem. 4.8], $H_{A,i}(S_\theta; \beta) = \mathbb{C}[x_{\overline{\theta}}] \otimes_{\mathbb{C}} H_{A',i}(S_{\theta'}; \beta')$ and hence

$$\mathrm{gr}^L(H_{A,i}(S_\theta; \beta)) = \mathbb{C}[x_{\overline{\theta}}] \otimes_{\mathbb{C}} \mathrm{gr}^{L'}(H_{A',i}(S_{\theta'}; \beta')).$$

For $j \in \tau \smallsetminus \theta$, ∂_j is both annihilator and unit on $\mathrm{gr}^L(H_{A,i}(S_\theta; \beta))_{P_\tau}$ and hence $\tau \subseteq \theta$ if $\mu^{L,\tau}_{A,i}(S_\theta; \beta) \neq 0$. Assume now that $\tau \subseteq \theta$. Then

$$W_{P_\tau} \otimes_W \mathrm{gr}^L(H_{A,i}(S_\theta; \beta)) = \mathbb{C}(x_{\overline{\theta}}) \otimes_{\mathbb{C}} \Big((W_{A'})_{P_\tau} \otimes_{W_{A'}} \mathrm{gr}^{L'}(H_{A',i}(S_{\theta'}; \beta'))\Big)$$

where the symbol P_τ is used for the two ideals in W_A and $W_{A'}$ from Definition 3.7 induced by $\tau \in \Phi^L_A$ and $\tau \in \Phi^{L'}_{A'}$. In particular, the length of $W_{P_\tau} \otimes_W \mathrm{gr}^L(H_{A,i}(S_\theta; \beta))$ over W_{P_τ} is the same as the length of $(W_{A'})_{P_\tau} \otimes_{W_{A'}} \mathrm{gr}^{L'}(H_{A',i}(S_{\theta'}; \beta'))$ over $(W_{A'})_{P_\tau}$. $\square$

Theorem 4.16. *For any toric module N and for every parameter $\beta \in \mathbb{C}^d$, if $\mu^{L,\tau}_{A,i}(N; \beta) > 0$ for some $i > 0$ then $\mu^{L,\tau}_{A,0}(N; \beta) > 0$.*

Proof. Suppose there is a counterexample (A, β, L, τ, N) to the theorem. This means that $\mu^{L,\tau}_{A,i}(N; \beta) \neq 0$ for some $i > 0$ while $\mu^{L,\tau}_{A,0}(N; \beta) = 0$. We choose a minimal counterexample in the sense that

1. A has minimal number of columns,
2. for a fixed such A, $\dim(N)$ is minimal, and
3. no quotient of N is part of such a counterexample.

This last choice can be made since toric modules are Noetherian. Let $\theta \in \Phi^{\mathbf{0}}_A$ such that I^θ_A is an associated prime of N. Then there is a short exact sequence

$$0 \longrightarrow S_\theta(-\mathbf{a}) \longrightarrow N \longrightarrow N' \longrightarrow 0$$

where $\mathbf{a} \in \mathbb{Z}^d$ is the degree of the image of $1 \in S_\theta$ within N. The long exact Euler–Koszul homology sequence shows that $H_{A,0}(N'; \beta)$ is a quotient of $H_{A,0}(N; \beta)$.

Recall that by Lemma 4.1 the L-characteristic cycle is additive. Since by hypothesis $\mu^{L,\tau}_{A,0}(N;\beta)=0$, we conclude that $\mu^{L,\tau}_{A,0}(N';\beta)=0$ as well.

By assumption, (A,β,L,τ,N') is not a counterexample and so $\mu^{L,\tau}_{A,i}(N';\beta)=0$ for all i. It follows that the $\mu^{L,\tau}_{A,i}$-vanishing patterns of N and of $S_\theta(-\mathbf{a})$ coincide. In particular, $S_\theta(-\mathbf{a})$ is a counterexample. Since $\dim(S_\theta)\le\dim(N)$, and since S_θ is a domain, $S_\theta(-\mathbf{a})$ is actually a minimal counterexample. Since $H_{A,i}(S_\theta(-\mathbf{a});\beta)=H_{A,i}(S_\theta;\beta-\mathbf{a})$ up to a $\mathbb{Z}^d$-shift we may assume that $N=S_\theta$. But if $\theta\neq A$ then Lemma 4.15 yields a counterexample whose matrix A' has strictly fewer columns than A. Therefore $\theta=A$ by minimality of A and (A,β,L,τ,S_A) is a minimal counterexample.

By Corollary 4.13, $\mu^{L,\tau}_{A,0}(\tilde S_A;\beta)>0$ where $\tilde S_A$ is the normalization of S_A. For appropriate $\mathbf{a}\in\mathbb{Z}^d$, there is a short exact sequence

$$0\longrightarrow \tilde S_A(-\mathbf{a})\longrightarrow S_A\longrightarrow Q\longrightarrow 0$$

where Q is not a counterexample as $\dim(Q)<d$. By additivity of the L-characteristic cycle, $\mu^{L,\tau}_{A,0}(S_A;\beta)=0$ implies $\mu^{L,\tau}_{A,0}(Q;\beta)=0$ and so, as Q is not a counterexample, $\mu^{L,\tau}_{A,i}(Q;\beta)=0$ for all i. This yields $\mu^{L,\tau}_{A,i}(S_A;\beta)=\mu^{L,\tau}_{A,i}(\tilde S_A;\beta)$ for all i which is a contradiction for $i=0$. $\square$

Finally, from Theorem 4.16, Corollary 4.12 and Theorem 4.11, we obtain the L-characteristic variety of the A-hypergeometric system and, as a consequence, the slopes of the A-hypergeometric system along coordinate varieties at the origin.

