DUALITY AND MONODROMY REDUCIBILITY OF A-HYPERGEOMETRIC SYSTEMS

ULI WALTHER

Abstract. We study hypergeometric systems $H_A(\beta)$ in the sense of Gelfand, Kapranov and Zelevinsky under two aspects: the structure of their holonomically dual system, and reducibility of their rank module.

We prove in the first part that rank-jumping parameters always correspond to reducible systems. We show further that the property of being reducible is "invariant modulo the lattice", and obtain as a special instance a theorem of Alicia Dickenstein and Timur Sadykov on reducibility of Mellin systems. In the second part we study a conjecture of Nobuki Takayama which states that the holonomic dual of $H_A(\beta)$ is of the form $H_A(\beta')$ for suitable β'. We prove the conjecture for all matrices A and generic parameter β, exhibit an example that shows that in general the conjecture cannot hold, and present a refined version of the conjecture.

Questions on both duality and reducibility have been quite difficult to answer with classical methods. This paper may be seen as an example of the usefulness, and scope of applications, of the homological tools for A-hypergeometric systems developed in [6].

1. Introduction

Let A be a $d \times n$-matrix with integer entries and distinct nonzero columns $\{\boldsymbol{a}_j\}$; we abuse notation and denote by A also the set of its columns. We assume throughout that $\mathbb{Z}A = \mathbb{Z}^d$, and that A is pointed: the semigroup $\mathbb{N}A$ is to contain no units besides the origin. We denote by $x_A = \{x_j \,|\, \boldsymbol{a}_j \in A\}$ a set of n indeterminates, and we let $\partial_A = \{\partial_j \,|\, \boldsymbol{a}_j \in A\}$ be the corresponding partial differentiation operators. Let D_A be the Weyl algebra $\mathbb{C}\langle x_A, \partial_A\rangle$, denote by R_A the polynomial subring $\mathbb{C}[\partial_A]$, and let $\mathcal{O}_A$ stand for $\mathbb{C}[x_j \,|\, \boldsymbol{a}_j \in A]$.

Gelfand, Kapranov and Zelevinsky discuss in [3] the following D_A-module $H_A(\beta)$. Let I_A be the kernel of the map $R_A \longrightarrow \mathbb{C}[t_1, \ldots, t_d]$ sending ∂_j to $t^{\boldsymbol{a}_j}$ where here and henceforth we freely use multi-index notation. For $1 \leq i \leq d$ put $E_i = \sum_{j=1}^n a_{i,j} x_j \partial_j$ and let $\beta \in \mathbb{C}^d$. Then by [3] the *A-hypergeometric system* (or *GKZ-system*) is

$$H_A(\beta) = D_A / D_A \cdot (I_A, \{E_i - \beta_i\}_{i=1}^d)\,.$$

This module is holonomic by [3], and it has regular singularities if and only if I_A defines a projective variety (i.e., if I_A is homogeneous in the usual sense), [5, 9].

There is a $(0,1)$-filtration (the *order filtration*) on D_A that leads to an associated graded ring isomorphic to a ring of polynomials in $2n$ variables where the image

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of each x_j has degree zero, and where the image of each ∂_j has degree one. This filtration relates to the holonomic rank by the formula (see [8])

$$\mathrm{rk}(H) = \dim_{\mathbb{C}(x)}(\mathrm{gr}_{(0,1)}(H)(x)) = \dim_{\mathbb{C}(x)}(H(x))$$

for any holonomic module H. It is known by [8] that the holonomic rank of $H_A(\beta)$ is bounded from below by the simplicial volume of A, the quotient of the Euclidean volume of the convex hull of A and the origin by the Euclidean volume of the standard simplex in $\mathbb{Z}A = \mathbb{Z}^d$. This bound is sharp for all parameters in a Zariski open set of $\mathbb{C}^d$, and the structure of the exceptional, *rank-jumping*, parameters is described in [6].

In this paper we study $H_A(\beta)$ from two aspects of D-module theory: duality and reducibility.

1.1. **Duality.** On D_A there is a standard transposition $\tau\colon x^{\boldsymbol{u}}\partial^{\boldsymbol{v}} \mapsto (-\partial)^{\boldsymbol{v}}x^{\boldsymbol{u}}$; it induces an equivalence between the categories of left and right D_A-modules. The *holonomic dual* of a holonomic left D_A-module M is by definition

$$\mathbb{D}(M) = \tau(\mathrm{Ext}^n_{D_A}(M, D_A))$$

considered as a complex concentrated in cohomological degree zero. Let $\mathcal{O}^{an}_{\mathbb{C}^n}$ be the analytic structure sheaf on $\mathbb{C}^n$ and $\mathcal{D}^{an}_{\mathbb{C}^n}$ its sheaf of $\mathbb{C}$-linear differential operators. The holonomic duality functor is carried into Verdier duality by the *solution functor* $M \mapsto \mathbb{R}\,\mathscr{H}om_{\mathcal{D}^{an}_{\mathbb{C}^n}}(\mathcal{D}^{an}_{\mathbb{C}^n} \otimes_{D_A} M, \mathcal{O}^{an}_{\mathbb{C}^n})$.

N. Takayama conjectured that the holonomic dual of a GKZ-system $H_A(\beta)$ is again a GKZ-system. We discuss this conjecture and prove that for any A and for generic parameters Takayama's conjecture holds. We give on the other hand a class of counterexamples to the conjecture. More precisely, we prove:

- If $\mathbb{N}A$ is a normal semigroup then the set of parameters β for which $\mathbb{D}(H_A(\beta))$ fails to be a GKZ-system is contained in a finite subspace arrangement of codimension at least three with rational coefficients.
- For arbitrary A, the set of parameters β for which $\mathbb{D}(H_A(\beta))$ is not a GKZ-system is contained in a finite hyperplane arrangement with rational coefficients.
- If $\dim(S_A) = 2$ and β is A-exceptional in the sense that $\mathrm{rk}(H_A(\beta))$ is larger than usual (see [6] and below for details) then $\mathbb{D}(H_A(\beta))$ is not a GKZ-system.

1.2. **Reducibility.** A D_A-module M is said to have *irreducible monodromy representation* if the *rank module* $M(x) := \mathbb{C}(x_A)\otimes_{\mathbb{C}[x_A]} M$ is an irreducible module over $D_A(x) := (\mathbb{C}(x_A)\otimes_{\mathbb{C}[x_A]} D_A)$. Gelfand, Kapranov and Zelevinsky proved in [4] that an A-hypergeometric system $H_A(\beta)$ has irreducible monodromy representation if the parameter β is generic. In this note we prove that if β has the property that $H_A(\beta)$ has irreducible monodromy representation then all $\gamma \in \beta+\mathbb{Z}^d$ have the same property; hence irreducibility descends to the quotient torus of $\mathbb{C}^d$ modulo $\mathbb{Z}A$. In particular, all $\beta \in \mathbb{Z}^d$ give reducible systems (unless A is square).

The basic principle of proof for these facts are the *contiguity operators*

$$\partial_j\colon H_A(\beta) \longrightarrow H_A(\beta + \boldsymbol{a}_j)$$

used in conjunction with the mechanisms of Euler–Koszul homology from [6]. In the process of the proof we define an ideal in the ring R_A that varies with the parameter β. This ideal contains some important information about $H_A(\beta)$ and is a useful tool.

We further show with similar methods that if β is a rank-jumping parameter for A then $H_A(\beta)$ must have reducible monodromy. We show this by producing a natural morphism from $H_A(\beta)$ whose kernel exhibits a nontrivial submodule of $\mathbb{C}(x_A) \otimes_{\mathbb{C}[x_A]} H_A(\beta)$.

Mellin systems are naturally associated to a polynomial f with generic coefficients. Their solutions contain the roots of the polynomial when interpreted as functions in the coefficients. We use our monodromy results to prove that Mellin systems are always reducible.

The paper is divided in four sections: this introduction, a section on the mechanisms of Euler–Koszul homology and related topics, and one each on monodromy and on duality.

2. Toric modules and Euler–Koszul homology

In this section we recall some notions and results from [6] regarding toric modules, Euler–Koszul homology, and generalized A-hypergeometric systems. If the vector $(1, 1, \dots, 1)$ is in the rowspan of A one calls A *projective*. We stress that projectivity of A is not assumed in this article.

Definition 2.1. We $\mathbb{Z}A$-grade R_A and D_A according to $x_j \mapsto \boldsymbol{a}_j$, $\partial_j \mapsto -\boldsymbol{a}_j$. The i-th *degree* function we denote by $\deg_i(-)$ so that

$$\deg(-) = (\deg_1(-), \dots, \deg_d(-))$$

and the degree of $\partial^{\boldsymbol{u}} \in R_A$ is $A \cdot \boldsymbol{u}$. The ideal $\mathfrak{m}_A = \langle \partial_1, \dots, \partial_n \rangle$ of R_A is maximal as A is pointed. We shall use "graded" and "homogeneous" to mean "$\mathbb{Z}A$-graded" and "homogeneous with respect to the $\mathbb{Z}A$-grading". Both $\mathfrak{m}_A$ and

$$I_A = \langle \partial^{\boldsymbol{u}} - \partial^{\boldsymbol{v}} \mid \boldsymbol{u}, \boldsymbol{v} \in \mathbb{N}^n,\ A \cdot \boldsymbol{u} = A \cdot \boldsymbol{v} \rangle$$

are graded, but unless A is projective, I_A will not define a projective variety. The *canonical degree* is $\varepsilon_A = \sum \boldsymbol{a}_j$, the degree of the generator of the canonical module of R_A.

