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Cohomology, stratifications and parametric Gröbner bases in characteristic zero

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Abstract

Let $P_K(n,d)$ be the set of polynomials in n variables of degree at most d over the field K of characteristic zero. We show that there is a number $c_{n,d}$ such that if $f \in P_K(n,d)$ then the algebraic de Rham cohomology group $H^i_{dR}(K^n \setminus \mathrm{Var}(f))$ has rank at most $c_{n,d}$. We also show the existence of a bound $c_{n,d,l}$ for the ranks of de Rham cohomology groups of complements of varieties in n-space defined by the vanishing of l polynomials in $P_K(n,d)$. In fact, if $\beta_i : P_K(n,d)^l \to \mathbb{N}$ is the i-th Betti number of the complement of the corresponding variety, we establish the existence of a $\mathbb{Q}$-algebraic stratification on $P_K(n,d)^l$ such that β_i is constant on each stratum.

The stratifications arise naturally from parametric Gröbner basis computations; we prove for parameter-insensitive weight orders in Weyl algebras the existence of specializing Gröbner bases.

Let $P_K(n,d)$ be the space of polynomials in n variables of degree at most d over the field K of characteristic zero. Fukuda proved in [3] a variety of topological results, one of which amounts to saying that there is a stratification on $P_K(n,d)^l$ on which the complements of the corresponding varieties have constant homotopy type. His methods include Morse theory and related results of Thom. Because of that, they are not constructive and it is not clear what field the stratification might be defined over.

In [6], Lyubeznik proved the existence of an algebraic stratification on $P_K(n,d)$ such that on the strata the Bernstein-Sato polynomial is constant. He proved this from algebraic properties of the Bernstein-Sato polynomial, and asked whether one could say something about the field of definition and whether the stratification might in fact be computable.

Leykin answered both results in the affirmative in his paper [5] and proved that the stratification is in fact rational. Leykin's main tool is a statement about the

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behavior of Gröbner bases for a term order in a Weyl algebra under algebraic maps induced by the specialization of one or more variables to a K-algebraic value, and he used it to give a Gröbner basis driven algorithm to compute a $\mathbb{Q}$-stratification along which Bernstein-Sato polynomials are constant.

In this note we take up these ideas and prove that Leykin's stratification can be refined such that

- on each stratum of $P_K(n,d)^l$ the local cohomology groups of the corresponding variety are deformations of each other and can be viewed to form a constant sheaf on the stratum.
- on each stratum the de Rham cohomology groups of the complement of the corresponding variety are of constant rank and generated by differential forms with parameters defined on the entire stratum.

The stratification remains rational under this refinement.

As in Leykin's work, we use Gröbner bases to prove the existence of such stratifications. We generalize and make precise his result on Gröbner bases under specializations by including the case of non-term orders and weight orders.

Weispfenning proved in [17] the existence of *comprehensive* Gröbner bases in the commutative setting. These are systems of generators for an ideal which are a Gröbner basis under every specialization of the parameters. Their existence has been stated in [4] in the setting of solvable type algebras, and a complicated method of computation is hinted at. We establish the existence of a weaker object (analogous to the constructions in [5]) in the Weyl algebra and give a straightforward algorithm for its computation. The same algorithm also works in solvable type algebras.

1. Introduction and Notation

In this section we introduce concepts and notation and present the main problem.

NOTATION 1.1: Throughout, we assume that $\mathbb{Q} \subseteq K \subseteq \mathbb{C}$. For any of the orders we consider (on numbers, monomials or other structures alike), the symbols $\prec$ and $<$ are reserved for strict inequalities only. If we allow for equality, we always use $\preceq$ and $\leq$.

As a general reference for background and details on the material of this section will serve the excellent book [13]. Throughout we will abbreviate $a_1, \dots, a_r$ to $\boldsymbol{a}$, $y_1, \dots, y_m$ to $\boldsymbol{y}$, $x_1, \dots, x_n$ to $\boldsymbol{x}$ and $\partial_1, \dots, \partial_n$ to $\boldsymbol{\partial}$. Let C be the ring of polynomials $K[\boldsymbol{a}]$, put D_n the Weyl algebra in $\boldsymbol{x}$ with derivations $\boldsymbol{\partial}$, set $R_m = K[\boldsymbol{y}]$ and let $D = C \otimes_K R_m \otimes_K D_n$. We call $\{a_1, \dots, a_r\}$ the *parameters* and $\{x_1, \dots, \partial_n, y_1, \dots, y_m\}$ the *variables*. For a collection G of elements in a module over either of those rings, $\langle G \rangle$ will denote the left module generated by G.

Elements of D have a *right normal form*,

$$P = \sum c_{\alpha,\eta,\xi,\delta} \boldsymbol{a}^\alpha \boldsymbol{y}^\eta \boldsymbol{x}^\xi \boldsymbol{\partial}^\delta,$$

where $c_{\alpha,\eta,\xi,\delta} \in K$, $\alpha \in \mathbb{N}^r$, $\xi, \delta \in \mathbb{N}^n$ and $\eta \in \mathbb{N}^m$ and we use multi-index notation. Each $\boldsymbol{a}^\alpha \boldsymbol{y}^\eta \boldsymbol{x}^\xi \boldsymbol{\partial}^\delta$ is called a *power product*. We assume that every expression we consider is expanded in right normal form. For any two power products $M_1 = \boldsymbol{a}^\alpha \boldsymbol{y}^\eta \boldsymbol{x}^\xi \boldsymbol{\partial}^\delta, M_2 = \boldsymbol{a}^{\alpha'} \boldsymbol{y}^{\eta'} \boldsymbol{x}^{\xi'} \boldsymbol{\partial}^{\delta'}$ with $(\alpha, \eta, \xi, \delta) \leq (\alpha', \eta', \xi', \delta')$ componentwise we say that M_1 *divides* M_2, despite the fact that this is only true in $K[\boldsymbol{a}, \boldsymbol{y}, \boldsymbol{x}, \boldsymbol{\partial}]$, which via the right normal form is $K[\boldsymbol{a}, \boldsymbol{y}]$-module isomorphic to D.

Definition 1.2: A *multiplicative monomial order* $\prec$ on D is a total order on the power products in D that satisfies

- $1 \prec x_i\partial_i$, $1 \prec y_j$, $1 \prec a_k$ for all i, j, k (so the largest power product of $PQ - QP$ is smaller than the largest power product in PQ);
- $M_1 \prec M_2 \Rightarrow M_3M_1 \prec M_3M_2$ for any three power products. (Note that products of power products may not be power products. This relation then applies to the largest power products of the right normal forms of the two products involved. In this spirit $\prec$ induces a relation on polynomials which however is not an order.)

A *term order* is a multiplicative monomial order $\prec$ that additionally satisfies $1 \prec x_i$, $1 \prec \partial_i$ for all i.

A *weight order* $<_w$ is the partial order defined on the set of power products by the weight vector $w = (w_\alpha, w_\eta, w_\xi, w_\delta)$ and the rule

$$\boldsymbol{a}^\alpha \boldsymbol{y}^\eta \boldsymbol{x}^\xi \boldsymbol{\partial}^\delta <_w \boldsymbol{a}^{\alpha'} \boldsymbol{y}^{\eta'} \boldsymbol{x}^{\xi'} \boldsymbol{\partial}^{\delta'}$$

if and only if $w \cdot (\alpha - \alpha', \eta - \eta', \xi - \xi', \delta - \delta') < 0$ (scalar product). We shall only consider weight vectors w for which $w_\xi + w_\delta$ is componentwise nonnegative.