Corollary 4.17. *For all $\tau\in\Phi^L_A$ and $\beta\in\mathbb{C}^d$, $\mu^{L,\tau}_{A,0}(\beta)>0$. In consequence, the L-characteristic variety of $M_A(\beta)$ consists for every β precisely of all conormal closures of the torus orbits indexed by the faces in the (A,L)-umbrella:*

$$\mathrm{Ch}^L(M_A(\beta))=\bigcup_{\tau\in\Phi^L_A}\bar C^\tau_A. \qquad\square$$

Corollary 4.18. *For $L=pF+qV$ as in* (3.1.2)*, $L'=p'/q'$ is a slope of $M_A(\beta)$ along Y at $0\in Y$ if and only if Φ^L_A jumps at $L=L'$.* $\square$

4.5. **Euler–Koszul characteristic and volume.** In this section we develop an explicit combinatorial formula for the Euler–Koszul characteristic $\mu^{L,\tau}_A$ for all $\tau\in\Phi^L_A$, Theorem 4.21 below. For generic β, this formula determines the characteristic cycle of $M_A(\beta)$ as stated in Corollary 4.17 below, but in general it provides only a lower bound.

Note that $\mathrm{rk}(M_A(\beta))=\mu^{F,\emptyset}_{A,0}(\beta)$, and that $\mu^{L,\tau}_{A,0}(\beta)=\mu^{L,\tau}_A$ if β is not a rank-jumping parameter by [MMW05, Thm. 6.6]. It is a classical result that $\mathrm{rk}(M_A(\beta))=\mathrm{vol}_{\mathbb{Z}^d}(\Delta^F_A)$ for generic β (see [GZK89, Ado94]). Theorem 4.21 below contains this result as a special case when $\tau=\emptyset$ and $L=F$. Our generalization of the fact that generic rank equals volume reads

$$\mu^{L,\emptyset}_A=\mathrm{vol}_{\mathbb{Z}^d}\left(\bigcup_{\tau\in\Phi^{L,d-1}_A}(\Delta^F_0\smallsetminus\mathrm{conv}(\tau))\right).$$

This paragraph outlines our strategy towards Theorem 4.21. By Theorem 4.11, $\mu_A^{L,\tau}$ is an intersection multiplicity and hence additive. Applying $\mu_A^{L,\tau}$ to the composition chain (2.7.1) of S_A^L yields

$$
\begin{aligned}
\mu_A^{L,\tau} = \chi^\tau(W/\langle E\rangle, S_A^L) &= \sum_{i=1}^{l} \chi^\tau(W/\langle E\rangle, S_{\tau_i}) \\
&= \sum_{\tau\subseteq\tau'\in\Phi_A^{L,d-1}} \nu_A^{L,\tau'}\cdot\chi^\tau(W/\langle E\rangle, S_{\tau'}) \\
&= \sum_{\tau\subseteq\tau'\in\Phi_A^{L,d-1}} [\mathbb{Z}^d : \mathbb{Z}\tau']\cdot\mu_A^{L,\tau}(S_{\tau'}).
\end{aligned} \tag{4.5.1}
$$

For the third equality note that $(R/I_A^{\tau'})_{I_A^\tau} = 0$ unless $\tau\subseteq\tau'$, and then use $\dim(S_{\tau'}) = \dim(\tau') + 1$ in Theorem 4.11. Proposition 2.16 gives the last equality. By equation (4.5.1) it is sufficient to know the multiplicities $\mu_A^{L,\tau}(S_{\tau'})$ for $\tau\subseteq\tau'\in\Phi_A^{L,d-1}$. To determine these we first reduce to $\tau' = A$ and then to $\tau = \emptyset$ while controlling the appropriate intersection multiplicity. Finally, we give a combinatorial description in that case. Some of our arguments are similar to [GZK89, §§2.3–2.4] and [Ado94, §5].

Let $\Phi_A^L \ni \tau\subseteq\tau'\in\Phi_A^{L,d-1}$ and abbreviate $\mu := \mu_A^{L,\tau}(S_{\tau'}) = \chi^\tau(W/\langle E\rangle, S_{\tau'})$. Since E is a regular sequence in W, μ can be computed from the Koszul complex of E on $W_{P_\tau}\otimes_R S_{\tau'}$. Since $\partial_{\overline{\tau'}}$ is zero on $S_{\tau'}$ one can erase the $\partial_{\overline{\tau}}$-terms in E. Now the (canonical basis of the) ambient lattice $\mathbb{Z}^d = \mathbb{Z}A$ can be replaced by (a lattice basis of) $\mathbb{Z}\tau'$; this leaves the quantities W, $\langle E\rangle$, I_τ and $S_{\tau'}$ invariant. We have thus reduced the problem to the case $\tau' = A$.

We now reduce to the case $\tau = \emptyset$. By the Fourier transform we may use ∂ as base variables and identify $-x_j\in W$ with the (symbol of the) partial derivation along ∂_j. Since χ^τ is determined at a generic point of C_A^τ, we may replace W by $W' = W[\partial_\tau^{-1}]$ whose spectrum T^*U is the cotangent space of $U = \operatorname{Spec}(R[\partial_\tau^{-1}])$.

We now modify the pair (τ',τ) into a more convenient one, while keeping track of μ. A row operation $(\hat\tau',\hat\tau) = g(\tau',\tau)$ defined by $g\in\mathrm{Gl}(\mathbb{Z},d)$ amounts to a basis change in the d-torus $\mathbb{T}$, so $\mu = \mu_{\hat\tau'}^{L,\hat\tau}(S_{\hat\tau'})$. On the other hand, for a triple $j\in\tau'$, $i\in\tau$, $m\in\mathbb{Z}$, the elementary column operation $\hat{\mathbf{a}}_j = \mathbf{a}_j + m\mathbf{a}_i$ transforming τ' into $\hat\tau'$ corresponds to the automorphism $\hat\partial_j = \partial_j\partial_i^m$ of U. The induced map on W' obviously transforms $I_{\tau'}$ into $I_{\hat\tau'}$. Note that $x_i\partial_i = \hat x_i\hat\partial_i + m\hat x_j\hat\partial_j$ and $x_k\partial_k = \hat x_k\hat\partial_k$ for $k\neq i$. Thus, $\mu = \mu_{\hat\tau'}^{L,\hat\tau}(S_{\hat\tau'})$. In other words, all elementary column operations $\mathbf{a}_j\mapsto\mathbf{a}_j + m\mathbf{a}_i$ with $i\in\tau$, all column switches $\mathbf{a}_i\leftrightarrow\mathbf{a}_j$ with $i,j\in\tau$ or $i,j\notin\tau$, and all $\mathbb{Z}$-invertible row operations leave μ invariant.