Let S_A be the toric ring R_A/I_A; since I_A is homogeneous, it inherits the grading from R_A. A graded R_A-module $M = \bigoplus_{\beta \in \mathbb{Z}^d} M_\beta$ is called a *toric module* if it has a finite filtration by graded R_A-modules such that each filtration quotient is a finitely generated S_A-module. We say that $\beta \in \mathbb{Z}^d$ is a *true degree* of the graded R_A-module M if β is contained in the set of degrees of M:

$$[\beta \in \operatorname{tdeg}(M)] \Longleftrightarrow [M_\beta \neq 0].$$

Moreover, we let the *quasi-degrees* of M be the points $\operatorname{qdeg}(M)$ in the Zariski closure of $\operatorname{tdeg}(M) \subseteq \mathbb{Z}^d \subseteq \mathbb{C}^d$. For every toric module the quasi-degrees form a finite subspace arrangement where each participating subspace is a shift of a complexified face of $\mathbb{R}_{\geq 0}A$ by a lattice element.

The toric modules with $\mathbb{Z}A$-homogeneous maps of degree zero form the category A-mods which is closed under graded subquotients and extensions.

For all $\beta \in \mathbb{C}^d$ and for any toric R_A-module M one can define a collection of d commuting endomorphisms denoted $E_i - \beta_i$, $1 \leq i \leq d$, on $D_A \otimes_{R_A} M$ which operate on a homogeneous element $m \in D_A \otimes_{R_A} M$ by $m \mapsto (E_i - \beta_i) \circ m$, where

$$(E_i - \beta_i) \circ m = (E_i - \beta_i - \deg_i(m))m\,.$$

There is an *Euler–Koszul functor* $\mathcal{K}_\bullet(\beta; -) = \mathcal{K}_\bullet(E - \beta; -)$ from the category of graded R_A-modules to the category of complexes of graded D_A-modules; its output are complexes concentrated between homological degrees 0 and d. It sends M to the Koszul complex defined by all morphisms $E_i - \beta_i$. The functor restricts to the category of toric modules on which it returns complexes with holonomic homology. By [6], $-\beta$ is not a quasi-degree of M if and only if $\mathcal{K}_\bullet(\beta; M)$ is exact, while if $\dim(M) = d$ and M is Cohen–Macaulay then $\mathcal{K}_\bullet(\beta; M)$ is a resolution of its 0-th homology module. A short exact sequence

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

in A-mods induces a short exact sequence of Euler–Koszul complexes

$$0 \longrightarrow \mathcal{K}_\bullet(\beta; M') \longrightarrow \mathcal{K}_\bullet(\beta; M) \longrightarrow \mathcal{K}_\bullet(\beta; M'') \longrightarrow 0$$

which in turn produces a long exact sequence of *Euler–Koszul homology*

$$\cdots \longrightarrow \mathcal{H}_i(\beta; M'') \longrightarrow \mathcal{H}_{i-1}(\beta; M') \longrightarrow \mathcal{H}_{i-1}(\beta; M) \longrightarrow \mathcal{H}_{i-1}(\beta; M'') \longrightarrow \cdots$$

where we have put $\mathcal{H}_i(\beta; -) = H_i(\mathcal{K}_\bullet(\beta; -))$. A module of the form $\mathcal{H}_0(\beta; M)$ is called a *generalized GKZ-system*; if $M = S_A$ then $\mathcal{H}_0(\beta; M)$ is a *proper* (or *classical*) GKZ-system.

An abstract module M may be toric for many different semigroup rings. In parsing the expressions $\mathcal{K}_\bullet(\beta; M)$ and $\mathcal{H}_i(\beta; M)$, the underlying matrix is implicitly understood to be A, unless expressly indicated otherwise.

The grading interacts with Euler–Koszul homology as follows: for all $\alpha \in \mathbb{Z}A$,

$$\mathcal{H}_i(\beta; M(\alpha)) = \mathcal{H}_i(\beta - \alpha; M)(\alpha)$$

where the left hand side is Euler–Koszul homology of the shifted module M while the right hand side is the shifted Euler–Koszul homology of M with shifted parameter.

Finally, for all toric modules M there is a *duality spectral sequence*

$$\mathcal{H}_i(-\beta - \varepsilon_A; \operatorname{Ext}^j_{R_A}(M, R_A)) \Longrightarrow \mathbb{D}(\mathcal{H}_{j+d-i-n}(\beta; M))^\vee, \tag{1}$$

where $M^\vee$ is the pullback of M along the coordinate change $x \longrightarrow -x$.

3. Reducibility of hypergeometric systems

For any $\mathbb{C}[x]$-module H let $H(x)$ denote $\mathbb{C}(x) \otimes_{\mathbb{C}[x]} H$ and recall that $(-)(x)$ is an exact functor. We use similar notation for morphisms, if $\phi\colon M \longrightarrow N$ then $\phi(x)\colon M(x) \longrightarrow N(x)$. In this section we investigate when the module $H_A(\beta)(x)$ has nontrivial submodules; in the absence of such submodules one says that $H_A(\beta)$ has *irreducible monodromy.* If $H_A(\beta)(x)$ is irreducible and if $\phi \in \operatorname{Hom}_{D_A}(H_A(\beta), \mathbb{C}\{x - c\})$ is any nonzero solution to $H_A(\beta)$ at the generic point $c \in \mathbb{C}^n$, then the D_A-annihilator of $\phi(1)$ annihilates every single solution of $H_A(\beta)$. It is in this sense that the monodromy is irreducible: analytic continuation of any solution germ discovers all the others (up to formation of linear combinations).

Example 3.1. Suppose that the simplicial volume of A is greater than one. Then the system $H_A(0)$ has reducible monodromy.

Namely, both I_A and $E-0$ are both contained in the ideal $\langle\partial_A\rangle = \langle\partial_1,\ldots,\partial_n\rangle$. Hence there is a surjection

$$0 \longrightarrow K \longrightarrow H_A(0) \longrightarrow\!\!\!\!\!\rightarrow D_A/\langle\partial_A\rangle \longrightarrow 0$$

with kernel K. As rank is additive in short exact sequences and since $\mathrm{rk}(H_A(0)) > 1 = \mathrm{rk}(D_A/\langle\partial_A\rangle)$, the rank of K must be positive; hence $H_A(\beta)$ has reducible monodromy.

3.1. **Reducibility and rank jumps.** In this subsection we show that if a GKZ-system is rank-jumping then it must have reducible monodromy. We do this by showing that the natural inclusion of S_A into its normalization $\tilde{S}_A$ exhibits a non-trivial submodule of $H_A(\beta)(x)$. To this end recall that $\tilde{S}_A$ is the semigroup ring to the saturation of $\mathbb{N}A$ and hence A-toric.

Notation 3.2. If F is a finite set within a lattice Λ then we write $\mathrm{vol}_\Lambda(F)$ for the *simplicial* or *lattice volume* of F in Λ, determined by the quotient of the Euclidean volume (taken in $\Lambda\otimes_{\mathbb{Z}}\mathbb{R}$) of the convex hull of F and zero divided by the volume of the unit simplex in Λ.

Lemma 3.3. *Consider the generalized hypergeometric system $\mathcal{H}_0(\beta;\tilde{S}_A)$. Its rank is equal to $\mathrm{vol}_{\mathbb{Z}A}(A)$, independently of β.*

Proof. We use the short exact sequence

$$0 \longrightarrow S_A \longrightarrow \tilde{S}_A \longrightarrow C \longrightarrow 0$$

where C is the $\mathbb{Z}A$-graded cokernel of the natural embedding of S_A into $\tilde{S}_A$. Normal semigroup rings being Cohen–Macaulay we obtain a natural 4-term exact sequence of Euler–Koszul homology

$$0 \longrightarrow \mathcal{H}_1(\beta;C) \longrightarrow \mathcal{H}_0(\beta;S_A) \xrightarrow{\psi} \mathcal{H}_0(\beta;\tilde{S}_A) \longrightarrow \mathcal{H}_0(\beta;C) \longrightarrow 0\,. \tag{2}$$

Since S_A contains an ideal isomorphic to a graded shift of $\tilde{S}_A$, $\dim(C) < \dim(S_A)$. In particular, for generic parameters β one has $-\beta\notin \mathrm{qdeg}(C)$, the outer terms of the 4-term sequence are zero, and the map ψ is an isomorphism. This means that for generic parameters, $\mathrm{rk}(\mathcal{H}_0(\beta;\tilde{S}_A)) = \mathrm{rk}(\mathcal{H}_0(\beta;S_A)) = \mathrm{vol}_{\mathbb{Z}A}(A)$.

As $\tilde{S}_A$ is a Cohen–Macaulay ring, Corollary 9.2 in [6] implies that $\tilde{S}_A$ has no exceptional parameters. Hence all β give the same value for $\mathrm{rk}(H_A(\beta))$, and as generic ones give rank $\mathrm{vol}_{\mathbb{Z}A}(A)$, all parameters must. □

Now let β be rank-jumping for A, $\mathrm{rk}(H_A(\beta)) > \mathrm{vol}_{\mathbb{Z}A}(A)$. With $C = \tilde{S}_A/S_A$, there is an exact sequence of type (2). By our lemma and the rank-jumping assumption, $\ker(\psi)(x)\neq 0$. We shall show that $\mathrm{im}(\psi)(x)\neq 0$ and thus prove

Theorem 3.4. *If β is rank-jumping for A then $H_A(\beta)$ has reducible monodromy; that is, $H_A(\beta)(x)$ has a nontrivial composition chain.*

The proof of this theorem will occupy the remainder of this subsection.

Proof. Suppose the theorem is false and let (A,β) be a counterexample with minimal number of columns. Then $\mathrm{rk}(H_A(\beta)) > \mathrm{vol}_{\mathbb{Z}A}(A)$ and $H_A(\beta)(x)$ is irreducible. In the light of the fact that $\ker(\psi)(x)\neq 0$, this gives an equality $\mathcal{H}_1(\beta;C)(x) = \mathcal{H}_0(\beta;S_A)(x)$ in the sequence (2).

Since $\dim(C) < d$, C has a composition chain whose factors are isomorphic to toric rings associated to proper faces of A. In particular, the S_A-module C

is annihilated by some monomial $\partial^{\boldsymbol{u}} \in R_A$. That implies that the submodule of $\mathcal{H}_1(\beta; C)(x) \cong \mathcal{H}_0(\beta; S_A)(x)$ generated by $\partial^{\boldsymbol{u}}$ is the zero module.