Notation 1.3: Let $\prec$ be a multiplicative monomial order. For $Q \in D$, $Q = \sum c_iM_i$ with $M_{i+1} \prec M_i$ power products and $K \ni c_i \neq 0$ for all i, we denote by

- $\mathrm{inM}(Q) \in D$ the *initial* (*largest*) *power product* M_1 of Q under this order;
- $\mathrm{inC}(Q) \in K$ the *initial coefficient* c_1 of Q;
- $\mathrm{in}(Q)$ the *initial term* $\mathrm{inC}(Q) \cdot \mathrm{inM}(P)$ of Q.

For two power products M_1, M_2 and $c_1, c_2 \in K \setminus \{0\}$ we define $\mathrm{lcm}(c_1M_1, c_2M_2)$ as the smallest power product L such that there exist power products L_1, L_2 with

$$\mathrm{inM}(L_1M_1) = \mathrm{inM}(L_2M_2) = L.$$

("Smallest" refers here to the partial order induced by divisibility rather than $\prec$.) With such L_1, L_2, define the *s-pair factors* $\mathrm{spf}(c_1M_1, c_2M_2) = c_2L_1$ and $\mathrm{spf}(M_2, M_1) = c_1L_2$. S-pair factors of pairs of arbitrary elements of D are those of the corresponding initial terms. The *s-pair* of two elements of D is of course

$$\mathrm{sp}(P, Q) := \mathrm{spf}(P, Q)P - \mathrm{spf}(Q, P)Q;$$

so $\mathrm{sp}(P, Q) \prec \mathrm{lcm}(\mathrm{inM}(P), \mathrm{inM}(Q))$.

Now we introduce the type of orders that we consider.

DEFINITION 1.4: A weight order $\prec_w$ is called $\boldsymbol{a}$*-insensitive* if $w(\boldsymbol{a}) = 0$. We say that the multiplicative monomial order $\prec$ *is* $\boldsymbol{a}$*-insensitive* if $\boldsymbol{a}^\alpha \boldsymbol{y}^\eta \boldsymbol{x}^\xi \boldsymbol{\partial}^\delta \prec \boldsymbol{a}^{\alpha'} \boldsymbol{y}^{\eta'} \boldsymbol{x}^{\xi'} \boldsymbol{\partial}^{\delta'}$ only

1. if $\boldsymbol{y}^\eta \boldsymbol{x}^\xi \boldsymbol{\partial}^\delta \prec \boldsymbol{y}^{\eta'} \boldsymbol{x}^{\xi'} \boldsymbol{\partial}^{\delta'}$, or
2. if $\boldsymbol{y}^\eta \boldsymbol{x}^\xi \boldsymbol{\partial}^\delta = \boldsymbol{y}^{\eta'} \boldsymbol{x}^{\xi'} \boldsymbol{\partial}^{\delta'}$ and $\boldsymbol{a}^\alpha \prec \boldsymbol{a}^{\alpha'}$.

If $\prec$ is an $\boldsymbol{a}$-insensitive term order, then it eliminates $\boldsymbol{y}, \boldsymbol{x}, \boldsymbol{\partial}$. There are $\boldsymbol{a}$-insensitive non-term orders for which variables are smaller than parameters.

Let $\prec$ be an $\boldsymbol{a}$-insensitive multiplicative monomial order and suppose that $Q = \sum A_i M_i \in D$ (where $0 \neq A_i \in C$, $M_i \in R_m \otimes_K D_n$ are power products, and $M_{i+1} \prec M_i$ for all i). We set

- $\mathrm{inM}_*(Q) = M_1 \in R_m \otimes_K D_n$;
- $\mathrm{inC}_*(Q) = A_1 \in C$;
- $\mathrm{in}_*(Q) = \mathrm{inC}_*(Q) \cdot \mathrm{inM}_*(Q) \in D$;
- $\mathrm{spf}_*(P, Q) = \mathrm{inC}_*(Q) \cdot \mathrm{spf}(\mathrm{inM}_*(P), \mathrm{inM}_*(Q))$, and
- $\mathrm{sp}_*(P, Q) = \mathrm{spf}_*(P, Q) \cdot P - \mathrm{spf}_*(Q, P) \cdot Q$.

As C is central one has the fundamentally important relation

$$\mathrm{sp}_*(P, Q) \prec \mathrm{lcm}(\mathrm{inM}_*(P), \mathrm{inM}_*(Q)).$$

Now let Λ be a finite set, $F = \bigoplus_{\lambda \in \Lambda} D \cdot e_\lambda$ be a free D-module and take a submodule $N \subseteq F$. Multiples of e_λ form the λ*-component* of F. Suppose one has a total order $<$ on the index set Λ and an order $\prec$ on D. Introduce the *p.o.t.* (position over term) *order on F induced by* $\prec$ by defining $P_\lambda \cdot e_\lambda \prec P_{\lambda'} \cdot e_{\lambda'}$ if either $\lambda < \lambda'$, or $\lambda = \lambda'$ and $P_\lambda \prec P_{\lambda'}$. If $\prec$ is a term order, the corresponding p.o.t. order satisfies the descending chain condition.

All previous definitions can be applied nearly verbatim to the module N rather than an ideal I. Power products then come with a generator e_λ of F, and lcm, spf and sp are defined to be zero if the power products under consideration are not in the same λ-component. The s-pair factor, of course, is even in the module case in D, and contains no e_λ.

DEFINITION 1.5: Let $\mathfrak{p}$ be a prime ideal of C. We call $\mathfrak{p} \in \mathrm{Spec}(C)$ a *specialization* and identify it with the homomorphism $\mathfrak{p} : C \to \mathfrak{p}(C) := (C/\mathfrak{p})_\mathfrak{p}$. For a C-module N contained in the free C-module F, and $P \in N$, we denote by $\mathfrak{p}(N)$ and $\mathfrak{p}(P)$ the submodule of $\mathfrak{p}(C) \otimes_C F$ generated by $1 \otimes N$, and (the coset of) the element $1 \otimes P \in \mathfrak{p}(N)$ respectively.

Any (multiplicative monomial, or weight) order $\prec$ induces in the obvious way an order $\prec_{\mathfrak{p}}$ on the $\mathfrak{p}(C)$-algebra $\mathfrak{p}(D)$ by ignoring the $\boldsymbol{a}^\alpha$ in $\boldsymbol{a}^\alpha \boldsymbol{y}^\eta \boldsymbol{x}^\xi \boldsymbol{\partial}^\delta$. This gives a term order if $\prec$ is a term order. If $\prec$ is $\boldsymbol{a}$-insensitive, we have (unless $\mathfrak{p}(\mathrm{inC}_*(Q)) = 0$)

$$P \preceq Q \Longrightarrow \mathfrak{p}(P) \preceq_{\mathfrak{p}} \mathfrak{p}(Q).$$

Right normal form sets up an isomorphism between $\mathfrak{p}(D)$ and $\mathfrak{p}(C)[\boldsymbol{y}, \boldsymbol{x}, \boldsymbol{\partial}]$ over $\mathfrak{p}(C)[\boldsymbol{y}]$. For $Q \in D$ we set

- $\mathrm{inM}_{\mathfrak{p}}(\mathfrak{p}(Q)) \in \mathfrak{p}(D)$ the initial power product of $\mathfrak{p}(Q)$;
- $\mathrm{inC}_{\mathfrak{p}}(\mathfrak{p}(Q)) \in \mathfrak{p}(C)$ its initial coefficient;
- $\mathrm{in}_{\mathfrak{p}}(\mathfrak{p}(Q)) = \mathrm{inC}_{\mathfrak{p}}(\mathfrak{p}(Q)) \cdot \mathrm{inM}_{\mathfrak{p}}(\mathfrak{p}(Q)) \in \mathfrak{p}(D)$.