After a suitable sequence of such transformations, we may assume that

$$
\tau = \begin{pmatrix} I & 0\\ 0 & 0\end{pmatrix},\quad I = \begin{pmatrix} k_1 & & 0\\ & \ddots & \\ 0 & & k_e\end{pmatrix}
$$

where $k_i\in\mathbb{N}\smallsetminus\{0\}$. Note that the product $k_1\cdots k_e$ is the index $[(\mathbb{Z}\tau'\cap\mathbb{Q}\tau):\mathbb{Z}\tau]$ of $\mathbb{Z}\tau$ in $\mathbb{Z}\tau'\cap\mathbb{Q}\tau$.

We next consider the k_1-fold covering space $\kappa_1\colon U\to U$ induced by $\hat\partial_1^{k_1} = \partial_1$. Let $(\hat\tau,\hat\tau')$ be the matrices obtained from (τ,τ') by dividing the first column by k_1

and let $\hat{E}$ be the Euler vector fields of $\hat{\tau}$. As one checks, $\hat{x}_1\hat{\partial}_1 = k_1x_1\partial_1$ and so $\kappa_1(\mathrm{Var}(\hat{E})) = \mathrm{Var}(E)$, $\kappa_1(\mathrm{Var}(I_{\hat{\tau}'})) = \mathrm{Var}(I_{\tau'})$, and $\kappa_1(C_A^{\hat{\tau}}) = C_A^{\tau}$. Moreover, the degree of κ_1 is equal to k_1 on C_A^{τ}, and equal to 1 on both $\mathrm{Var}(E)$ and $\mathrm{Var}(I_{\tau'})$. By Example 8.2.5 in [Ful98],

$$\mu = k_1 \cdot \chi^{\hat{\tau}}(W/\langle\hat{E}\rangle, S_{\hat{\tau}'}).$$

Analogous transformations for $k_2, \ldots, k_e$ followed by suitable elementary column operations as above yield

$$\hat{\tau} = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}, \quad \hat{\tau}' = \begin{pmatrix} I & 0 & 0 \\ 0 & 0 & \check{\tau} \end{pmatrix}, \quad I = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix},$$

where now

$$\mu = k_1 \cdots k_e \cdot \chi^{\hat{\tau}}(W/E_{\hat{\tau}'}, S_{\hat{\tau}'}) = [(\mathbb{Z}\tau' \cap \mathbb{Q}\tau) : \mathbb{Z}\tau] \cdot \chi^{\hat{\tau}}(W/E_{\hat{\tau}'}, S_{\hat{\tau}'}).$$

Note that $\check{\tau}$ is pointed since τ is a face of τ'. Let $\check{E}$ be the Euler vector fields of $\check{\tau}$. Then on T^*U, $\langle E\rangle = \langle \hat{x}_1\hat{\partial}_1, \ldots, \hat{x}_e\hat{\partial}_e, \check{E}\rangle = \langle \hat{x}_1, \ldots, \hat{x}_e, \check{E}\rangle$ as well as $I_{\hat{\tau}'} = \langle \hat{\partial}_{e+1}, \ldots, \hat{\partial}_{|\tau|}\rangle + I_{\check{\tau}}$. This establishes the announced reduction to the case $\tau = \emptyset$,

$$\mu = [(\mathbb{Z}\tau' \cap \mathbb{Q}\tau) : \mathbb{Z}\tau] \cdot \chi^{\emptyset}(W/\langle\check{E}\rangle, S_{\check{\tau}}) = [(\mathbb{Z}\tau' \cap \mathbb{Q}\tau) : \mathbb{Z}\tau] \cdot \mu_{\check{\tau}}^{L,\emptyset}(S_{\check{\tau}}).$$

The constructed lattice $\mathbb{Z}\check{\tau} \subseteq \mathbb{Z}\tau' \subseteq \mathbb{Z}^d$ corresponds to a splitting of the natural projection

$$\pi_{\tau,\tau'} \colon \mathbb{Z}\tau' \twoheadrightarrow \mathbb{Z}\tau'/(\mathbb{Z}\tau' \cap \mathbb{Q}\tau)$$

which identifies the positive semigroups $\mathbb{N}\check{\tau}$ and $\pi_{\tau,\tau'}(\mathbb{N}\tau')$.

Definition 4.19. In a lattice Λ, the volume function vol_Λ is normalized so that the unit simplex of Λ has volume 1. We abbreviate $\mathrm{vol}_{\tau,\tau'} := \mathrm{vol}_{\pi_{\tau,\tau'}(\mathbb{Z}\tau')}$.