Definition 3.5. For a fixed matrix A and a parameter β we set $\mathfrak{a}_\beta$ to be the subset of R_A given by

$$\begin{aligned} [P \in \mathfrak{a}_\beta] \quad &\Leftrightarrow \quad [P \in R_A \,|\, P = 0 \text{ in } \mathcal{H}_0(\beta; S_A)(x)] \\ &\Leftrightarrow \quad [P \in R_A \,|\, \exists f \in \mathcal{O}_A, fP = 0 \text{ in } H_A(\beta)] \\ &\Leftrightarrow \quad [P \in R_A \,|\, \exists f \in \mathcal{O}_A, fP \in D_A \cdot (I_A, \{E_i - \beta_i\}_{i=1}^d)] \end{aligned}$$

and note that $\mathfrak{a}_\beta \supseteq I_A$ for all β.

We now study general properties of $\mathfrak{a}_\beta$.

Lemma 3.6. *If $P \in \mathfrak{a}_\beta$ and $Q \in D_A$ then $QP = 0$ in $H_A(\beta)(x)$. In particular, the set $\mathfrak{a}_\beta$ is an ideal of R_A.*

Proof. Obviously, $\mathfrak{a}_\beta$ is a $\mathbb{C}$-vector space. Let $Q \in D_A$ and pick $P \in \mathfrak{a}_\beta$ with $fP \in D_A(I_A, \{E_i - \beta_i\}_{i=1}^n)$. We shall prove that $f^{\mathrm{ord}(Q)+1}QP = 0$ in $\mathcal{H}_0(\beta; S_A)$, which in particular implies the lemma.

The claim is clearly true if the order of Q is zero since then f and Q commute. Assume that Q is of order $k > 0$ and the claim has been proved for operators Q up to order $k - 1$. We have

$$fQP = [f, Q]P + QfP = [f, Q]P$$

in $\mathcal{H}_0(\beta; S_A)$. Now the operator $[f, Q]$ has order smaller than k. It follows then by induction on the order of Q that

$$0 = f^{k-1+1}[f, Q]P = f^{k+1}QP$$

in $\mathcal{H}_0(\beta; S_A)$. In particular, if $Q \in R_A$ then $QP \in \mathfrak{a}_\beta$. ☐

If $P \in \mathfrak{a}_\beta$ then we can write it as a finite sum $P = \sum_{\alpha \in \mathbb{Z}A} P_\alpha$ where P_α is homogeneous of degree $\alpha = (\alpha_1 \ldots, \alpha_d)$. Then since P and $E_i - \beta_i$ are zero in $H_A(\beta)(x)$ we get that

$$H_A(\beta)(x) \ni 0 = (E_i - \beta_i)P - P(E_i - \beta_i) = [E_i, P] = \sum_{\alpha \in \mathbb{Z}A} \alpha_i P_\alpha.$$

In particular, $\sum_{\alpha \in \mathbb{Z}A} \alpha_i P_\alpha \in \mathfrak{a}_\beta$. Repeated iteration implies that if $P \in \mathfrak{a}_\beta$, then $\sum_{\alpha \in \mathbb{Z}A} {\alpha_i}^k P_\alpha \in \mathfrak{a}_\beta$. The non-vanishing of the appropriate Vandermonde determinant assures that for all integers a the subsum $\sum_{\alpha \in \mathbb{Z}A \,|\, \alpha_i = a} P_\alpha$ is in $\mathfrak{a}_\beta$. Repeating this argument for all i it follows that $P_\alpha \in \mathfrak{a}_\beta$. In particular, $\mathfrak{a}_\beta$ is a $\mathbb{Z}A$-graded ideal containing I_A.

Since in $S_A = R_A/I_A$ homogeneous elements of the same degree are constant multiples of each other, $\mathfrak{a}_\beta$ is an ideal of R_A generated by I_A and monomials.

Proposition 3.7. *Suppose $H_A(\beta)(x)$ is an irreducible module. Then there exists a monomial $\partial^{\boldsymbol{u}}$ with $A \cdot \boldsymbol{u} = \alpha$ such that $\mathfrak{a}_{\beta-\alpha}$ is a prime ideal containing $\mathfrak{a}_\beta$.*

Proof. Let $\boldsymbol{e}_k \in \mathbb{Z}^n$ be the unit vector in direction k. We shall argue by contradiction. Since I_A is prime, we may assume that $\mathfrak{a}_\beta$ strictly contains I_A and that there is a product $\partial_k \cdot \partial^{\boldsymbol{v}-\boldsymbol{e}_k} = \partial^{\boldsymbol{v}} \in \mathfrak{a}_\beta$ such that neither ∂_k nor $\partial^{\boldsymbol{v}-\boldsymbol{e}_k}$ is in $\mathfrak{a}_\beta$. There is a toric exact sequence

$$0 \longrightarrow S_A(\boldsymbol{a}_k) \xrightarrow{\partial_k} S_A \longrightarrow S_A/\langle \partial_k \rangle \longrightarrow 0.$$

We study the associated long exact sequence of Euler–Koszul homology:

$$\cdots \longrightarrow \mathcal{H}_1(\beta; S_A/\langle\partial_k\rangle) \longrightarrow H_A(\beta - \boldsymbol{a}_k) \xrightarrow{\psi} H_A(\beta) \longrightarrow \mathcal{H}_0(\beta; S_A/\langle\partial_k\rangle) \longrightarrow 0.$$

The image of $\psi(x)$ is generated by the coset of ∂_k in $H_A(\beta)(x)$. Therefore, since ∂_k is not in $\mathfrak{a}_\beta$, $\psi(x)$ is not the zero map and since furthermore $H_A(\beta)(x)$ is irreducible, $\psi(x)$ is onto. This in turn implies that $\mathcal{H}_0(\beta; S_A/\langle\partial_k\rangle)$ has zero rank. By Proposition 5.3 in [6], $\mathcal{H}_i(\beta; S_A/\langle\partial_k\rangle) = 0$ for all i. Hence ψ and $\psi(x)$ are isomorphisms. Since ψ is right multiplication by ∂_k, any monomial $\partial^{\boldsymbol{w}}$ is in $\mathfrak{a}_{\beta-\boldsymbol{a}_k}$ if and only if $\partial^{\boldsymbol{w}} \cdot \partial_k$ is in $\mathfrak{a}_\beta$. In other words, $\mathfrak{a}_{\beta-\boldsymbol{a}_k}$ is the ideal quotient

$$\mathfrak{a}_{\beta-\boldsymbol{a}_k} \quad = \quad \mathfrak{a}_\beta : \langle\partial_k\rangle \supseteq \mathfrak{a}_\beta \tag{3}$$

and by construction we have $\partial^{\boldsymbol{v}-\boldsymbol{e}_k} \in \mathfrak{a}_{\beta-\boldsymbol{a}_k} \smallsetminus \mathfrak{a}_\beta$.

As ψ is an isomorphism, $H_A(\beta - \boldsymbol{a}_k)$ is irreducible. Iterating this procedure with $\beta - \boldsymbol{a}_k$ replacing β we get an increasing sequence of ideals in R_A. Since R_A is Noetherian we arrive eventually at a parameter $\beta - \alpha \in \beta - \mathbb{N}A$ such that for every $\partial^{\boldsymbol{v}-\boldsymbol{e}_k} \cdot \partial_k = \partial^{\boldsymbol{v}} \in \mathfrak{a}_{\beta-\alpha}$ either $\partial_k \in \mathfrak{a}_{\beta-\alpha}$ or $\partial^{\boldsymbol{v}-\boldsymbol{e}_k} \in \mathfrak{a}_{\beta-\alpha}$. It follows that the ideal $\mathfrak{a}_{\beta-\alpha}$ is prime. □

Remark 3.8. Closer inspection of the proof shows that the following hold whenever $H_A(\beta)$ is irreducible:

(1) if $\mathfrak{a}_\beta$ is prime then the map $\partial^{\boldsymbol{u}} \colon H_A(\beta - A\cdot\boldsymbol{u}) \longrightarrow H_A(\beta)$ is an isomorphism for all $\boldsymbol{u} \in \mathbb{N}^n$ with $\partial^{\boldsymbol{u}} \notin \mathfrak{a}_\beta$;

(2) for a suitable $\boldsymbol{u}$, the ideal $\mathfrak{a}_{\beta-\alpha} = \mathfrak{a}_\beta : \langle\partial^{\boldsymbol{u}}\rangle$ is an associated prime of $\mathfrak{a}_\beta$.

Problem 3.9. Describe for fixed β the function $\mathbb{Z}^d \ni \boldsymbol{u} \mapsto \mathfrak{a}_{\beta-A\boldsymbol{u}}$, and more generally the function $\mathbb{C}^d \ni \beta \mapsto \mathfrak{a}_\beta$.

Example 3.10. Let $A = (1)$ and $\beta \in \mathbb{N}$. One easily checks that $\mathfrak{a}_\beta = \langle\partial_1^{\beta+1}\rangle$. While $\partial_1 \colon H_A(0) \longrightarrow H_A(1) \longrightarrow H_A(2) \longrightarrow \cdots$ and $\partial_1 \colon \cdots H_A(-3) \longrightarrow H_A(-2) \longrightarrow H_A(-1)$ are all isomorphisms, $\partial_1 \colon H_A(-1)(x) \longrightarrow H_A(0)(x)$ is not since $\mathfrak{a}_0 = \langle\partial_1\rangle$.

We now introduce notation for each subset F of A.