S-pair factors and s-pairs are defined analogously to those over D. Note that $\mathfrak{p}(\mathrm{inC}_*(Q)) = \mathrm{inC}_{\mathfrak{p}}(\mathfrak{p}(Q))$ if $\mathrm{inC}_*(Q) \notin \mathfrak{p}$.

For multiplicative monomial, term, or weight orders $\prec$ and an ideal I in D we call a *$\prec$-Gröbner basis* a finite collection G of elements in I such that for all $f \in I$ there is a presentation

$$f = \sum_{g_i \in G} h_i g_i \quad \text{with } \mathrm{in}(h_i g_i) \preceq \mathrm{in}(f) \text{ for all } i.$$

Of course, this implies $I = \langle G \rangle$. A Gröbner basis G is *reduced* if for all $g_i, g_j \in G$ the initial power product of g_i never divides any power product of g_j.

Term orders satisfy the descending chain condition on initial terms. Hence Buchberger's algorithm will work, and terminate with a Gröbner basis (reduced, if desired). This is not true for multiplicative monomial orders, and even less for weight orders. Although there are Gröbner bases for multiplicative monomial and weight orders, it is not always true that reduced Gröbner bases exist for such orders.

One may produce Gröbner bases for weight vectors from Gröbner bases for suitable multiplicative monomial orders. On the other hand one can use term orders in the *homogenized Weyl algebra* to compute Gröbner bases for multiplicative monomial orders, see Chapter 1 in [13] and the remarks at the end of the following section.

In the next section we are going to solve

Problem 1.6 (Parameter Gröbner Basis, PGB): Let $\prec$ be any $\boldsymbol{a}$-insensitive (weight, multiplicative monomial, or term) order on D. Let $F = \bigoplus_{\lambda \in \Lambda} D \cdot e_\lambda$ be a free D-module on the finite totally ordered set Λ. Prove that for every left module $N \subseteq F$ there is a finite set of Gröbner bases $\{G_i\}$, $G_i \subseteq F$, such that for all specializations $\mathfrak{p} \in \mathrm{Spec}(C)$ at least one of the G_i satisfies

> $\mathfrak{p}(G_i)$ is a Gröbner basis for $\mathfrak{p}(N)$ in $\mathfrak{p}(F)$ under the p.o.t. order $\prec_{\mathfrak{p}}$ induced on F by $\prec$.

The motivation behind this question is the fact that once it is known to be true, all rational quantities that are computable through Gröbner bases from a fixed input involving parameters can only take a finite number of values for varying parameters.

2. Parametric Gröbner bases and Weyl algebras

We start to investigate Problem 1.6 for term orders. At the end of the section we shall discuss the more general case of a multiplicative monomial order, which will in turn contain the solution for weight orders.

One observes immediately that if I is the extension to D of an ideal of C then the parametric Gröbner basis problem is trivial: the elements of any reduced Gröbner basis provide either a unit in $\mathfrak{p}(D)$ (if $D\cdot\mathfrak{p} \not\supseteq I$) or are all zero in $D\cdot\mathfrak{p}$ (if $D\cdot\mathfrak{p} \supseteq I$). Similar statements can be made for nice submodules $N \subseteq \bigoplus_{\lambda\in\Lambda} D\cdot e_\lambda$, which we shall indicate later.

The following result is inspired by Theorem 1.7.4 of [1] and follows the proof given there closely.

LEMMA 2.1: *Let $\prec$ be a term order, let F be the free D-module on the finite ordered set Λ, and denote by $\prec$ the p.o.t. order induced on F by $\prec$ as well.*

Let $G = \{g_i\}_1^t \subseteq F$ be a finite set with the property that for each s-pair $s_{i,j} = \operatorname{spf}(g_i, g_j)g_i - \operatorname{spf}(g_j, g_i)g_j$ there is a presentation

$$s_{i,j} = \sum q_{i,j,k} g_k \tag{1}$$

with $q_{i,j,k} \in D$ and $q_{i,j,k}g_k \prec \operatorname{lcm}(\operatorname{in}(g_i), \operatorname{in}(g_j))$ for all i, j, k. Then G is a Gröbner basis.

Proof: Before we engage in the proof, let us point out that in the standard Gröbner basis result of this type the hypothesis is that $q_{i,j,k}g_k \preceq s_{i,j}$, in which case one speaks of a *standard presentation* with respect to G. Our assumption is less stringent. We call (1) a *quasi-standard presentation* for $s_{i,j}$.

Let N be the submodule of F generated by G and let $f \in N$. Then choose a presentation

$$f = \sum_{i=1}^{t} h_i g_i, \tag{2}$$

in which $\max_{1\le i\le t}\{\operatorname{inM}(h_i g_i)\} =: X \cdot e_\lambda$ is minimal. We must show that $X \cdot e_\lambda = \operatorname{inM}(f)$. So assume to the contrary that $\operatorname{inM}(f) \prec X \cdot e_\lambda$. Let S be the set of indices in (2) for which $\operatorname{inM}(h_i g_i) = X \cdot e_\lambda$. Write $X_i = \operatorname{inM}(h_i)$, $c_i = \operatorname{inC}(h_i)$, and set

$$g = \sum_{i\in S} c_i X_i g_i.$$

Thus, $\mathrm{inM}(X_i g_i) = X \cdot e_\lambda$ for all i, but $\mathrm{inM}(g) \prec X \cdot e_\lambda$. If g can be written as a sum of multiples of the g_i each of which has its leading power product smaller than $X \cdot e_\lambda$, then so can f.

Let $\{i_1 < \cdots < i_l\} = S$; clearly $l > 1$. Write $c'_{i_j} X'_{i_j} = \mathrm{spf}(g_{i_j}, g_{i_1})$ for $j > 1$ where $0 \neq c'_{i_j} \in K$, and X'_{i_j} is a power product. As $X = \mathrm{inM}(X_{i_j} g_{i_j})$ for all j, X_{i_j} is divided by $X'_{i_j} = \mathrm{inM}(\mathrm{spf}(g_{i_j}, g_{i_1}))$. Hence for $j > 1$ there is a power product X''_{i_j} with $X''_{i_j} X'_{i_j} = X_{i_j}$ plus $\prec$-smaller terms. Therefore one can rewrite

$$\begin{aligned}
\sum_{i \in S} c_i X_i g_i &= c_{i_1} X_{i_1} g_{i_1} + \sum_{i_1 \neq i \in S} c_i \left(X_i - X''_i X'_i\right) g_i + \sum_{i_1 \neq i \in S} c_i X''_i X'_i g_i \\
&= \sum_{j=2}^{l} c_{i_j} \left(X_{i_j} - X''_{i_j} X'_{i_j}\right) g_{i_j} \\
&\qquad + \sum_{j=2}^{l} \frac{c_{i_j}}{c'_{i_j}} X''_{i_j} \left(\mathrm{spf}(g_{i_j}, g_{i_1}) g_{i_j} - \mathrm{spf}(g_{i_1}, g_{i_j}) g_{i_1}\right) \\
&\qquad + \left(c_{i_1} X_{i_1} + \sum_{j=2}^{l} \frac{c_{i_j}}{c'_{i_j}} X''_{i_j} \,\mathrm{spf}(g_{i_1}, g_{i_j})\right) g_{i_1}.
\end{aligned}$$