We continue under the assumption that $\tau = \emptyset$, $\check{\tau} = \tau' = A$, and compute the intersection multiplicity of $\mathrm{Var}(E)$ with $\mathrm{Var}(I_A)$ along $C_A^{\emptyset} = X$ inside $T^*U = \mathrm{Spec}(W')$,

$$\check{\mu} := \mu_{\check{\tau}}^{L,\emptyset}(S_{\check{\tau}}) = \chi^{\emptyset}(W/\langle\check{E}\rangle, S_{\check{\tau}}).$$

Since for $y \in (\mathbb{C}^*)^{|\check{\tau}|}$ the linear polynomials $x_1 - y_1, \ldots, x_{|\check{\tau}|} - y_{|\check{\tau}|}$ form a regular sequence on $W/\langle E\rangle$, on $S_{\check{\tau}}$ and on $W/P_\emptyset$, we may replace W by R, E_i by $\bar{E}_i := \sum_j a_{i,j}y_j\partial_j$, and $C_A^{\emptyset}$ by $\{0\} = \mathrm{Var}(\partial_{\check{\tau}})$. By the sequence (4.3.1), we may further replace $S_{\check{\tau}}$ by its normalization $\tilde{S}_{\check{\tau}}$. Thus, we have reduced to computing

$$\check{\mu} = \chi^{\{0\}}(R/\langle\bar{E}\rangle, \tilde{S}_{\check{\tau}}).$$

For generic y, the function on $\mathrm{Spec}(\tilde{S}_{\check{\tau}})$

$$f := \sum_{j=1}^{|\check{\tau}|} y_j\partial_j = \sum_{j=1}^{|\check{\tau}|} y_j t^{\mathbf{a}_j}$$

is by [Kou76, Thm. 6.1] and its proof Newton nondegenerate at the origin. We can interpret $\bar{E}$ as functions on $\mathrm{Spec}(\tilde{S}_{\check{\tau}})$,

$$\bar{E}_i = t_i\frac{\partial f}{\partial t_i}.$$

Let $\mathfrak{m}_{\check\tau} \subseteq S_{\check\tau}$ and $\tilde{\mathfrak{m}}_{\check\tau} \subseteq \tilde S_{\check\tau}$ be the maximal ideals at 0. In the special case where $\check\tau$ is the unit matrix, and where hence $\tilde S_{\check\tau}$ is a polynomial ring, [Kou76, Thm. 2.8] states that the Koszul complex of $\bar E$ on $\tilde S_{\check\tau}$ is acyclic in positive dimension when completed at $\tilde{\mathfrak{m}}_{\check\tau}$. The $\mathfrak{m}_{\check\tau}$- and $\tilde{\mathfrak{m}}_{\check\tau}$-adic topologies on $\tilde S_{\check\tau}$ are equivalent and completion at $\tilde{\mathfrak{m}}_{\check\tau}$ can be replaced by completion at $\mathfrak{m}_{\check\tau}$. By faithful flatness of completion this implies that the Koszul complex of $\bar E$ on $(\tilde S_{\check\tau})_{\mathfrak{m}_{\check\tau}}$ is acyclic in positive dimension and hence

$$\check\mu = \dim_{\mathbb{C}}((\tilde S_{\check\tau})_{\mathfrak{m}_{\check\tau}}/\langle \bar E\rangle).$$

Since this dimension is finite, localization can be replaced by completion at $\mathfrak{m}_{\check\tau}$ or equivalently at $\tilde{\mathfrak{m}}_{\check\tau}$. Then [Kou76, Thm. A.I] states that this dimension equals

$$\check\mu = \mathrm{vol}_{\mathbb{Z}\check\tau}(\Gamma_-(f)), \quad \Gamma_-(f) = \mathrm{conv}(\check\tau \cup \{0\}) \smallsetminus \mathrm{conv}(\check\tau).$$

One checks that Kouchnirenko's result applies to a general normal semigroup ring $\tilde S_{\check\tau}$ for which $\check\tau$ is pointed (see [Kou76, §2.12]). The following definition serves to interpret this volume in terms of the original matrix A, and mirrors the one given in [GZK89, §2.1].

Definition 4.20. For $\Phi_A^L \ni \tau \subseteq \tau' \in \Phi_A^{L,d-1}$, define the polyhedra

$$P_{\tau,\tau'} := \mathrm{conv}(\pi_{\tau,\tau'}(\tau' \cup \{0\})), \quad Q_{\tau,\tau'} := \mathrm{conv}(\pi_{\tau,\tau'}(\tau' \smallsetminus \tau)).$$

We are now ready to give the promised multiplicity formula.

Theorem 4.21. *For all $\tau \in \Phi_A^L$, the multiplicity of $\bar C_A^\tau$ in $\mathrm{CC}^L(K_{A,\bullet}(S_A;\beta))$ is*

$$\mu_A^{L,\tau} = \sum_{\tau\subseteq\tau'\in\Phi_A^{L,d-1}} [\mathbb{Z}^d : \mathbb{Z}\tau'] \cdot [(\mathbb{Z}\tau' \cap \mathbb{Q}\tau) : \mathbb{Z}\tau] \cdot \mathrm{vol}_{\tau,\tau'}(P_{\tau,\tau'} \smallsetminus Q_{\tau,\tau'}). \qquad \square$$

Corollary 4.22. *If $\tau = \emptyset$, then the local degree of S_A^L at the origin equals*

$$\mu_A^{L,\emptyset} = \mathrm{vol}_{\mathbb{Z}^d}\left(\bigcup_{\tau'\in\Phi_A^{L,d-1}} \left(\Delta_{\tau'}^F \smallsetminus \mathrm{conv}(\tau')\right)\right) \le \mathrm{vol}_{\mathbb{Z}^d}(\Delta_A^F).$$

Proof. Let $\tau' \in \Phi_A^{L,d-1}$. For $\tau = \emptyset$, $\mathrm{vol}_{\tau,\tau'} = \mathrm{vol}_{\mathbb{Z}\tau'}$, $P_{\tau,\tau'} = \Delta_{\tau'}^F$, and $Q_{\tau,\tau'} = \mathrm{conv}(\tau')$. Clearly, $[(\mathbb{Q}\tau \cap \mathbb{Z}\tau') : \mathbb{Z}\tau] = 1$ and $\mathrm{vol}_{\mathbb{Z}\tau'}(\Delta_{\tau'}^F) = \mathrm{vol}_{\mathbb{Z}^d}(\Delta_{\tau'}^F)/[\mathbb{Z}^d : \mathbb{Z}\tau']$. Then Theorem 4.21 implies the equality while the inequality is obvious. $\square$

Remark 4.23. If all facets τ' of the (A,L)-umbrella are F-homogeneous then the volumes $\mathrm{vol}_{\mathbb{Z}^d}(\mathrm{conv}(\tau'))$ are zero and one obtains

$$\mu_A^{L,\emptyset} = \mathrm{vol}_{\mathbb{Z}^d}\left(\bigcup_{\tau'\in\Phi_A^{L,d-1}} \Delta_{\tau'}^F\right).$$

In particular, $\mu_A^{F,\emptyset} = \mathrm{vol}_{\mathbb{Z}^d}(\Delta_A^F)$.