Definition 3.11. Let F be a subset of A. We write $\partial_F = \{\partial_k \mid \boldsymbol{a}_k \in F\}$, set $R_F = \mathbb{C}[\partial_F]$, put $x_F = \{x_k \mid \boldsymbol{a}_k \in F\}$, let $\mathcal{O}_F = \mathbb{C}[x_F]$, and denote by D_F the ring of $\mathbb{C}$-linear differential operators on $\mathcal{O}_F$. The *lattice of* F is the group $\mathbb{Z}F$, and we let E_F stand for the truncated Euler operators $\{\sum_{\boldsymbol{a}_k\in F} a_{i,k}x_k\partial_k\}_{i=1}^d$.

If F is the set of all columns $\boldsymbol{a} \in A$ such that $L(\boldsymbol{a}) = 0$ for some linear functional $L \colon \mathbb{Z}^d \longrightarrow \mathbb{Z}$ with $L(A) \geq 0$ then we say that F is a *prime face* and define the ideals $I_F = \ker(R_F \longrightarrow \mathbb{C}[\mathbb{N}F]) \subseteq R_F$, and $I_A^F = \langle\partial_k \mid \boldsymbol{a}_k \notin F\rangle + R_A \cdot I_F \subseteq R_A$. Note the natural isomorphism $R_A/I_A^F \cong R_F/I_F$. The collection of all ideals I_A^F (where F runs through all prime faces of A) is identical with the collection of all $\mathbb{Z}A$-graded prime ideals of R_A that contain I_A, which is in turn identified with the faces of the cone $\mathbb{R}_+A$.

Definition 3.12. We say that A is a *pyramid* over the prime face F if $|A \smallsetminus F| = 1$. In such a case, $\mathrm{rk}(F) = \mathrm{rk}(A) - 1$. We call A a *t-fold iterated pyramid* if there is a prime face F of A such that $|A \smallsetminus F| = t$ and $\mathrm{rk}(A) = \mathrm{rk}(F) + t$. This is equivalent to the existence of a sequence of prime faces $A = F_t \supseteq \ldots \supseteq F_0 = F$ such that F_{i+1} is a pyramid over F_i for all i. Note that if $A = \{\boldsymbol{a}_1\} \sqcup F$ is a pyramid then there is a change of coordinates in $\mathbb{Z}^d$ that sends $\boldsymbol{a}_1$ to the first unit vector in $\mathbb{Z}^d$ and F to a matrix whose top row is zero.

We now return to our hypothetical irreducible rank-jumping module $H_A(\beta)(x)$ of minimal column number. By Proposition 3.7,

$$H_A(\beta)(x) = \mathcal{H}_0(\beta; S_A)(x) \cong \mathcal{H}_0(\beta - \alpha; S_A)(x)$$

where $\mathfrak{a}_{\beta-\alpha}$ is prime and the isomorphism is given by right multiplication of $\mathcal{H}_0(\beta - \alpha; S_A)$ by $\partial^{\boldsymbol{u}}$ with $A \cdot \boldsymbol{u} = \alpha$.

Since $\mathfrak{a}_{\beta-\alpha}$ is a prime $\mathbb{Z}A$-graded ideal in R_A then $\mathfrak{a}_{\beta-\alpha} = I_A^F$ for some prime face F where F is obtained from A by erasing the columns whose variables are linear generators of $\mathfrak{a}_{\beta-\alpha}$. In consequence,

$$\begin{aligned} H_A(\beta-\alpha)(x) &= D_A(x)/\langle I_A, E-\beta+\alpha\rangle \\ &= D_A(x)/\langle I_A^F, E-\beta+\alpha\rangle \qquad \text{since } I_A^F = \mathfrak{a}_{\beta-\alpha} \\ &\cong D_F(x_F)/\langle I_F, E_F-\beta+\alpha\rangle \otimes_{\mathbb{C}} D_{A\smallsetminus F}(x_{A\smallsetminus F})/\langle \partial_{A\smallsetminus F}\rangle \\ &= H_F(\beta-\alpha)(x_F) \otimes_{\mathbb{C}} \mathbb{C}(x_{A\smallsetminus F}). \end{aligned}$$

We note that the second factor is irreducible of rank one and consider now the hypergeometric system associated to F and $\beta - \alpha$.

By the *dimension* $\dim(F)$ of a subset of $\mathbb{Z}^d$ we mean the rank of the smallest subgroup of $\mathbb{Z}^d$ that contains F. Equivalently, this is the dimension of the semigroup ring generated by $\mathbb{N}F$. We also denote by $\operatorname{conv}(F)$ the convex hull of F in $\mathbb{R}^d$.

Lemma 3.13. *Let F be a subset of A. Then* $\operatorname{vol}_{\mathbb{Z}F}(F) \le \operatorname{vol}_{\mathbb{Z}A}(A)$. *Moreover, if F is a prime face of A such that equality holds in this estimate then A is an iterated pyramid over F.*

Proof. Let $F_0 = F$, let L_0 be the lattice spanned by F and let V_0 be the simplicial volume of F_0. Let $\boldsymbol{a}_{j_1}, \ldots, \boldsymbol{a}_{j_t}$ be the columns of $A \smallsetminus F$.

For each triple (F_i, L_i, V_i) already constructed, put $F_{i+1} = F_i \cup \{\boldsymbol{a}_{j_i}\}$, and let L_{i+1} and V_{i+1} be lattice and simplicial volume of F_{i+1} respectively.

If $\dim(F_i) < \dim(F_{i+1})$ then $L_{i+1} \cong \mathbb{Z} \cdot \boldsymbol{a}_{j_{i+1}} \oplus L_i$. Hence $V_i = V_{i+1}$. On the other hand, if $\dim(F_i) = \dim(F_{i+1})$ then either $L_i = L_{i+1}$, or L_{i+1} contains L_i as a finite index subgroup. In the former case the simplicial volumes V_i and V_{i+1} are computed with reference to the same lattice, and since F_{i+1} contains F_i we see that $V_i \le V_{i+1}$. In the second case even $[L_{i+1} : L_i] \cdot V_i \le V_{i+1}$ follows. Hence $V_0 \le V_1 \le \cdots \le V_t$ and the first part of the lemma follows.

If now F is a prime face with $\operatorname{vol}_{\mathbb{Z}F}(F) = \operatorname{vol}_{\mathbb{Z}A}(A)$ then $V_0 = V_t$ and the equality must hold throughout. We claim that this proves the lemma. For, assume that there is $i \ge 0$ such that F_{i+1} is not a pyramid over F_i, and pick the smallest value i of that nature. Since F_{i+1} is not a pyramid over F_i we must have $\dim(F_i) = \dim(F_{i+1})$. As $V_i = V_{i+1}$ we have $L_i = L_{i+1}$ and $\boldsymbol{a}_{j_{i+1}} \in L_i \cap \operatorname{conv}(F_i \cup \{0\})$. However, since F_i is an iterated pyramid over F then the only points in the intersection of L_i with the convex hull of F_i and the origin are contained in $L_0 \cup \{\boldsymbol{a}_{j_1}, \ldots, \boldsymbol{a}_{j_i}\}$. Since F is a prime face of A, $\boldsymbol{a}_{j_{i+1}}$ cannot lie in L_0, and the lemma follows. □

Thus, returning to $H_F(\beta - \alpha)$ in the theorem,

$$\begin{aligned} \operatorname{rk}(H_F(\beta-\alpha)) &= \dim_{\mathbb{C}(x_F)}(H_F(\beta-\alpha)(x_F)) \\ &= \dim_{\mathbb{C}(x_A)}(\mathbb{C}(x_A) \otimes_{\mathbb{C}[x_F]} H_F(\beta-\alpha)) \\ &= \dim_{\mathbb{C}(x_A)}(H_A(\beta)(x_A)) \\ &= \operatorname{rk}(H_A(\beta)) > \operatorname{vol}_{\mathbb{Z}A}(A) \ge \operatorname{vol}_{\mathbb{Z}F}(F). \end{aligned}$$

In order to use the minimality condition of our supposed counterexample we need to modify $(F, \beta - \alpha)$ to a case of full rank. Let (A', β') be constructed as follows. First apply a generic element g of $GL(d, \mathbb{Z})$ to the matrix $(A, \beta - \alpha)$. The effect is that now the $\mathrm{rk}(F)$ top rows A' of $g(F)$ are linearly independent, and the other $\mathrm{rk}(A) - \mathrm{rk}(F)$ rows of $g(F)$ are $\mathbb{Z}$-linear combinations of the rows of A'. The latter implies that the simplicial volumes of A' and of F are identical (although, of course, their convex hulls live in spaces of different dimension). Put β' to be the top $\mathrm{rk}(F)$ rows of $g(\beta - \alpha)$. Then the rows of $(F, \beta - \alpha)$ are in the rowspan of (A', β') and so $H_{A'}(\beta') = H_F(\beta - \alpha)$.

Above we have seen that $\mathrm{rk}(H_{A'}(\beta')) = \mathrm{rk}(H_F(\beta - \alpha)) > \mathrm{vol}_{\mathbb{Z}F}(F) = \mathrm{vol}_{\mathbb{Z}A'}(A')$. Hence $H_{A'}(\beta')$ is rank-jumping with fewer columns than A. It follows that (A', β') is not a counterexample to the theorem and hence $H_{A'}(\beta')(x_F)$ has a nontrivial submodule. Any associated nontrivial sequence

$$0 \longrightarrow M \longrightarrow H_{A'}(\beta')(x_F) \longrightarrow N \longrightarrow 0,$$

when tensored over $\mathbb{C}$ with $\mathbb{C}(x_{A \smallsetminus F})$, gives a nontrivial submodule for $H_A(\beta - \alpha)(x) \cong H_A(\beta)(x)$, a contradiction. This concludes the proof of Theorem 3.4. $\square$

3.2. **Reducibility for varying β.** In this subsection we consider to what extent the property of $H_A(\beta)(x)$ being irreducible is preserved under variation of the parameter.