Each $(X_{i_j} - X''_{i_j} X'_{i_j}) g_{i_j}$ is strictly smaller than $X \cdot e_\lambda$ by the choice of X''_{i_j}. By hypothesis, each of the s-pairs in the second sum has a quasi-standard presentation, i.e. $X''_{i_j} \,\mathrm{sp}(g_{i_j}, g_{i_1})$ may be replaced by a linear combination in G such that each summand is strictly smaller than $\mathrm{inM}(X''_{i_j} \cdot \mathrm{lcm}(\mathrm{in}(g_{i_1}), \mathrm{in}(g_{i_j}))) = X \cdot e_\lambda$. Since the left hand side is known to be strictly less than $X \cdot e_\lambda$, the final term on the right must be so as well and so $c_{i_1} X_{i_1} + \sum_{j=2}^{l} \frac{c_{i_j}}{c'_{i_j}} X''_{i_j} \,\mathrm{spf}(g_{i_1}, g_{i_j}) \prec X_{i_1}$. This proves that g has a presentation relative to G with largest term smaller than $X \cdot e_\lambda$. The same follows for f and by contradiction we are done. □

We now come to the main technical result, which gives ideas of Leykin a different interpretation [5]. We consider the free module $F = \bigoplus_{\lambda \in \Lambda} D \cdot e_\lambda$ and assume that Λ is totally ordered by $<$.

Definition 2.2: Let $N \subseteq F$ be a submodule of the free D-module F. We denote by $\mathfrak{q}_N$ the C-ideal $\{c \in C : cF \subseteq N\}$.

It is to be noted that although $\mathfrak{q}_N$ heavily depends on F, we do not make this explicit in order to avoid cumbersome notation.

Lemma 2.3: *Let $G = \{g_i\}_{i=1}^{t}$ be the reduced Gröbner basis for the left module $N \subseteq F$ under the p.o.t. order induced by the $\boldsymbol{a}$-insensitive term order $\prec$. Let $h = \prod_{g_i \notin \mathfrak{q}_N F} \mathrm{inC}_*(g_i) \in C$ and $\mathfrak{p} \in \mathrm{Spec}(C)$ a specialization. If $h \notin \mathfrak{p}$ then $\{\mathfrak{p}(g_i)\}$ is a Gröbner basis in $\mathfrak{p}(F)$.*

Proof: If $\mathfrak{p}$ does not contain $\mathfrak{q}_N$ then G contains for all λ a g_i with $\mathrm{in}(g_i) = c \cdot e_\lambda$ and $c \notin \mathfrak{p}$. Thus, in that case these g_i will be a Gröbner basis for $\mathfrak{p}(N) = \mathfrak{p}(F)$.

Assume thus that $\mathfrak{p} \supseteq \mathfrak{q}_N$ and $h \notin \mathfrak{p}$. We are going to show that all s-polynomials of the $\mathfrak{p}(g_i)$ formed inside $\mathfrak{p}(F)$ have a quasi-standard reduction to zero relative to $\mathfrak{p}(G)$.

If either g_i or g_j is in $\mathfrak{q}_N F$, we are looking at the s-polynomial of two elements at least one of which is zero in $\mathfrak{p}(D)$, so this s-polynomial will be trivial.

Assume hence that $g_i, g_j \in G \setminus \mathfrak{q}_N F$ with lead term in the same λ-component. Note that then $\mathrm{inC}_*(g_i) \notin \mathfrak{q}_N$ because else $g_i - \mathrm{inC}_*(g_i)\,\mathrm{inM}_*(g_i) \in N$ could be reduced to zero modulo G, contradicting the reducedness of G. Hence $\mathrm{inM}_{\mathfrak{p}}(g_i) = \mathfrak{p}(\mathrm{inM}_*(g_i))$ and $\mathrm{inM}_{\mathfrak{p}}(g_j) = \mathfrak{p}(\mathrm{inM}_*(g_j))$ as $h \notin \mathfrak{p}$. Consider the s_*-polynomial derived from $g_i, g_j \in G \setminus \mathfrak{q}_N F$:

$$t_{i,j} \;=\; \mathrm{spf}_*(g_i, g_j) g_i - \mathrm{spf}_*(g_j, g_i) g_j \in N.$$

As $\prec$ is $\boldsymbol{a}$-insensitive and C is central,

$$\mathrm{in}(t_{i,j}) \prec \mathrm{lcm}(\mathrm{inM}_*(g_i), \mathrm{inM}_*(g_j)).$$

Moreover, $\mathfrak{p}(t_{i,j}) = \mathrm{sp}(\mathfrak{p}(g_i), \mathfrak{p}(g_j))$ up to a unit in $\mathfrak{p}(C)$ since $h \notin \mathfrak{p}$. Since G is Gröbner, we can write

$$t_{i,j} \;=\; \sum_{k=1}^{t} q_{i,j,k} g_k, \tag{3}$$

where $q_{i,j,k} g_k \preceq \mathrm{in}(t_{i,j}) \prec \mathrm{lcm}(\mathrm{inM}_*(g_i), \mathrm{inM}_*(g_j))$.

Now read equation (3) as one in $\mathfrak{p}(F)$:

$$\mathfrak{p}(t_{i,j}) = \sum \mathfrak{p}(q_{i,j,k}) \mathfrak{p}(g_k)$$

with $\mathfrak{p}(q_{i,j,k} g_k) \prec_{\mathfrak{p}} \mathrm{lcm}(\mathrm{inM}_{\mathfrak{p}}(g_i), \mathrm{inM}_{\mathfrak{p}}(g_j))$. (Note that $\mathfrak{p}(q_{i,j,k} g_k) \preceq_{\mathfrak{p}} \mathfrak{p}(t_{i,j})$ does not necessarily hold because of possible cancellation of the lead term modulo $\mathfrak{p}$.) Since $\mathfrak{p}(\mathrm{inC}_*(g_i)), \mathfrak{p}(\mathrm{inC}_*(g_j)) \neq 0$, $\mathrm{sp}(\mathfrak{p}(g_i), \mathfrak{p}(g_j))$ has a quasi-standard presentation with respect to G.

One now applies Lemma 2.1 to the ring $\mathfrak{p}(D)$ in order to conclude that $\mathfrak{p}(G)$ is Gröbner. This finishes the proof of Lemma 2.3. $\square$

We shall employ the previous lemma to solve the parameter Gröbner basis problem by reduction of N to simpler submodules of F. Our method requires an algorithm for factorization in rings of polynomials. Such algorithms (and much more) are described for $K = \mathbb{Q}$ in [2]. If $h \in C$, we write $\mathrm{ass}(h \cdot C)$ for the set of prime ideals of C defined by the irreducible factors of h.

Algorithm 2.4 (Replacement):
Input: An $\boldsymbol{a}$-insensitive term order $\prec$ on D; a finite totally ordered set Λ; generators for a submodule $N \subseteq F = \bigoplus_{\lambda \in \Lambda} D \cdot e_\lambda$.
Output: A set $\mathcal{I}$ of submodules of F, a replacement for N relative to the p.o.t. order to $\prec$ on F.