Example 4.24. We continue Example 3.12 and investigate characteristic cycles. The following table, whose rows are indexed by the three weight vectors $L = F + tV$ considered in Example 3.12, lists in its columns the nonzero multiplicities $\mu_{A,0}^{L,\tau}(\beta)$ for generic β. Genericity implies that $\mu_{A,i}^{L,\tau}(\beta) = 0$ for $i > 0$, so that $\mu_{A,0}^{L,\tau}(\beta) = \mu_A^{L,\tau}$.

The matrix A permits two rank-jumping parameters, $\mathcal{E}(A) = \{(2,1),(3,1)\}$. In both cases, $H_{A,1}(S_A;\beta) \cong D/\langle\partial_1,\dots,\partial_n\rangle$ while $H_{A,2}(S_A;\beta) \cong 0$. As $\langle\partial\rangle = P_\emptyset$,

FIGURE 3. Euler–Koszul multiplicities in Example 4.24

t \ τ	$\emptyset$	$\{1\}$	$\{2\}$	$\{3\}$	$\{4\}$	$\{1,3\}$	$\{1,4\}$	$\{2,3\}$	$\{2,4\}$	$\{3,4\}$
0	13	12	1		13		12		1	
1	11	3	1	10	8	3			1	7
4	5	3	2	5		3		2		

with $\beta \in \mathcal{E}(A)$ one has $\mu_{A,1}^{L,\tau}(\beta) = 1$ if $\tau = \emptyset$, and $\mu_{A,1}^{L,\tau}(\beta) = 0$ otherwise. (This holds for *all* $L \in \mathbb{Q}^{2n}$ as long as $L_x + L_\partial > 0$ since $\mathrm{gr}^L(\langle \partial_1, \ldots, \partial_n \rangle)$ is independent of L). In accordance with Corollary 4.13, the multiplicities $\mu_{A,0}^{L,\tau}(\beta)$ for $\beta \in \mathcal{E}(A)$ are given by the data in the table above, with $\mu_{A,0}^{L,\emptyset}(\beta)$ incremented by one. This behavior is typical in dimension two as the following proposition shows.

Proposition 4.25. *If $d = 2$ then for every rank-jumping β and for all L with $L_x + L_\partial > 0$ one has $\mu_{A,1}^{L,\tau}(\beta) = 1$ if $\tau = \emptyset$ and $\mu_{A,1}^{L,\tau}(\beta) = 0$ otherwise.*

Proof. Recall that $\tilde{S}_A$ is the Cohen–Macaulay S_A-module $\bigoplus_{\mathbf{a} \in \mathbb{Q}_+ A \cap \mathbb{Z}^d} t^{\mathbf{a}}$. We first construct the minimal toric submodule of $\tilde{S}_A$ that contains S_A and satisfies Serre's condition S_2. To that end, apply the local cohomology functor $H_{\mathfrak{m}}^{\bullet}(-)$ to the sequence (4.3.1); we obtain $H_{\mathfrak{m}}^0(Q) = H_{\mathfrak{m}}^1(S_A)$. Let C be the (toric) preimage of $H_{\mathfrak{m}}^0(Q)$ under the projection $\tilde{S}_A \to Q$; as $C \subseteq \tilde{S}_A$ we have $H_{\mathfrak{m}}^0(C) = 0$. Applying $H_{\mathfrak{m}}^{\bullet}(-)$ to

$$0 \longrightarrow S_A \longrightarrow C \longrightarrow H_{\mathfrak{m}}^0(Q) \longrightarrow 0 \tag{4.5.2}$$

we find an exact piece $0 \to H_{\mathfrak{m}}^0(H_{\mathfrak{m}}^0(Q)) \to H_{\mathfrak{m}}^1(S_A) \to H_{\mathfrak{m}}^1(C) \to H_{\mathfrak{m}}^1(H_{\mathfrak{m}}^0(Q)) = 0$. As the second arrow is an isomorphism by construction, the two-dimensional module C has depth two and is hence Cohen–Macaulay.

Application of the Euler–Koszul functor to the short exact sequence (4.5.2) shows that $H_{A,1}(S_A; \beta) \cong H_{A,2}(H_{\mathfrak{m}}^0(Q); \beta)$. Since $H_{\mathfrak{m}}^0(Q)$ is a toric Artinian quotient of C, it has a toric filtration whose quotients are of the form $R/\langle \partial_1, \ldots, \partial_n \rangle \cdot t^\beta$ where $\beta \in \mathcal{E}(A)$ and each such β occurs exactly once (see [MMW05, Thm. 6.6,Thm .9.1]). Hence, $H_{A,1}(S_A; \beta) \cong D/\langle \partial_1, \ldots, \partial_n \rangle$ if $\beta \in \mathcal{E}(A)$ and zero otherwise. It follows as in the example above that $\mathrm{gr}^L(H_{A,1}(S_A; \beta)) \cong W/\langle \partial_1, \ldots, \partial_n \rangle$ in the nonvanishing case. □

Remark 4.26. (1) The module C constructed in the above proof is actually a ring, the *ideal transform* of S_A relative to $\mathfrak{m}$ from [BS98, Thm. 2.2.4].

(2) It is suggestive, particularly in the light of Example 4.24, to view $\mu_A^{L,\emptyset}$ as "L-rank". Specifically, one might speculate whether the L-rank detects rank jumps in the sense that $\mu_{A,i}^{L,\emptyset}(\beta) = 0$ for all $i > 0$ implies that β is not rank-jumping. We think that this is plausible.