Corollary 3.14 (to the proof of Theorem 3.4)**.** *If $H_A(\beta)(x)$ is irreducible then there is a face F of A (this may be the trivial face $F = A$) and $\boldsymbol{u} \in \mathbb{N}^n$ with $A \cdot \boldsymbol{u} = \alpha \in \mathbb{N}A$ such that*

(1) A is an iterated pyramid over F,
(2) $\partial^{\boldsymbol{u}} \colon H_A(\beta - \alpha)(x) \longrightarrow H_A(\beta)(x)$ is an isomorphism,
(3) $\mathfrak{a}_\beta \subseteq \mathfrak{a}_{\beta - \alpha} = I_A^F$ and $\{P \in R_F \mid P = 0$ in $H_A(\beta - \alpha)(x)\} = I_F$.

Proof. Suppose $H_A(\beta)(x)$ is irreducible and $\mathfrak{a}_\beta \neq I_A$. In Proposition 3.7 we have proved that there is $\alpha \in \mathbb{N}A$ such that $H_A(\beta) \cong H_A(\beta - \alpha)$, the isomorphism is given by right multiplication on $H_A(\beta - \alpha)$ by $\partial^{\boldsymbol{u}}$ with $A \cdot \boldsymbol{u} = \alpha$, and $\mathfrak{a}_{\beta - \alpha} = I_A^F$ is the prime ideal attached to the prime face F.

Let $\overline{F} = A \smallsetminus F$. Our hypotheses imply that

$$\begin{aligned} H_A(\beta - \alpha)(x) &= D_A / D_A(I_A, E - \beta + \alpha)(x) \\ &= D_A / D_A(I_A^F, E - \beta + \alpha)(x) \\ &= D_F / D_F(I_F, E_F - \beta + \alpha)(x_F) \otimes_{\mathbb{C}} \mathbb{C}(x_{\overline{F}}). \end{aligned}$$

As at the end of the proof of Theorem 3.4, left multiply $(A, \beta - \alpha)$ by a generic element of $GL(d, \mathbb{Z})$. Then the top $\dim(F)$ rows (A', β') of $(gF, g\beta - g\alpha)$ give a GKZ-system $H_{A'}(\beta')$ that is isomorphic to $H_F(\beta - \alpha)$ and where A' has full rank. In particular, $H_{A'}(\beta')(x_F) \otimes_{\mathbb{C}} \mathbb{C}(x_{\overline{F}}) \cong H_A(\beta)$ and $\mathrm{rk}(H_{A'}(\beta')) = \mathrm{rk}(H_A(\beta))$. Now $H_A(\beta)(x)$, and hence $H_{A'}(\beta')(x_F)$, is irreducible and therefore (A, β) and (A', β') are not rank jumps by Theorem 3.4. So $\mathrm{vol}_{\mathbb{Z}F}(F) = \mathrm{vol}_{\mathbb{Z}A'}(A') = \mathrm{rk}(H_{A'}(\beta')) = \mathrm{rk}(H_A(\beta)) = \mathrm{vol}_{\mathbb{Z}A}(A)$.

By Lemma 3.13, A is thus an iterated pyramid over F. Finally, by construction $\mathfrak{a}_{\beta - \alpha} = I_A^F$ and $I_A^F \cap R_F = I_F$. $\square$

We now state the main theorem in this subsection.

Theorem 3.15. *If $H_A(\beta)(x)$ is irreducible then $H_A(\gamma)(x)$ is irreducible for all $\gamma \in \beta + \mathbb{Z}^d$.*

Proof. Let F be the smallest prime face of A such that A is an iterated pyramid over F. Let $t = \dim(A) - \dim(F)$. Suppose that the columns of F are the final columns of A. By means of an element $g \in GL(\mathbb{Z}^d)$ we can transform (A, β) into a matrix with block decomposition $\begin{pmatrix} \mathrm{id}_t & F'' & \beta'' \\ 0 & F' & \beta' \end{pmatrix}$ where id_t is the $t \times t$ identity matrix. Since A is an iterated pyramid over F, gA is an iterated pyramid over gF. In particular, F'' is in the rowspan of F' and so g can be chosen in such a way that $F'' = 0$. With this particular g we consider $H_{gA}(g\beta)$. Since $H_A(\beta) \cong H_{gA}(g\beta)$ it is sufficient to prove the theorem under the assumption that $g = 1$ and A is already in the block form $\begin{pmatrix} \mathrm{id}_t & 0 \\ 0 & F \end{pmatrix}$.

There is an isomorphism $H_A(\beta) = D_t/D_t \cdot (\{x_i\partial_i - \beta_i\}_{i=1}^t) \otimes_{\mathbb{C}} H_F(\beta_F)$ where D_t is the Weyl algebra in $x_1, \ldots, x_t$, and where β_F are the last $\dim(F)$ rows of β. The module $D_t/D_t \cdot (\{x_i\partial_i - \beta_i\}_{i=1}^t)(x_F)$ is always irreducible, so all reducibility issues of $H_A(\beta)$ are determined by those of $H_F(\beta_F)$. According to our assumption, $H_F(\beta_F)(x_F)$ is irreducible, and we wish to show that $H_F(\gamma_F)(x_F)$ is irreducible for all $\gamma \in \beta + \mathbb{Z}F$. This reduces the proof of the theorem to the case that A is not a pyramid at all.

We now assume that A is not a pyramid and that $H_A(\beta)(x)$ is irreducible. Let $\gamma \in \beta + \mathbb{Z}^d$; we shall show that $H_A(\gamma)(x)$ is irreducible as well. Suppose $\gamma' = \gamma + \alpha$ is a third parameter such that $\alpha \in \mathbb{N}A$ and $H_A(\gamma')(x)$ is irreducible. By Corollary 3.14, A is a pyramid unless $\mathfrak{a}_{\gamma'} = I_A$. We conclude from Remark 3.8.(1) that all maps $H_A(\gamma)(x) = H_A(\gamma' - \alpha)(x) \longrightarrow H_A(\gamma')(x)$ given by right multiplication by $\partial^{\boldsymbol{u}}$ with $\alpha = A \cdot \boldsymbol{u}$ are isomorphisms. Thus in this case $H_A(\gamma)(x) \cong H_A(\gamma')(x)$ is, in particular, irreducible.

It is hence sufficient to show that $H_A(\gamma')(x)$ is irreducible for some $\gamma' \in \gamma + \mathbb{N}A$. Consider the intersection of $\beta + \mathbb{N}A$ and $\gamma + \mathbb{N}A$. Since $\mathbb{N}A$ generates $\mathbb{Z}^d$ as a group, this intersection will contain a subset of the form $\gamma'' + \mathbb{N}A$. Since the set of exceptional parameters is a Zariski closed subset of positive codimension in $\mathbb{C}^d$, we can choose $\gamma' \in (\gamma + \mathbb{N}A) \cap (\beta + \mathbb{N}A)$ such that γ' is not a rank jump. Let $\boldsymbol{u}$ be an element of $\mathbb{N}^n$ such that $\beta + A \cdot \boldsymbol{u} = \gamma'$ and put $A \cdot \boldsymbol{u} = \alpha$. We show that $H_A(\gamma')(x)$ is irreducible.

Consider the short exact sequence

$$0 \longrightarrow S_A(\alpha) \xrightarrow{\partial^{\boldsymbol{u}}} S_A \longrightarrow S_A/\langle \partial^{\boldsymbol{u}} \rangle \longrightarrow 0$$

and the following part of the long exact Euler–Koszul sequence with parameter γ':

(4)

$$\mathcal{H}_1(\gamma'; S_A/\langle \partial^{\boldsymbol{u}} \rangle) \longrightarrow \mathcal{H}_0(\gamma' - \alpha; S_A) \xrightarrow{\psi} \mathcal{H}_0(\gamma'; S_A) \longrightarrow \mathcal{H}_0(\gamma'; S_A/\langle \partial^{\boldsymbol{u}} \rangle) \longrightarrow 0.$$

Since $H_A(\gamma' - \alpha)(x) = H_A(\beta)(x)$ is irreducible, the map $\psi(x)$ is either zero or injective.

We now prove that $\psi(x)$ cannot be zero. For, assume otherwise. Then $H_A(\beta)(x)$ is a quotient of $\mathcal{H}_1(\gamma'; S_A/\langle \partial^{\boldsymbol{u}} \rangle)(x)$. Consider the Euler–Koszul complex on the module $S_A/\langle \partial^{\boldsymbol{u}} \rangle$ and let $\kappa_1, \ldots, \kappa_d$ be the canonical generators for the module $\mathcal{K}_1(\gamma'; S_A/\langle \partial^{\boldsymbol{u}} \rangle) \cong \bigoplus_{i=1}^d D_A \otimes_{R_A} S_A/\langle \partial^{\boldsymbol{u}} \rangle$. As $\partial^{\boldsymbol{u}} \kappa_k = 0$, for all k and for all

$P = x^{\boldsymbol{v}}\partial^{\boldsymbol{w}} \in D_A$ one has $\partial^{\boldsymbol{u}} P\kappa_k = [\partial^{\boldsymbol{u}}, P]\kappa_k$. Of course, the x-degree of $[\partial^{\boldsymbol{u}}, P]$ is less than that of $x^{\boldsymbol{v}}$. By iteration one sees that the coset of $P\kappa_k$ is $\partial^{\boldsymbol{u}}$-torsion. It follows that $\mathcal{H}_1(\gamma'; S_A/\langle\partial^{\boldsymbol{u}}\rangle)(x)$, and hence its image in $H_A(\beta)(x)$, is $\partial^{\boldsymbol{u}}$-torsion. If $\psi(x)$ is zero, the coset of 1 in $H_A(\beta)$ is annihilated by the product $f\partial^{k\boldsymbol{u}}$ of a nonzero polynomial $f = f(x_1, \ldots, x_n)$ and a high power of $\partial^{\boldsymbol{u}}$. Since $H_A(\beta)(x)$ is irreducible and A is not a pyramid, the coset of $\partial^{k\boldsymbol{u}}$ generates $H_A(\beta)(x)$ for all $k \in \mathbb{N}$. Since $f\partial^{k\boldsymbol{u}}$ is zero in $H_A(\beta)$, the support of $H_A(\beta)$ is then contained in the variety of f. In that case, its rank must be zero since rank is measured in a generic point, hence not in a point of $\mathrm{Var}(f)$. But $\mathrm{rk}(H_A(\beta)) \geq \mathrm{vol}_{\mathbb{Z}A}(A) \geq \mathrm{vol}_{\mathbb{Z}^0}(\{0\}) = 1$. This contradiction shows that $\psi(x)$ is nonzero and hence injective.