1. If $N = \mathfrak{q}_N F$, return $\mathcal{I}(N) := \{N\}$.
2. Compute a reduced $\prec$-Gröbner basis $G = \{g_i\}$ for N.
3. Set $h = \prod_{g_i \in G \setminus \mathfrak{q}_N F} \mathrm{inC}_*(g_i)$.
4. Return $\mathcal{I}(N) := \{J : J = N + \mathfrak{p} \cdot F, \mathfrak{p} \in \mathrm{ass}(h \cdot C)\}$.

End.

If $\{N_k\}$ is a family of submodules, we denote by $\mathcal{I}(\{N_k\})$ the union of all $\mathcal{I}(N_k)$. Thus one may iterate the procedure: set $\mathcal{I}^{k+1}(\{N_k\}) = \mathcal{I}(\mathcal{I}^k(\{N_k\}))$ for $k \geq 1$.

Note that for all $\mathfrak{p}$ we have $\mathfrak{p}(N) = \mathfrak{p}(N + \mathfrak{p} \cdot F)$. A direct consequence of the algorithm is the following

Lemma 2.5: *The parameter Gröbner basis problem can be solved for N if it can be solved for each $J \in \mathcal{I}(N)$.*

Proof: By construction of $\mathcal{I}(N)$, we may assume that $N \neq \mathfrak{q}_N F$.

Let G be a $\prec$-Gröbner basis for N and let $\mathfrak{p}$ be a specialization. We compute $h = \prod_{g_i \in G \setminus \mathfrak{q}_N F} \mathrm{inC}_*(g_i)$. If $\mathfrak{p} \not\ni h$ then by Lemma 2.3, $\mathfrak{p}(G)$ will be a $\prec_{\mathfrak{p}}$-Gröbner basis for $\mathfrak{p}(N)$. If $\mathfrak{p} \ni h$ then necessarily $\mathfrak{p}$ contains at least one irreducible factor of h and thus $N + \mathfrak{p} \cdot F$ contains a $J \in \mathcal{I}(N)$. Moreover, in that case $\mathfrak{p}(N + \mathfrak{p} \cdot F) = \mathfrak{p}(J) = \mathfrak{p}(N)$ so that a $\prec_{\mathfrak{p}}$-Gröbner basis for $\mathfrak{p}(J)$ is also one for $\mathfrak{p}(N)$. □

Our algorithm will converge:

Lemma 2.6: *If $\mathcal{I}(N) \ni N$ then $N = \mathfrak{q}_N F$. For any N, $\mathcal{I}^k(N)$ consists for large k only of submodules N' for which $N' = \mathfrak{q}_{N'} F$.*

Proof: Assume $N = J \in \mathcal{I}(N)$, assume also that $N \neq \mathfrak{q}_N F$, and let G be the reduced Gröbner basis for N that was used to produce $\mathcal{I}(N)$. Then for some $g_i \in G \setminus \mathfrak{q}_N F$ an irreducible component of $\mathrm{inC}_*(g_i)$ is in $\mathfrak{q}_N$ and so $\mathrm{inC}_*(g_i)\,\mathrm{inM}_*(g_i) \in N$. As G is reduced, $g_i - \mathrm{inC}_*(g_i)\,\mathrm{inM}_*(g_i)$ must be zero, contradicting $g_i \in G \setminus \mathfrak{q}_N F$. Thus $\mathcal{I}(N) \ni N$ implies that $G \subseteq \mathfrak{q}_N F$ and so $N = \mathfrak{q}_N(F)$ and in fact $\mathcal{I}(N) = \{N\}$.

To prove the second part we need (in view of the first part) to show that for all N there is a $\sigma \in \mathbb{N}$ such that any sequence

$$N = J_0 \subseteq J_1 \subseteq J_2 \subseteq \cdots$$

with $J_{k+1} \in \mathcal{I}(J_k)$ has no more than σ strict inclusions. This would obviously follow if we could show that $\bigcup_{k \in \mathbb{N}} \mathcal{I}^k(N)$ is a finite set. Since the replacement algorithm splits only into finitely many branches in any iteration, $\mathcal{I}^k(N)$ is finite for all k. In order to have an infinite union over all k, there must hence be a sequence of submodules J_k as shown above where all J_k are *distinct*. As F is Noetherian, this is impossible. □

We conclude

Theorem 2.1 (PGB): *Suppose that K is a field for which factorization in C can be computed. Let $N \subseteq F = \bigoplus_{\lambda\in\Lambda} D\cdot e_\lambda$ where Λ is finite and totally ordered. Let $\prec$ be any $\boldsymbol{a}$-insensitive term order.*

One can compute a finite number of Gröbner bases $\{G_S\}_{S\in\mathfrak{S}}$ for submodules of F such that at least one of them specializes to a $\prec_\mathfrak{p}$-Gröbner basis in $\mathfrak{p}(F)$ for all specializations $\mathfrak{p} \in \operatorname{Spec}(C)$. In fact,

$$\{D\cdot G_S : S \in \mathfrak{S}\} = \bigcup_{k\in\mathbb{N}} \mathcal{I}^k(N).$$

Proof: By Lemma 2.6, $\mathcal{I}^k(N)$ consists for large k of submodules N' with $N' = \mathfrak{q}_{N'}F$. Then if $\mathfrak{p} \subseteq \mathfrak{q}_{N'}$ we have $\mathfrak{p}(N') = 0$ while if $\mathfrak{p} \not\subseteq \mathfrak{q}_{N'}$ then $\mathfrak{p}(N') = \mathfrak{p}(F)$. By Lemma 2.5 and induction, our problem can then be solved for N itself. $\square$

Corollary 2.7: *For $i = 0, 1, \ldots$ let $\prec_i$ be an $\boldsymbol{a}$-insensitive term order on D. Let $N \subseteq F = \bigoplus_{\lambda\in\Lambda} D\cdot e_\lambda$ where Λ is finite and totally ordered.*

With the assumptions of the preceding theorem in power, there is for every $m \in \mathbb{N}$ a collection of D-resolutions (of finitely generated free modules) of given length m

$$F_S^{-m} \to \cdots \to F_S^0 \to F/N \to 0, \tag{4}$$

indexed by the finite set $\mathfrak{S}$, such that for all specializations $\mathfrak{p}$ there exists $S \in \mathfrak{S}$ such that specializing the S-th resolution (4) above to $\mathfrak{p}$ leads to a $\mathfrak{p}(D)$-resolution (of finitely generated free modules) for $\mathfrak{p}(F)/\mathfrak{p}(N)$, and at each step $-m \le i \le -1$ in the resolution the generators of F_S^i map to a $\prec_{i,\mathfrak{p}}$-Gröbner basis inside F_S^{i-1}.

Proof: We argue by induction. If $m = 0$ set $F^0 = F$. The desired resolution is $F^0 \to F/N \to 0$. If $m = 1$, apply Theorem 2.1 to $N^0 = N$. This gives a collection of Gröbner bases $\{G_S\}$ in F^0. For each G_S, let $F_S^{-1} = \bigoplus_{g\in G_S} D\cdot e_g$ and define N_S^{-1} to be the kernel of the obvious map $F_S^{-1} \to F_S^0$. For each $\mathfrak{p}$ there is $S_\mathfrak{p} \in \mathfrak{S}$ such that $\{\mathfrak{p}(g)\}_{g\in G_{S_\mathfrak{p}}}$ is a $\prec_{0,\mathfrak{p}}$-Gröbner basis for $\mathfrak{p}(N^0)$ and thus $\mathfrak{p}(F_{S_\mathfrak{p}}^{-1}) \to \mathfrak{p}(F^0) \to \mathfrak{p}(F)/\mathfrak{p}(N) \to 0$ is exact.