4.6. **Generic Euler–Koszul homology.** In [GZK89], I.M. Gel'fand, M.M. Kapranov and A.V. Zelevinskiĭ proved that in the projective case the rank of $M_A(\beta)$ is always at least equal to $\mathrm{vol}_{\mathbb{Z}^d}(\Delta_A^F) = \mu_A^{F,\emptyset}$, cf. Remark 4.23; this was generalized by A. Adolphson in [Ado94]. In this section, we prove that every Euler–Koszul multiplicity $\mu_{A,0}^{L,\tau}(\beta)$ is bounded from below by the Euler–Koszul characteristic $\mu_A^{L,\tau}$ for all β. This yields some evidence for the following conjecture.

Conjecture 4.27. *For fixed A, L and τ, $\mu^{L,\tau}_{A,0}(\beta)$ is upper semicontinuous in β.*

A proof of this would generalize [MMW05, Thm. 2.6, Thm. 7.5] which correspond to $\tau = \emptyset$ and $L = F$. However, we do not know how to approach this question since it involves a flat deformation in combination with a specialization. The case where $\tau = \emptyset$ and $L = F$ is much easier since there one may skip the computation of the graded by the Cauchy–Kovalevskaya–Kashiwara Theorem (see [SST00, Thm. 1.4.14, Thm. 1.4.19]).

Theorem 4.28. *For any $\tau \in \Phi^L_A$ and $\beta \in \mathbb{C}^d$, $\mu^{L,\tau}_{A,0}(\beta) \geq \mu^{L,\tau}_A$. Equality holds if β is generic (more precisely, not rank-jumping).*

Proof. The sequence (4.3.1) yields a four-step exact sequence

$$0 \longrightarrow H_{A,1}(Q;\beta) \longrightarrow H_{A,0}(S_A;\beta) \longrightarrow H_{A,0}(\tilde{S}_A;\beta) \longrightarrow H_{A,0}(Q;\beta) \longrightarrow 0\ .$$

By Corollary 4.13 and Lemma 4.29 below,

$$\mu^{L,\tau}_{A,0}(S_A;\beta) \geq \mu^{L,\tau}_{A,0}(\tilde{S}_A;\beta) = \mu^{L,\tau}_A.$$

The second claim is part of Corollary 4.12. ☐

Lemma 4.29. *For any toric module N with $\dim(N) < d$, $\tau \in \Phi^L_A$, and $\beta \in \mathbb{C}^d$,*

$$\mu^{L,\tau}_{A,0}(N;\beta) \leq \mu^{L,\tau}_{A,1}(N;\beta).$$

Proof. Let $(A', \beta') = (gA, g\beta)$ for some $g \in \mathrm{Gl}(\mathbb{Z}, d)$. Then the Euler–Koszul complexes $K_{A,\bullet}(N;\beta)$ and $K_{A',\bullet}(N;\beta')$ are homotopic for any toric module N. Hence we may replace (A, β) by (A', β') and choose g generic. Fix the toric module N of dimension less than d. Then the Koszul complex $K_{A,\bullet}(N;\beta')$ of the endomorphisms $E' - \beta' := (E_1 - \beta_1, \dots, E_{d-1} - \beta_{d-1})$ on the $\mathbb{Z}^d$-graded left D-module $D \otimes_R N$, defined as in Definition 4.6, is homologically L-holonomic. To see this it is by the long Euler–Koszul homology sequence enough to consider a length-one toric module $N = S_\theta$ for some $\theta \in \Phi^{\mathbf{0}}_A$. Then, using Proposition 3.8, genericity of g assures that $W\langle I^L_\theta, E'\rangle$ is an n-dimensional ideal and hence $\dim \mathrm{Ch}^L(H_{A,i}(N;\beta')) \leq n$ by the spectral sequence

$$H_i(\mathrm{gr}^L(K_{A,\bullet}(N;\beta'))) \Longrightarrow \mathrm{gr}^L(H_{A,i}(N;\beta')).$$

We interpret $K_{A,\bullet}(N;\beta)$ as the complex induced by $E_d - \beta_d$ on $K_{A,\bullet}(N;\beta')$ and abbreviate $H_i := H_{A,i}(N;\beta)$ and $H'_i := H_{A,i}(N;\beta')$. There is a double complex spectral sequence abutting to $H_\bullet$ with E_1-term $H'_\bullet$ and differential d_1 induced by $E_d - \beta_d$ which collapses at the E_2-term. In particular, $H_0 = H'_0/(E_d - \beta_d)H'_0$ and there is a short exact sequence of holonomic D-modules

$$0 \longrightarrow \ker_{H'_0}(E_d - \beta_d) \longrightarrow H_1 \to H'_1/(E_d - \beta_d)H'_1 \longrightarrow 0\ .$$

Using the exact sequence

$$0 \longrightarrow \ker_{H'_0}(E_d - \beta_d) \longrightarrow H'_0 \xrightarrow{E_d - \beta_d} H'_0 \longrightarrow H'_0/(E_d - \beta_d)H'_0 \longrightarrow 0$$

it follows that

$$\begin{aligned}
\mathrm{CC}^L(H_0) &= \mathrm{CC}^L(H'_0/(E_d - \beta_d)H'_0) \\
&= \mathrm{CC}^L(\ker_{H'_0}(E_d - \beta_d)) \\
&= \mathrm{CC}^L(H_1) - \mathrm{CC}^L(H'_1/(E_d - \beta_d)H'_1).
\end{aligned}$$

We conclude that $\mathrm{CC}^L(H_1) - \mathrm{CC}^L(H_0) = \mathrm{CC}^L(H_1'/(E_d - \beta_d)H_1')$ is nonnegative which implies the claim. $\square$

Appendix A. On spectral sequences

Let $(K^\bullet, d)$ be a complex of groups equipped with a descending filtration F of subgroups such that $d(F^pK^\bullet) \subseteq F^pK^{\bullet+1}$. We stress that there are no boundedness conditions on the filtration. We abbreviate $\mathrm{gr} = \mathrm{gr}_F$ and freely use induced filtrations on subgroups and quotient groups.