Since $\psi(x)$ is injective, the rank of $H_A(\beta)$ is a lower bound for the rank of $H_A(\gamma')$. However, $H_A(\gamma')$ is not rank-jumping and so $\mathrm{rk}(H_A(\beta)) = \mathrm{rk}(H_A(\gamma'))$. By the sequence (4) the rank of $\mathcal{H}_0(\gamma'; S_A/\langle\partial^{\boldsymbol{u}}\rangle)$ is zero which by Proposition 5.3 of [6] means that $\mathcal{H}_i(\gamma'; S_A/\langle\partial^{\boldsymbol{u}}\rangle) = 0$ for all i. The long exact sequence (4) gives now that $H_A(\beta)(x) \cong H_A(\gamma')(x)$ is irreducible. This concludes the proof. ☐

Remark 3.16. It would be interesting to know how irreducibility is encoded in M. Saito's invariants $E_\tau(\beta)$ discussed in [7].

Remark 3.17. Let $z = z(c_0, \ldots, c_{n+1})$ be the algebraic multi-valued function defined by the equation

$$c_0 z^m + c_1 z^{m_1} + \cdots + c_n z^{m_n} + c_{n+1} = 0 \tag{5}$$

where $m > m_1 > \cdots > m_k > 0$. Consider also the A-hypergeometric system $H_A(\beta)$ where $\beta = (0, -1)$ and

$$A = \begin{pmatrix} 1 & 1 & \cdots & 1 & 1 \\ m & m_1 & \cdots & m_k & 0 \end{pmatrix}.$$

We assume that $\mathbb{Z}^2 = \mathbb{Z}A$; this is equivalent with

$$\gcd(m, m_1, \ldots, m_k) = 1.$$

The solution space of $H_A(\beta)$ is spanned by the leaves of the function z from above as long as $m_1 = m - 1$, see [10]. For example,

$$\rho_{1,2} = \frac{-c_1 \pm \sqrt{c_1^2 - 4c_0c_2}}{2c_0}$$

are solutions to $H_A(\beta)$ in the case $c_0z^2 + c_1z + c_2 = 0$. In the case $m_1 < m - 1$, the roots are (still solutions of $H_A(\beta)$ but) linearly dependent and $H_A(\beta)$ has extra logarithmic solutions, compare [1] for the general $2 \times n$ case.

Note that $m = \mathrm{rk}(H_A(\beta))$ exceeds 1 apart from the degenerate case $m = 1$ when (5) is a linear equation. By Example 3.1 the monodromy of $H_A(0)$ is reducible, and thus by Theorem 3.15 the monodromy of $H_A(\beta)$ is reducible as well. This generalizes the fact that for arbitrary m, if $m_1 = m - 1$, the sum of the roots $-c_1/c_0$ is monodromy invariant and nonzero.

There is a second system of PDE's naturally associated to Equation (5). Introducing $y = (-c_0/c_{n+1})^{1/m} z$ the defining equation becomes

$$y^m + x_1 y^{m_1} + \ldots + x_n y^{m_n} = 1.$$

The function $y = y(x_1, \ldots, x_n)$ satisfies the associated *Mellin system* of n partial differential equations in n variables (cf. [2]):

$$\prod_{k=0}^{m_j-1}\left(mk+1+\sum_{t=1}^{n} m_t\theta_t\right)\prod_{k=0}^{m'_j-1}\left(mk-1+\sum_{t=1}^{n} m'_t\theta_t\right)\bullet y = (-1)^{m_j} m^m \partial_j^m \bullet y$$

where $j = 1, \ldots, n$, $\theta_j = x_j\partial_j$, and $m'_j = m - m_j$.

In [2], A. Dickenstein and T. Sadykov show with ingenious methods that every Mellin system has reducible monodromy. Our methods can be used to demonstrate this irreducibility whenever they apply—that is, whenever the columns of A generate $\mathbb{Z}^2$.

Namely, removing the two built-in homogeneities from the solutions f of $H_A(\beta)$ by setting

$$g(x_1, \ldots, x_n) = f(1, x_1, \ldots, x_n, -1)$$

is a $\mathbb{C}$-linear injection resulting in (a not necessarily complete set of) solutions g for the Mellin system ([2, Theorem 3.3]). Since homogenization preserves monodromy invariance, reducibility of $H_A(\beta)$ implies reducibility of the Mellin system.

4. Duality for hypergeometric systems

In this section we investigate holonomic duality of GKZ-systems. We show that in the Gorenstein case holonomic duality is a functor that preserves the set of GKZ-systems. More generally, holonomic duals of GKZ-systems are GKZ-systems for all A and generic β, but for special A, β this may be false.

We begin with the simplest possible case, that of a Gorenstein toric variety. Recall that $\varepsilon_A = \sum_{j=1}^n \boldsymbol{a}_j$ is the canonical degree of R_A.

Proposition 4.1. *Suppose that A is such that S_A is Gorenstein. If c_A is such that $\operatorname{Ext}^{n-d}_{R_A}(S_A, R_A) = S_A(-c_A)$ then $\mathbb{D}(H_A(\beta)) \cong H_A(-\beta - \varepsilon_A + c_A)$ is a proper GKZ-system for all β.*

Proof. We consider the duality spectral sequence (1). Since A is Cohen–Macaulay, it collapses and we obtain duality

$$\mathbb{D}(H_A(\beta))^\vee = \mathcal{H}_0(-\beta - \varepsilon_A, \operatorname{Ext}^{n-d}_{R_A}(S_A, R_A)).$$

Since $\operatorname{Ext}^{n-d}_{R_A}(S_A, R_A) = S_A(-c_A)$, $\mathbb{D}(H_A(\beta))^\vee \cong H_A(-\beta - \varepsilon_A + c_A)$. □

It is clear that any time the duality spectral sequence is used, we obtain an identification of the dual of a GKZ-system with another GKZ-system only up to the coordinate transformation $x \mapsto -x$. In the projective case, this transformation can be omitted since it induces an isomorphism.

Example 4.2. If S_A is Gorenstein, then its *a-invariant* $a(S_A)$ is by definition the degree of the socle of $H^d_{\mathfrak{m}_A}(S_A)$. The shift $\varepsilon_A - c_A$ in Proposition 4.1 coincides with the a-invariant. Namely, $\operatorname{Ext}^{n-d}_{R_A}(S_A, R_A) = S_A(-c_A)$ implies that $\operatorname{Ext}^{n-d}_{R_A}(S_A, R_A(\varepsilon_A)) = S_A(\varepsilon_A - c_A)$ and $R_A(\varepsilon_A)$ is the canonical module of R_A. By graded local duality, $H^d_{\mathfrak{m}_A}(S_A)$ has its socle in degree $\varepsilon_A - c_A = a(S_A)$.

Consider the curve defined by $A = (a_1, \ldots, a_n)$ with $0 \le a_i \le a_{i+1}$ for all i. Assume that $\gcd(a_1, \ldots, a_n) = 1$ and that S_A is Gorenstein. In this case, the number c_A in Proposition 4.1 is the largest integer in $\mathbb{N} \smallsetminus \mathbb{N}A$. For example, $A = (2, 5)$ and $A' = (3, 4)$ have $n = 2$, $d = 1$ and $\varepsilon = (7)$, but the c-values differ

(3 versus 5). Both S_A and $S_{A'}$ are graded complete intersections of degrees 2×5 and 3×4 respectively. Hence the duality shifts are $3 = -7 + 10$ and $5 = -7 + 12$ respectively as claimed.

A natural approach to showing that $\mathbb{D}(H_A(\beta))$ is a hypergeometric system is to exploit the spectral sequence (1) as follows. Suppose one has a toric map $\phi\colon S_A(-c) \longrightarrow \operatorname{Ext}^{n-d}_{R_A}(S_A, R_A) =: X_{n-d}$ with cokernel C. If one picks a parameter with $\beta + \varepsilon_A$ outside $\operatorname{qdeg}(C)$, then $\mathcal{K}_\bullet(-\beta - \varepsilon_A; C)$ is exact. Hence $\mathcal{H}_0(-\beta - \varepsilon_A; X_{n-d}) \cong H_A(-\beta - \varepsilon_A + c)$. If, in addition, $-\beta - \varepsilon_A$ is not a quasi-degree of $\operatorname{Ext}^i_{R_A}(S_A, R_A)$ for any $i > n - d$, the spectral sequence collapses and we obtain $\mathbb{D}(H_A(\beta)) \cong (H_A(-\beta - \varepsilon_A + c))^\vee$ is a proper hypergeometric system up to the coordinate transformation $x \mapsto -x$. From this point of view, $H_A(-\beta - \varepsilon_A + c)$ is a natural candidate to equal $\mathbb{D}(H_A(\beta))$, if the latter is a proper GKZ-system at all.

However, even if S_A is a Cohen–Macaulay ring, and hence the spectral sequence collapses for all β, there is no natural choice for ϕ unless S_A is Gorenstein. To illustrate this point we consider the following example that contains most of the core ideas for the proof of Proposition 4.6.