For general m assume that we already have a finite set of resolutions (4) of length $m-1$. To prove it for m, it is in analogy to $m = 1$ sufficient to show that for each such resolution Problem 1.6 can be solved for $N_S^{1-m} = \ker(F_S^{1-m} \to F_S^{2-m})$ and $\prec_{i-m}$. But this follows again from Theorem 2.1. $\square$

Thus far we have dealt with term orders and throughout we have used this property by assuming the existence of reduced $\prec$-Gröbner bases. Let now $\prec$ be any $\boldsymbol{a}$-insensitive weight or multiplicative monomial order on D. By homogenizing the Weyl algebra relation $x_i\partial_i + 1 = \partial_i x_i$ to $x_i\partial_i + h^2 = \partial_i x_i$ one arrives at a new C-algebra which is called *homogenized Weyl algebra* D^h. Orders on D^h are required to satisfy $h^2 \prec x_i\partial_i$ for all i (resp. $2w(h) \leq w(x_i) + w(\partial_i)$ if it is a weight order). This is needed to make the normal form algorithm for D^h of [13] terminate. In [13] it is shown that any multiplicative monomial order $\prec$ on D_n induces a term order $\prec'$ on D_n^h. Modifying their ideas, assume that

- $\prec$ is an $\boldsymbol{a}$-insensitive multiplicative monomial order on D and
- $\prec$ is a *term* order on C.

Define an order $\prec'$ on D^h by $h^\chi \boldsymbol{a}^\alpha \boldsymbol{y}^\eta \boldsymbol{x}^\xi \boldsymbol{\partial}^\delta \prec' h^{\chi'} \boldsymbol{a}^{\alpha'} \boldsymbol{y}^{\eta'} \boldsymbol{x}^{\xi'} \boldsymbol{\partial}^{\delta'}$ if

- $\chi + |\eta| + |\xi| + |\delta| < \chi' + |\eta'| + |\xi'| + |\delta'|$ or
- $\chi + |\eta| + |\xi| + |\delta| = \chi' + |\eta'| + |\xi'| + |\delta'|$ and $\boldsymbol{a}^\alpha \boldsymbol{y}^\eta \boldsymbol{x}^\xi \boldsymbol{\partial}^\delta \prec \boldsymbol{a}^{\alpha'} \boldsymbol{y}^{\eta'} \boldsymbol{x}^{\xi'} \boldsymbol{\partial}^{\delta'}$ in D.

This is certainly a total order on the power products in D^h. Moreover, as $\prec$ is $\boldsymbol{a}$-insensitive then so is $\prec'$.

An infinite $\prec'$-descending chain of power products must contain an infinite $\prec'$-descending chain of power products with the same exponents in the variables $h, \boldsymbol{y}, \boldsymbol{x}, \boldsymbol{\partial}$, because there are only finitely many different such exponents of limited total degree. Since $\prec$ is term on C, any such descending chain must stabilize. Hence $\prec'$ is term on D^h, the normal form algorithm of [13] terminates for $\prec'$, and the Buchberger algorithm computes reduced $\prec'$-Gröbner bases in D^h.

If $I \subseteq D$ is an ideal, and G^h is a $\prec'$-Gröbner basis for the homogenized ideal I^h of [13], then Theorem 1.2.4 of loc. cit. shows that $G^h|_{h=1}$ is a $\prec$-Gröbner basis for I. (The Theorem is stated there in a less general form, but the proof carries over verbatim.)

Proposition 2.8: *The conclusions of Corollary 2.7 are valid for $\boldsymbol{a}$-insensitive multiplicative monomial orders $\prec$ that induce a term order on C.*

Proof: It is left to the interested reader to verify that the earlier results of this section, particularly Lemma 2.1 and Lemma 2.3, are valid, mutatis mutandem, for modules over D^h and $\boldsymbol{a}$-insensitive term orders $\prec'$. Since $\boldsymbol{a}$-insensitive multiplicative monomial orders $\prec$ on D that induce a term order on C produce an $\boldsymbol{a}$-insensitive term order $\prec'$ on D^h, one can construct a finite set of $\prec'$ −resolutions of D^h-modules which after specialization are $\prec'_{\mathfrak{p}}$-resolutions over $\mathfrak{p}(D^h)$ satisfying the conclusions of Corollary 2.7. Hence setting h to 1 in these produces by [13] a $\prec_{\mathfrak{p}}$-resolution. □

Corollary 2.9: *The conclusions of Corollary 2.7 hold for $\boldsymbol{a}$-insensitive weight orders.*

Proof: Let w be an $\boldsymbol{a}$-insensitive weight. Define a multiplicative monomial order on D by breaking ties using an $\boldsymbol{a}$-insensitive term order. The result is $\boldsymbol{a}$-insensitive. By the previous result and Theorem 1.1.6 of [13] we are done. □

3. Stratifying by cohomology

Let $(f_1,\dots,f_l) \in P_K(n,d)^l$ define the subvariety V of K^n determined by the vanishing of all elements of the ideal $\mathfrak{a} = \langle f_1,\dots,f_l\rangle \subseteq R_n = K[\boldsymbol{x}]$. In this section we investigate the stratifications induced by Gröbner bases attached to two cohomological functors, local cohomology and the de Rham functor. Let for now $K=\mathbb{Q}$; then algorithms for factorization in C are known to exist and the results from Section 2 apply.

3.1. Local cohomology

Let $\boldsymbol{R}\Gamma_{\mathfrak{a}}(-)$ denote the derived local cohomology functor with support in V, and set $H^i_{\mathfrak{a}}(-) = H^i(\boldsymbol{R}\Gamma_{\mathfrak{a}}(-))$. We want to prove that one can compute a finite number of complexes over $C\otimes_K D_n$ such that for each $\mathfrak{p}\in \operatorname{Spec}(C)$ at least one of the complexes specializes to a complex that is quasi-isomorphic to $\boldsymbol{R}\Gamma_{\mathfrak{a}}(R_n)$. This will induce a stratification on $P_K(n,d)^l$ according to which resolution works for a particular ideal $\mathfrak{a}$. This stratification will be algebraic and defined over $\mathbb{Q}$.

We recall one way of computing $\boldsymbol{R}\Gamma_{\mathfrak{a}}(M)$ for a D_n-module M, compare for example [7, 10, 14]. For $f\in R_n$ one defines the Čech complex

$$\check{C}^\bullet(f) = (0\to \underbrace{R_n}_{\text{degree }0} \hookrightarrow \underbrace{R_n[f^{-1}]}_{\text{degree }1}\to 0) \tag{5}$$

where the injection is the natural map sending $g\in R_n$ to $\frac{g}{1}\in R_n[f^{-1}]$ and "degree" refers to cohomological degree. For $f_1,\dots,f_l\in R_n$,

$$\check{C}^\bullet(f_1,\dots,f_l) = \bigotimes_{j=1}^{l} \check{C}^\bullet(f_j), \tag{6}$$

the tensor being over R_n, and for an R_n-module M

$$\check{C}^\bullet(M; f_1,\dots,f_l) = M\otimes_{R_n} \check{C}^\bullet(f_1,\dots,f_l). \tag{7}$$

Then $\check{C}^\bullet(M;f_1,\dots,f_l))$ is naturally quasi-isomorphic to $\boldsymbol{R}\Gamma_{\mathfrak{a}}(M)$.