By [God58, Ch. I, Thm. 4.2.1, Thm. 4.2.2] there is an associated spectral sequence involving the following data:

$$Z_r^{p,q} = F^pK^{p+q} \cap d^{-1}(F^{p+r}K^{p+q+1}),$$
$$B_r^{p,q} = F^pK^{p+q} \cap d(F^{p-r}K^{p+q-1}),$$
$$E_r^{p,q} = \mathrm{gr}^p \frac{d^{-1}(F^{p+r}K^{p+q+1})}{d(F^{p-r+1}K^{p+q-1})} = \mathrm{gr}^p \frac{Z_r^{p,q}}{B_{r-1}^{p,q}} = Z_r^{p,q}/\left(Z_{r-1}^{p+1,q-1} + B_{r-1}^{p,q}\right). \tag{A.1}$$

Note that $Z_r^{p,q} \cap F^{p+1}K^{p+q} = Z_{r-1}^{p+1,q-1}$, $d(Z_r^{p-r,q+r-1}) = B_r^{p,q}$, $E_0^{p,q} = \mathrm{gr}^p K^{p+q}$, and $E_1^{p,q} = H^{p+q}(\mathrm{gr}^p(K^\bullet), \mathrm{gr}^p(d))$.

For all p, q and r, d induces a group homomorphism $d_r^{p,q} : E_r^{p,q} \to E_r^{p+r,q-r+1}$ such that $d_r^{p,q} \circ d_r^{p-r,q+r+1} = 0$. More explicitly,

$$d_r^{p,q}\left(z \mod (Z_{r-1}^{p+1,q-1} + B_{r-1}^{p,q})\right) = d(z) \mod (Z_{r-1}^{p+1+r,q-r} + B_{r-1}^{p+r,q-r+1}) \tag{A.2}$$

and $E_r^{p,q} = (\ker d_r^{p,q})/(\operatorname{im} d_r^{p-r,q+r-1})$. Let

$$Z_\infty^{p,q} = F^pK^{p+q} \cap d^{-1}(0),$$
$$B_\infty^{p,q} = F^pK^{p+1} \cap d(K^{p+q-1}),$$
$$E_\infty^{p,q} = \mathrm{gr}^p H^{p+q}(K^\bullet) = \mathrm{gr}^p \frac{Z_\infty^{p,q}}{B_\infty^{p,q}} = Z_\infty^{p,q}/(Z_\infty^{p+1,q-1} + B_\infty^{p,q}). \tag{A.3}$$

In general, there is no relation between the terms $E_r^{p,q}$ for finite r and $E_\infty^{p,q}$. The purpose of this section is to prove the following theorem and its corollary below.

Theorem A.1. *Let D be a ring and $(K^\bullet, d)$ be a complex of D-modules, both equipped with a descending filtration F of subgroups subject to the hypotheses:*

(a) D is a filtered ring: $(F^pD) \cdot (F^{p'}D) \subseteq F^{p+p'}D$.
(b) $K^\bullet$ is a filtered D-module: $(F^pD) \cdot (F^{p'}K^n) \subseteq F^{p+p'}K^n$.
(c) d is compatible with F: $d(F^pK^{p+q}) \subseteq F^pK^{p+q+1}$.
(d) $E_0 = \mathrm{gr}(K^\bullet)$ is a Noetherian $\mathrm{gr}(D)$-module.
(e) F is exhaustive and separated on $K^\bullet$: $\bigcup_p F^pK^n = K^n$, $\bigcap_p F^pK^n = 0$.

Then one has the consequences:

(1) $d_r \colon E_r \to E_r$ are graded $\mathrm{gr}(D)$-linear maps of graded $\mathrm{gr}(D)$-modules with $\mathrm{gr}^{p'}(D) \cdot E_r^{p,q} \subseteq E_r^{p+p',q-p'}$.
(2) E_r is Noetherian and d_r is the zero map for all sufficiently large r.
(3) The stable value of E_r is isomorphic to E_∞.

Proof. We abbreviate $W = \mathrm{gr}(D)$.

(1) follows from (a), (b), and (c): E_0 is a graded W-module by (a) and (b). By induction, we may assume that E_r is a graded W-module and d_{r-1} is graded

W-linear. Let $\delta \in F^{p'}D$ and $c \in Z_r^{p,q}$ represent non zero elements $\bar\delta \in W^{p'}$ and $\bar c \in E_r^{p,q}$. By (A.2) and (c),

$$\begin{aligned}
d_r(\bar\delta \cdot \bar c) &= d_r\left((\delta \mod F^{p'+1}D)\cdot(c \mod (Z_{r-1}^{p+1,q-1}+B_{r-1}^{p,q}))\right)\\
&= d_r\left(\delta c \mod (Z_{r-1}^{p+p'+1,q-p'-1}+B_{r-1}^{p+p',q-p'})\right)\\
&= d(\delta c) \mod (Z_{r-1}^{p+p'+r+1,q-p'-r-1}+B_{r-1}^{p+p'+r,q-r-p'})\\
&= \delta d(c) \mod (Z_{r-1}^{p+p'+r+1,q-p'-r-1}+B_{r-1}^{p+p'+r,q-r-p'})\\
&= (\delta \mod F^{p'+1}D)\cdot\left(d(c) \mod (Z_{r-1}^{p+r+1,q-r-1}+B_{r-1}^{p+r,q-r})\right)\\
&= (\delta \mod F^{p'+1}D)\cdot d_r\left(c \mod (Z_{r-1}^{p+1,q-1}+B_{r-1}^{p,q})\right)\\
&= \bar\delta \cdot d_r(\bar c) \in E_r^{p+p',q-p'}.
\end{aligned}$$

Thus, d_r is graded W-linear and $\ker(d_r)$, $\operatorname{im}(d_r)$, and $E_r = \ker(d_r)/\operatorname{im}(d_r)$ are graded W-modules.