Example 4.3. Let $A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \end{pmatrix}$ and $\beta \in \mathbb{C}^2$; then

$$I_A = \langle \partial_1\partial_3 - \partial_2^2, \partial_2\partial_4 - \partial_3^2, \partial_1\partial_4 - \partial_2\partial_3 \rangle$$

is the ideal of the projective rational cubic, which is arithmetically Cohen–Macaulay. As $\varepsilon_A = (4, 6)$ and by the spectral sequence (1), up to a coordinate transformation $\mathbb{D}(H_A(\beta))$ is the quotient of $D_A \otimes_{R_A} \operatorname{Ext}^2_{R_A}(S_A, R_A)$ by the images of the endomorphisms

$$E_1 + \beta_1 + 4\colon m \mapsto (x_1\partial_1 + x_2\partial_2 + x_3\partial_3 + x_4\partial_4 + \beta_1 + 4 - \deg_1(m))m$$

and

$$E_2 + \beta_2 + 6\colon m \mapsto (x_2\partial_2 + 2x_3\partial_3 + 3x_4\partial_4 + \beta_2 + 6 - \deg_2(m))m.$$

By explicit computation one sees with $X_2 = \operatorname{Ext}^2_{R_A}(S_A, R_A)$ that

$$X_2 \cong R_A^2 / \langle (\partial_4, -\partial_3), (\partial_3, -\partial_2), (\partial_2, -\partial_1) \rangle .$$

Let m_1, m_2 be the cosets of $(1, 0)$ and $(0, 1)$, so that $\deg(m_1) = (3, 5)$ and $\deg(m_2) = (3, 4)$. Put $M' = X_2 / S_A \cdot m_1 \cong S_A(-3, -4)/\langle \partial_3, \partial_2, \partial_1 \rangle$. Then $\beta + \varepsilon_A \notin \operatorname{qdeg}(M')$ implies that $M/\operatorname{im}(E + \beta + \varepsilon_A) \cong H_A(-\beta - \varepsilon_A + \deg(m_1))$ is a classical GKZ-system generated by m_1.

On the other hand, put $M'' := X_2 / S_A \cdot m_2 = S_A(-3, -5)/\langle \partial_4, \partial_3, \partial_2 \rangle$. In similar fashion, $\beta + \varepsilon_A \notin \operatorname{qdeg}(M'') = \operatorname{Var}(\beta_2 - 5)$ implies that $M/\operatorname{im}(E_1 + \beta_1 + 4, E_2 + \beta_2 + 6) \cong H_A((-\beta - \varepsilon_A + \deg(m_2)))$ is a classical GKZ-system as well.

The exceptional lines $\operatorname{qdeg}(M') - \varepsilon_A$ and $\operatorname{qdeg}(M'') - \varepsilon_A$ are depicted in Figure 1 below.

The two embeddings $S_A \hookrightarrow X_2$ above cover every case except when $3\beta_1 - \beta_2 + 1 = 0 = \beta_2 + 1$. In that case, $\beta_1 = -2/3$ and $\beta_2 = -1$. We reverse the approach and identify X_2 with the S_A-ideal generated by ∂_2 and ∂_3, shifted by ε_A so that $\deg(\partial_2) = (3, 4)$ and $\deg(\partial_3) = (3, 5)$. Then

$$S_A(-\varepsilon_A)/X_2 \cong (R_A/\langle \partial_1\partial_4, \partial_2, \partial_3 \rangle)(-\varepsilon_A).$$

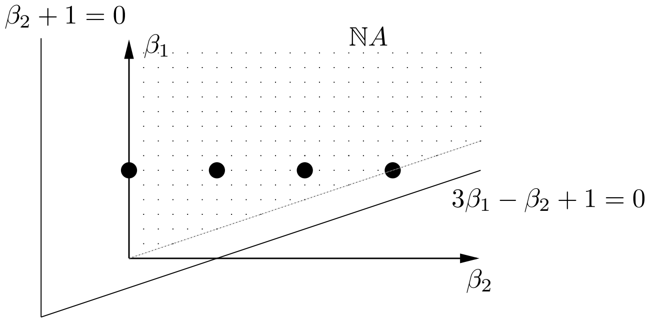

FIGURE 1. An arrangement of special duality parameters.

The quasi-degrees of this quotient are the union of the lines $\beta_1 - 3\beta_2 = 2$ and $\beta_2 = 6$. Since $(-2/3,-1)+(4,6)$ is not on either line, the dual of $H_A((-2/3,-1))$ is isomorphic to $H_A(-\beta+\varepsilon_A-\varepsilon_A) = H_A(-\beta)$.

The example shows that even if $\mathrm{Ext}^{n-d}_{R_A}(S_A,R_A)$ is not cyclic (as it is in the Gorenstein case), the dual of $H_A(\beta)$ may often be identified with a hypergeometric system. There is an obvious difficulty in picking the right morphism between S_A and the Ext-module, and the question lingers what to do with A-exceptional parameters (which have to be avoided irrespective of the chosen morphism).

We consider first the case where S_A is normal, generalizing the above example. Let $\{H_1 \geq 0,\dots,H_e \geq 0\}$ be the defining hyperplanes of $\mathbb{R}_{\geq 0}A \subseteq \mathbb{R}^d$.

Proposition 4.4. *Suppose S_A is normal. Then the set of all γ for which $\mathbb{D}(H_A(\gamma))$ is not a proper GKZ-system is contained in a finite subspace arrangement of codimension at least three. All participating subspaces are shifts of (complexified) faces of $\mathbb{R}_{\geq 0}A$ by lattice elements.*

Proof. The idea is to use several embeddings of $S_A(c) \hookrightarrow \mathrm{Ext}^{n-d}_{R_A}(S_A,R)A)$, study the quasi-degrees of the cokernel, and intersect these as an estimate for the parameters that might fail the proposition.

If A is normal then it is Cohen–Macaulay and hence the duality spectral sequence collapses: $\mathbb{D}(H_A(\gamma)) \cong \mathcal{H}_0(-\gamma-\varepsilon_A;\mathrm{Ext}^{n-d}_{R_A}(S_A,R_A))$ up to a coordinate change. Moreover, the canonical module $X_{n-d} = \mathrm{Ext}^{n-d}_{R_A}(S_A,R_A)$ is the interior ideal N_A of S_A, shifted by ε_A. We therefore want to show that for all choices β away from a codimension three arrangement the module $\mathcal{H}_0(\beta;N_A)$ is a proper GKZ-system.

Consider the natural inclusion $N_A \hookrightarrow S_A$ and the quotient Q. This is a toric sequence and hence there is a long exact sequence

$$\cdots \longrightarrow \mathcal{H}_1(\beta;Q) \longrightarrow \mathcal{H}_0(\beta;N_A) \longrightarrow \mathcal{H}_0(\beta;S_A) \longrightarrow \mathcal{H}_0(\beta;Q) \longrightarrow 0.$$

The quasi-degrees of Q are those elements of $\mathbb{C}^d$ that sit on a complexified defining hyperplane of $\mathbb{R}_{\geq 0}A$. Hence if $-\beta$ is not on any complexified defining hyperplane of $\mathbb{R}_{\geq 0}A$, then $\mathcal{H}_0(\beta;N_A)$ and $H_A(\beta) = \mathcal{H}_0(\beta;S_A)$ are isomorphic.

Let now $-\beta \neq 0$ sit on one such hyperplane. Then there is a smallest nonzero prime face F of A such that $-\beta$ is a quasi-degree of $\mathbb{N}F$ but not of any smaller face. Note that there is an infinite face N_F of $\mathrm{conv}(\deg(N_A))$ such that some shift $\mathbb{N}F+z_F$ of F, $z_F \in \mathbb{Z}^d$, is inside N_F while $\dim(N_F/(\mathbb{N}F+z_F)) < \dim(N_F) = \dim(F)$. Let f_F be a point in $\mathrm{tdeg}(N_A)$ sitting on $\mathbb{N}F+z_F$. Let δ_F be an element of $\mathbb{N}A$ in the relative interior of $\mathbb{N}F$. Then $f_F - l_F\delta_F + \mathbb{N}A$ will contain $\operatorname{tdeg}(N_A)$ for large $l_F \in \mathbb{N}$. We hence have an injection $N_A \hookrightarrow S_A(l_F\delta_F - f_F)$ and we let Q_F be the cokernel. Now if $t = t_1, \ldots, t_d$ are the torus variables then $t^{l_F\delta_F} \cdot S_A(l_F\delta_F - f_F) \subseteq N_A$. Hence all components of $\operatorname{qdeg}(Q_F)$ are lattice shifts of $\mathbb{C}F'$ where F' does not contain F.

This, and saturatedness of $\mathbb{N}A$, implies that the quasi-degrees of Q_F are contained in a finite union of hyperplanes of the form $H_j = r_{j,k,F}$ where H_j is a defining hyperplane of $\mathbb{R}_+A$ transverse to F, where $r_{j,k,F} \in \mathbb{Q}$, and where $H_j(f_F - l_F\delta_F) \leq r_{j,k,F} < \min\{r \in H_j(N_A)\}$. Note that this construction depends not on β but only on the minimal face F containing $-\beta$, so there are finitely many $r_{j,k,F}$ altogether.

On the other hand, for arbitrary β, consider the embedding $S_A(-f_F) \hookrightarrow N_A$. The true degrees, and hence the quasi-degrees, of the cokernel are contained in a union of hyperplanes of the form $H_j = s_{j,k,F}$ where $H_j(f_F) > s_{j,k,F} \geq \min\{s \in H_j(N_A)\}$ and H_j is transversal to F.

We conclude that for $\beta \neq 0$ the module $\mathcal{H}_0(\beta; Q)$ is zero unless

- β is on $H_j = 0$ for some defining hypersurface of $\mathbb{N}A$, and
- for the minimal face F with $-\beta \in \operatorname{qdeg}(F)$ and some H_j transverse to F with $H_j(\beta) \in H_j(\mathbb{Z}A)$ we have $H_j(f_F - l_F\delta_F) \leq H_j(\beta) < \min\{r \in H_j(N_A)\}$, and
- for the minimal face F with $-\beta \in \operatorname{qdeg}(F)$ and some H_j transverse to F with $H_j(\beta) \in H_j(\mathbb{Z}A)$ we have $H_j(f_F) > H_j(\beta) \geq \min\{s \in H_j(N_A)\}$.