Each module of this complex is a submodule of a finite direct sum of copies of $R_n[(f_1\cdots f_l)^{-1}]$. For any polynomial g in R_n the localization $R_n[g^{-1}]$ can be described as a D_n-module by $D_n[s]/(J(f^s)+\langle s-a\rangle)$ where s is a new variable and $J(f^s)$ consists of the operators in $D_n[s]$ that annihilate f^s via formal differentiation. Here a is a suitable integer determined by the condition that $a-m$ is never a root of the Bernstein-Sato polynomial $b_f(s)$ for natural m.

Each f_j can be written as $\sum_{x^\xi\in P_{\mathbb{Q}}(n,d)} c_{j,\xi}\boldsymbol{x}^\xi$. We consider hence the collection of indeterminate polynomials

$$g_j = \sum_{x^\xi\in P_{\mathbb{Q}}(n,d)} a_{j,\xi}x^\xi, \qquad 1\le j\le l$$

of degree at most d in the variables $x_1, \ldots, x_n$. The $a_{j,\xi}$ are our parameters, thus $C = \mathbb{Q}[\{a_{j,\xi}\}]$. The variety V above is defined by specializing $a_{j,\xi}$ to $c_{j,\xi}$, turning g_j into f_j. Denote by $\mathfrak{p}$ the kernel of this specialization, intersected with C. Let g be a product of some of the generic polynomials: $g = g_{j_1} \cdots g_{j_v}$, and let f the corresponding product of the f_j. Oaku [10] gave an algorithm to compute the ideal $J(f^s)$. Considering the $a_{j,\xi}$ as parameters it is as follows ($\partial_i \bullet (g)$ denotes $\frac{d}{dx_i}(g)$, the result of the natural action of D_n on R_n extended to $C \otimes_K R_n$):

1. Form $I = \langle t - g, \{\partial_i + \partial_i \bullet (g)\partial_t\}_{i=1}^n\rangle \subseteq C \otimes_K D_{n+1}$.
2. Put a weight on $C \otimes_K D_{n+1}[y_1, y_2]$ by $w(t) = -w(\partial_t) = w(y_1) = -w(y_2) = 1$. Homogenize I by replacing each t by ty_2 and each ∂_t by $\partial_t y_1$.
3. Compute an $\boldsymbol{a}$-insensitive Gröbner basis G for $I + \langle 1 - y_1 y_2\rangle$ eliminating y_1, y_2 and discard all basis elements not in $C \otimes_K D_{n+1}$.
4. For each $P \in G$ of weight β form $t^{\max(-\beta,0)}\partial_t^{\max(\beta,0)} P$ of weight zero. In each such operator replace $-\partial_t t$ by the new variable s. Let J_1 be the ideal generated by these operators, $J_1 \subseteq C[s] \otimes_{\mathbb{Q}} D_n$. Compute a Gröbner basis for J_1.

According to our results in Section 2, one may compute a finite number of J_1-Gröbner bases such that to every specialization $\mathfrak{p}$ there is one such basis that specializes to a set of generators for the annihilator ideal of $(\mathfrak{p}(g))^s$. The finiteness results by Lyubeznik [6] on the set of roots of Bernstein-Sato polynomials for $f \in P_K(n, d)$ for arbitrary $K \supseteq \mathbb{Q}$ imply that there is a number $a \in \mathbb{Z}$ such that for all at most l-fold products f of elements in $P_K(n, d)$ the localization $R_n[f^{-1}]$ is generated by f^a. Hence there is a finite collection of $C \otimes_{\mathbb{Q}} D_n$-ideals such that for each specialization $\mathfrak{p}$ one of their cokernels in $C \otimes_{\mathbb{Q}} D_n$ specializes to $R_n[(\mathfrak{p}(g))^{-1}]$.

Now observe that the maps in the Čech complex are inclusion maps and that for $-a \in \mathbb{N}$ the map

$$R_n[f_{i_1}^{-1}] = \frac{D_n[s]}{(J(f_{i_1}^s) + \langle s - a\rangle)} \to R_n[(f_{i_1} f_{i_2})^{-1}] = \frac{D_n[s]}{(J((f_{i_1} f_{i_2})^s) + \langle s - a\rangle)}$$

is on the level of D_n-modules simply given by right multiplication by $f_{i_2}^{-a} \in R_n$. Using multiplication by g^{-a} to form a parametric version of the Čech complex (with g varying through the various products of distinct g_i's) we have established

Theorem 3.1: *There is a finite collection of $C \otimes_{\mathbb{Q}} D_n$-complexes such that for each specialization $\mathfrak{p} \in \operatorname{Spec}(R_n)$ at least one of them specializes to the Čech complex associated to $\mathfrak{p}(g_1, \ldots, g_l)$. These complexes may be computed via Gröbner bases and contain only rational coefficients.*

Proof: Only the last statement need to be proved. It follows because the complexes are the output of Gröbner basis computations with rational input. □

3.2. de Rham cohomology

We put $\Omega = D_n/\langle\partial_1, \dots, \partial_n\rangle \cdot D_n$. and now investigate the behavior of the de Rham functor $\mathrm{dR}(-) = \Omega \otimes^L_{D_n} (-)$ when applied to the Čech complex. The cohomology groups are by [15] equal to the algebraic de Rham cohomology groups $H^i_{\mathrm{dR}}(\overline{K}^n \setminus \mathrm{Var}(f_1, \dots, f_r); K)$. We recall from [11, 15, 16] the notion of $\tilde{V}$-filtration and of b-function for integration. Let $M = D_n/I$ be a holonomic D_n-module, see [13].

If one defines a filtration on D_n by

$$\tilde{F}^k(D_n) = K\text{-span}\left\{ \boldsymbol{x}^\xi \boldsymbol{\partial}^\delta \in D_n : \sum_{j=1}^{n} (-\delta_j + \xi_j) \leq k \right\},$$

then the b-function $b(s)$ for integration of D_n/I along $\partial_1, \dots, \partial_n$ is the polynomial of smallest degree that satisfies

$$b(-x_1\partial_1 - \dots - x_n\partial_n - n) \in \tilde{F}^{-1}(D_n) + I.$$

The b-function can be determined by a Gröbner basis computation in $R_{2n} \otimes_{\mathbb{Q}} D_n$ as is explained in [12]. However, if $M = R_n[f^{-1}]$, it is quite unclear what roots $b(s)$ might have for varying f. Thus, consider the case where M is the localization of R_n at a homogeneous polynomial f of degree at most ld given as $D_n[s]/(J(f^s) + \langle s - a\rangle)$ for suitable a. By the above-mentioned results of Lyubeznik and Leykin, there is an a that works for all such f. Then of course $x_1\partial_1 + \dots + x_n\partial_n - ald$ is in the annihilator of the generator f^a of $R_n[f^{-1}]$. It follows that the b-function for integration of this localization is a divisor of $s + ald + n$. All submodules of $R_n[f^{-1}]$ that are themselves localizations of R_n by divisors of f have therefore a b-function for integration dividing $s + \rho$ where $\rho \geq ald + n$. In particular, the largest root of any of these b-functions is no larger than $-n - ald$.

The de Rham functor of a D_n-module $M = D_n/I$ may be algorithmically computed as follows.

1. Compute a $\tilde{V}$-strict finite free resolution $C^\bullet$ of M.
2. Compute the b-function of integration of M along $\partial_1, \dots, \partial_n$. Let ρ be its largest integral root.
3. Form the subcomplex of $\Omega \otimes_{D_n} C^\bullet$ consisting of all elements of $\tilde{V}$-degree at most ρ. This is a complex of finite-dimensional vector spaces and naturally quasi-isomorphic to $\mathrm{dR}(M)$.