(2) follows from (d): As a subquotient of E_r, E_{r+1} is Noetherian by induction. Let $\pi_r\colon \ker(d_r) \to \ker(d_r)/\operatorname{im}(d_r) = E_{r+1}$ denote the natural projection. Note that $\pi_r^{-1}(\operatorname{im}(d_{r+1})) \supseteq \pi_r^{-1}(0) = \operatorname{im}(d_r)$. Then the chain

$$\operatorname{im}(d_0) \subseteq \pi_0^{-1}(\operatorname{im}(d_1)) \subseteq \pi_0^{-1}(\pi_1^{-1}\operatorname{im}(d_2)) \subseteq \cdots$$

of submodules of E_0 must stabilize at some r_0: $\pi_r^{-1}(\operatorname{im}(d_{r+1})) = \operatorname{im}(d_r) = \pi_r^{-1}(0)$ and hence $\operatorname{im}(d_{r+1}) = 0$ for all $r \geq r_0$.

(3) follows from (e):

$$\begin{aligned}
B_\infty^{p,q} &= F^pK^{p+q} \cap d(K^{p+q-1}) = F^pK^{p+q} \cap d\left(\bigcup_r F^{p-r}K^{p+q-1}\right)\\
&= \bigcup_r (F^pK^{p+q} \cap d(F^{p-r}K^{p+q-1})) = \bigcup_r B_r^{p,q},\\
Z_\infty^{p,q} &= F^pK^{p+q} \cap d^{-1}(0) = F^pK^{p+q} \cap d^{-1}\left(\bigcap_r F^{p+r}K^{p+q+1}\right)\\
&= \bigcap_r (F^pK^{p+q} \cap d^{-1}(F^{p+r}K^{p+q+1})) = \bigcap_r Z_r^{p,q}.
\end{aligned}$$

By (A.2), $d_r = 0$ means that

$$B_r^{p,q} = d(Z_r^{p-r,q+r-1}) \subseteq Z_{r-1}^{p+1,q-1} + B_{r-1}^{p,q} \subseteq B_{r-1}^{p,q} + F^{p+1}K^{p+q} \tag{A.4}$$

and also

$$d(Z_r^{p,q}) \subseteq Z_{r-1}^{p+r+1,q-r} + B_{r-1}^{p+r,q-r+1} = Z_{r-1}^{p+r+1,q-r} + d(Z_{r-1}^{p+1,q-1})$$

which implies that

$$Z_r^{p,q} \subseteq Z_{r+1}^{p,q} + Z_{r-1}^{p+1,q-1} \subseteq Z_{r+1}^{p,q} + F^{p+1}K^{p+q}. \tag{A.5}$$

From (A.4) and (A.5) follows $\operatorname{gr}^p B_r^{p,q} = \operatorname{gr}^p B_{r-1}^{p,q}$ and $\operatorname{gr}^p Z_r^{p,q} = \operatorname{gr}^p Z_{r+1}^{p,q}$ for $r \geq r_0$. By (A.1) and (A.3) and exactness of gr for induced filtrations, we conclude that, for $r \geq r_0$,

$$E_r^{p,q} = \operatorname{gr}^p \frac{Z_r^{p,q}}{B_{r-1}^{p,q}} = \frac{\operatorname{gr}^p Z_r^{p,q}}{\operatorname{gr}^p B_{r-1}^{p,q}} = \frac{\operatorname{gr}^p\left(\bigcap_r Z_r^{p,q}\right)}{\operatorname{gr}^p\left(\bigcup_r B_{r-1}^{p,q}\right)} = \frac{\operatorname{gr}^p Z_\infty^{p,q}}{\operatorname{gr}^p B_\infty^{p,q}} = \operatorname{gr}^p \frac{Z_\infty^{p,q}}{B_\infty^{p,q}} = E_\infty^{p,q}.$$

□

Corollary A.2. *Under the hypotheses of Theorem A.1 assume that $W = \mathrm{gr}(D)$ is commutative, let $P \in \mathrm{Spec}(W)$ be a prime ideal and denote by W_P the local ring of* $\mathrm{Spec}(W)$ *at P. Then there is a spectral sequence*

$$H^p(\mathrm{gr}(K^\bullet) \otimes_W W_P) = {}'E_1 \Longrightarrow {}'E_\infty = (\mathrm{gr}(H^p(K^\bullet))) \otimes_W W_P.$$

This spectral sequence collapses eventually and converges: ${}'E_r = {}'E_{r+1} = {}'E_\infty$ for all large r.

Proof. The spectral sequence (E_r, d_r) of Theorem A.1 is a sequence of W-modules and W-morphisms. Since localization commutes with the formation of kernels and cokernels, the objects and morphisms $({}'E_r, {}'d_r) = (E_r \otimes_W W_P, d_r \otimes_W W_P)$ form a spectral sequence. Since (E_r, d_r) collapses, so does $({}'E_r, {}'d_r)$, converging to the localization of E_∞. □

Remark A.3. As far as we can see, the collapse of the sequence is essential for the corollary. If in general (E_r, d_r) is a spectral sequence of W-modules and W-morphisms converging to E_∞ in the sense that $E_\infty = \bigcap_r Z_r / \bigcup_r B_r$ then while $(\bigcup_r B_r) \otimes_W W_P = \bigcup_r (B_r \otimes_W W_P)$, it is not necessarily the case that $(\bigcap_r Z_r) \otimes_W W_P = \bigcap_r (Z_r \otimes_W W_P)$. The problem is that infinite intersections (in contrast to infinite unions) and homomorphisms (such as the localization morphism) are not necessarily interchangeable.

In the situation of the theorem it is probably not sufficient to replace (d) by the condition that $K^\bullet$ consists of Noetherian D-modules. For example, in [ST04] some rings appear in the theory of hypergeometric systems that are Noetherian, but the order filtration leads to non-Noetherian associated graded rings.

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