Vanishing of $\mathcal{H}_0(\beta; Q)$ implies isomorphy of $\mathcal{H}_0(\beta; S_A)$ and $\mathcal{H}_0(\beta; N_A)$. Hence, if $\beta \neq 0$ then $\mathcal{H}_0(\beta; N_A)$ is a proper GKZ-system unless β violates all three conditions listed above. The set of violating parameters of each separate condition is a hyperplane arrangement. Note that if β violates two of the above conditions then the hyperplanes witnessing the violation are not parallel. Since the participating hyperplanes of all three arrangements form the (shifted) facets of a polyhedral cone, any three of them intersect in expected codimension.

Note finally that the case $\beta = 0$ is covered by the paragraph directly before the list of conditions above, since the quasi-degrees of any proper quotient of N_A cannot contain 0.

It follows that the set of parameters for which $\mathbb{D}(H_A(\beta))$ is not a proper GKZ-system is contained in an arrangement of codimension three, and the origin is not part of this set. $\square$

Corollary 4.5. *If $\dim(S_A) \leq 2$ and S_A is normal then the holonomic duals of all $H_A(\beta)$ are proper GKZ-systems.* $\square$

Normal semigroups are Cohen–Macaulay by Hochster's theorem. We now show by example that absence of Cohen–Macaulayness may bring with it the failure of Takayama's conjecture.

Proposition 4.6. *Let $A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 4 \end{pmatrix}$ and $M = \mathbb{D}(H_A(\beta))$ with $\beta = (1, 2)$. Then M is not a classical GKZ-system: $M \neq H_B(\beta')$ for any B and β'.*

Proof. The only candidate for B is A itself, or a matrix that is obtained from A by an action of $GL(\mathbb{Z}, 2)$. We may hence assume that $B = A$. Denote by X_i the module $\operatorname{Ext}^i_{R_A}(S_A, R_A)$. So X_2 is the canonical module of the normalization of S_A.

It is well-known that A is not Cohen–Macaulay, and that $X_3 \cong \mathbb{C}$, generated in degree $(5,10)$.

By [11], $H_A(\beta)$ is of rank five and β is the only exceptional parameter. By duality, M is of rank five as well, so only $H_A(\beta)$ is a candidate for equaling M among all A-hypergeometric systems.

Since $\varepsilon_A = (4,8)$ is the canonical degree of R_A, and $(5,10)$ is the unique degree of X_3, $E+\beta+\varepsilon_A = E+(5,10)$ is a pair of zero morphisms on $D_A \otimes_{R_A} X_3$. Thus, $\mathcal{H}_i(-\beta-\varepsilon_A; X_3) \cong \bigwedge^i(\mathcal{O}_A \oplus \mathcal{O}_A)$. In the spectral sequence (1) there are hence four nonzero terms in the $E_2^{p,q}$-picture: $\mathcal{H}_0(-\beta-\varepsilon_A; X_2)$, and $\mathcal{H}_i(-\beta-\varepsilon_A; X_3)$ for $0 \le i \le 2$. The sequence converges to $M = M^\vee = \mathbb{D}(H_A(\beta)^\vee$. Since there are only two nonzero columns in (1), the spectral sequence collapses on the $E_2^{\bullet,\bullet}$-page and there is a "long" exact sequence

$$0 \longrightarrow \mathcal{H}_2(-\beta-\varepsilon_A; X_3) \longrightarrow \mathcal{H}_0(-\beta-\varepsilon_A; X_2) \longrightarrow M \longrightarrow \mathcal{H}_1(-\beta-\varepsilon_A; X_3) \longrightarrow 0.$$

The display shows that $M \longrightarrow \mathcal{O}_A \oplus \mathcal{O}_A \longrightarrow 0$ is exact and hence $\operatorname{Hom}_{D_A}(M, \mathcal{O}_A)$ is at least a two-dimensional vector space. However, by Proposition 3.4.11 in [8], $\mathcal{O}_A$ contains at most one polynomial solution to any $H_A(\beta')$. This contradiction shows that M is not a GKZ-system proper. $\square$

Remark 4.7. Let A be a $2 \times n$ matrix of rank two with $\mathbb{Z}A = \mathbb{Z}^2$. If $\mathbb{N}A$ is not Cohen–Macaulay, then $X_{n-1} = \operatorname{Ext}^{n-1}_{R_A}(S_A, R_A)$ is a finite-dimensional nonzero vector space. The exceptional set of A is the finite set $\operatorname{tdeg}(X_{n-1}) - \varepsilon_A$, see [1]. The module $X_{n-2} = \operatorname{Ext}^{n-2}_{R_A}(S_A, R_A)$ is Cohen–Macaulay, so has no higher Euler–Koszul homology.

For each such exceptional β the duality spectral sequence has the form as in the proposition. Namely, there are two nonzero columns, the $(n-2)$-nd and the $(n-1)$-st. The $(n-1)$-st contains three modules, $\mathcal{O}_A$, $\mathcal{O}_A^2$, $\mathcal{O}_A$, and the $(n-2)$-nd contains one module. The spectral sequence gives an epimorphism from $\mathbb{D}(H_A(\beta))$ onto $\mathcal{O}_A^2$. In particular, A-exceptional degrees give duals that are not A-hypergeometric systems.

It would be interesting to know what can be rescued if A is not normal but Cohen–Macaulay. However, no matter how bad A is, generic parameters satisfy Takayama's conjecture:

Theorem 4.8. *For all $A \in \mathbb{Z}^{d,n}$ there is a Zariski open set $U \subseteq \mathbb{C}^d$ of parameters such that $\mathbb{D}(H_A(\beta))$ is a proper GKZ-system for all $\beta \in U$.*

Proof. We note that $H_A(\beta) = \mathcal{H}_0(\beta; S_A)$ is for generic β isomorphic to $\mathcal{H}_0(\beta; \tilde{S}_A)$ since $\tilde{S}_A/S_A$ is an R_A-module of dimension at most $d-1$. Since $\tilde{S}_A$ is a Cohen–Macaulay S_A-module, the dual of $\mathcal{H}_0(\beta; \tilde{S}_A)$ equals $\mathcal{H}_0(-\beta-\varepsilon_A, \operatorname{Ext}^{n-d}_{R_A}(\tilde{S}_A, R_A))$ up to a coordinate transformation.

Let $\tilde{A} \supseteq A$ be a matrix composed of columns that generate the saturation of the semigroup $\mathbb{N}A$: $\mathbb{N}\tilde{A} = \mathbb{R}_{\geq 0}A \cap \mathbb{Z}A$. Let $R_{\tilde{A}} \supseteq R_A$ be the polynomial ring $\mathbb{C}[\partial_{\tilde{A}}]$ surjecting onto $\tilde{S}_A = S_{\tilde{A}}$. Then $\operatorname{Ext}^{n-d}_{R_A}(\tilde{S}_A, R_A))$ and $\operatorname{Ext}^{\tilde{n}-d}_{R_{\tilde{A}}}(\tilde{S}_A, R_{\tilde{A}})$ are isomorphic as R_A-modules. To see this, note first that the restriction from $R_{\tilde{A}}$-modules to R_A-modules preserves the degree. Then local duality over R_A shows that $\operatorname{Ext}^{n-d}_{R_A}(\tilde{S}_A, R_A)^* \cong H^d_{\mathfrak{m}_A}(\tilde{S}_A)$ over R_A. Here $(-)^*$ denotes the vector space dual on the graded pieces, followed by the shift by ε_A. Since the radical of $\mathfrak{m}_A\tilde{S}_A$

is $\mathfrak{m}_{\tilde{A}}$, $H^d_{\mathfrak{m}_A}(\tilde{S}_A) = H^d_{\mathfrak{m}_{\tilde{A}}}(\tilde{S}_A)$ as graded R_A-modules. By local duality over $R_{\tilde{A}}$, $H^d_{\mathfrak{m}_{\tilde{A}}}(\tilde{S}_A)$ is isomorphic to $\left(\operatorname{Ext}^{\tilde{n}-d}_{R_{\tilde{A}}}(\tilde{S}_A, R_{\tilde{A}})\right)^*$ shifted by $\varepsilon_A - \varepsilon_{\tilde{A}}$. Thus,

$$\operatorname{Ext}^{n-d}_{R_A}(\tilde{S}_A, R_A) \cong \operatorname{Ext}^{\tilde{n}-d}_{R_{\tilde{A}}}(\tilde{S}_A, R_{\tilde{A}})(\varepsilon_{\tilde{A}} - \varepsilon_A).$$

So $\mathbb{D}(H_A(\beta)) \cong \mathcal{H}_0(-\beta - \varepsilon_{\tilde{A}}, \operatorname{Ext}^{\tilde{n}-d}_{R_{\tilde{A}}}(\tilde{S}_A, R_{\tilde{A}}))$ for generic β where $\operatorname{Ext}^{\tilde{n}-d}_{R_{\tilde{A}}}(\tilde{S}_A, R_{\tilde{A}})$ is viewed as an R_A-module.

Since $\tilde{A}$ is normal, $\operatorname{Ext}^{\tilde{n}-d}_{R_{\tilde{A}}}(\tilde{S}_A, R_{\tilde{A}})$ agrees up to shift with the interior ideal $N_{\tilde{A}}$ of $\tilde{S}_A$. Now $N_{\tilde{A}}$, shifted appropriately, injects into S_A with a cokernel of dimension at most $d-1$. Hence for generic β, $\mathbb{D}(H_A(\beta)) \cong H_A(-\beta - \gamma; S_A))$ for suitable γ. ☐

Based on our results and computer experiments we close with

Conjecture 4.9. The set of parameters β for which the dual of $H_A(\beta)$ is no classical GKZ-system is the A-exceptional arrangement $\mathcal{E}_A = \bigcup_{i=0}^{d-1} \operatorname{qdeg}(H^i_{\mathfrak{m}}(S_A))$.

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Department of Mathematics, Purdue University, 150 N University Street, West Lafayette IN 47907, USA.
E-mail address: walther@math.purdue.edu