The final step is purely mechanical once the other two have been carried out. It relies on the fact that $\Omega \otimes^L_{D_n} M$ may be computed by resolving either factor. Elements of $\Omega^\bullet \otimes_{D_n} M$ represent M-valued differential forms on K^n (where $\Omega^\bullet$ is the obvious Koszul resolution of Ω as right D_n-module). Step 1 can be done in the usual manner of constructing resolutions through computing Gröbner bases.

In order to account for $\tilde{V}$-strictness, one needs to make Gröbner bases for the weight order w with $w(x_j) = 1, w(\partial_j) = -1$ for $1 \leq j \leq n$, [11, 12, 15].

To compute the de Rham functor of a complex $M^\bullet$ rather than a module, one proceeds in the same manner, first computing a $\tilde{V}$-strict complex $C^\bullet$ quasi-isomorphic to $M^\bullet$, next computing the b-function for integration of all $H^i(M^\bullet)$, and then selecting the elements in $\Omega \otimes_{D_n} C^\bullet$ of $\tilde{V}$-degree bounded by the largest integral root of all these b-functions.

Computing over $C \otimes_{\mathbb{Q}} D_n$ with the complexes from Theorem 3.1, there are, according to Corollary 2.9, finitely many complexes over $C \otimes_{\mathbb{Q}} D_n$ whose specializations create all possible $\mathrm{dR}(\check{C}^\bullet(f_1, \ldots, f_l))$ for homogeneous $(f_1, \ldots, f_l) \in P_K(n, d)^l$.

Therefore we have

THEOREM 3.2: *There is a finite number of C-complexes such that for every specialization $\mathfrak{p} \in \mathrm{Spec}(R_n)$ of $g_1, \ldots, g_l \in C \otimes_{\mathbb{Q}} R_n$ to* homogeneous *polynomials $f_1, \ldots, f_l$ at least one of the complexes is quasi-isomorphic to* $\mathrm{dR}(\check{C}^\bullet(f_1, \ldots, f_l))$. *In particular, the Betti numbers of the complement of the variety defined by $f_1 = \ldots = f_l = 0$ in $\mathbb{C}^n$ may take only finitely many different values.* □

This is in fact even true for non-homogeneous polynomials $f_1, \ldots, f_l$. Since $\mathbb{A}^n \setminus \mathrm{Var}(f_1, \ldots, f_l) = \mathbb{P}^n \setminus \mathrm{Var}(\tilde{f}_1 x_0, \ldots, \tilde{f}_l x_0)$ (with $\tilde{f}_j$ the x_0-homogenization of f_j), there is a fibration

$$\mathbb{A}^{n+1} \setminus \mathrm{Var}(\tilde{f}_1 x_0, \ldots, \tilde{f}_l x_0) \to \mathbb{A}^n \setminus \mathrm{Var}(f_1, \ldots, f_l)$$

with fiber $\mathrm{Spec}(K[x_0]) \setminus \{\langle x_0 \rangle\}$. Via the associated spectral sequence, the finiteness results on the total space then imply similar ones on the base. One does recover the de Rham cohomology groups (not just the factors of a filtration), because over fields all complexes split.

3.3. Stratifications

Suppose $K = \mathbb{Q}$. Each element in the collection of Gröbner bases computed in Algorithm 2.4 is by Lemma 2.3 and construction always valid for a locally open set of specializations $\mathfrak{p}$. Hence there is an induced $\mathbb{Q}$-algebraic stratification $\mathfrak{S}$ on $\mathrm{Spec}(C)$ such that on each stratum local cohomology is obtained by a flat deformation of the local cohomology in a neighborhood (within the stratum) of $\mathfrak{p}$.

We claim now that for any extension field K of $\mathbb{Q}$ the same finite set of complexes of Theorem 3.1 also determines the local cohomology groups for specializations $\mathfrak{P}$ of $\mathrm{Spec}(K \otimes_{\mathbb{Q}} C)$. To see this, let $\mathfrak{p} = \mathfrak{P} \cap C$ and $C_K = K \otimes_{\mathbb{Q}} C$. For the C-free module N set $N_K = K \otimes_{\mathbb{Q}} N$. If F is C-free, F_K is C_K-free. Let $N \subseteq F$ where F is C-free.

Then $\mathfrak{P}(C_K)$ is an extension field of $\mathfrak{p}(C)$ and $\mathfrak{P}(F_K/N_K) = \mathfrak{P}(C_K) \otimes_{\mathfrak{p}(C)} \mathfrak{p}(F/N)$. Extension of scalars between fields is an exact functor. So let $F_S^\bullet$ be the complex of free $C \otimes_{\mathbb{Q}} D_n$-modules from Theorem 3.1 that works for $\mathfrak{p}$ and hence computes $H^{\bullet}_{\mathfrak{a}_{\mathfrak{p}}}(\mathfrak{p}(C) \otimes_{\mathbb{Q}} R_n)$. Apply $\mathfrak{p}(C) \otimes_C (-)$, followed by $\mathfrak{P}(C_K) \otimes_{\mathfrak{p}(C)} (-)$.

The result is a complex whose cohomology is $\mathfrak{P}(C_K) \otimes_{\mathfrak{p}(C)} H^i_{\mathfrak{a}_{\mathfrak{p}}}(\mathfrak{p}(C) \otimes_{\mathbb{Q}} R_n)$ which by flat base change is $H^i_{\mathfrak{a}_{\mathfrak{P}}}(\mathfrak{P}(C \otimes_{\mathbb{Q}} K) \otimes_{\mathbb{Q}} R_n)$.

In the same manner it follows, that the de Rham cohomology of the complement of varieties in K^n is determined by the finitely many complexes of Theorem 3.2. The general principle is that invariants that commute with field extensions and that are determined by Gröbner basis computations over D with input defined over $\mathbb{Q}$ can be stratified over $\mathbb{Q}$. This follows from our results and the fact that field extensions do not destroy the Gröbner property.

One can further refine the stratifications to obtain strata on which the ring structure deforms. According to [9] it is finally possible to make the stratifications satisfy the Whitney criteria.

It is intuitively clear that there is a stratification on which the cohomology algebras are constant, but we do not know how to prove that. The structure constants are rational, but this may require a nontrivial coordinate change. One expects this coordinate change to work on locally open sets, but we have no proof for this.

Remark 3.1: In view of the difficulties of computing Bernstein-Sato polynomials with or without parameters, we do not expect that the methods outlined here will lead to actually computed stratifications in the near future. It is, however, intellectually satisfying to know that in principle they may be computed, and to have an algorithm. Also, Lemma 2.1 may find further applications than the here demonstrated existence of stratifying sets of Gröbner bases for $\boldsymbol{a}$-insensitive orders.

Remark 3.2: In [8], L. Matusevich asked for a constructive existence proof for comprehensive Gröbner bases. While we do not prove this here, our results are however sufficient to assure the constructibility of the exceptional set of an A-hypergeometric system.

Remark 3.3: S. Basu pointed out to the author that quantifier elimination may be used to determine a stratification of $P_{\mathbb{Q}}(n,d)^l$ such that the Betti numbers are constant on the strata. We do not know whether this would compare favorably to the methods suggested in this paper. Quantifier elimination seems to have complexity problems in the same way as Gröbner basis computations do.

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