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1 Introduction

This thesis deals with algorithmic algebraic geometry in characteristic zero. More specifically, it develops ideas that arise in the theory of linear differential operators and applies them to the construction of algorithms that compute cohomological functors and invariants associated to varieties or polynomials and modules.

The basic ideas and algorithms were introduced by T. Oaku in a series of papers. They include the computation of: Bernstein-Sato polynomials, restriction of holonomic modules to a hyperplane, localizations, and local cohomology at a hypersurface [32, 33, 34, 35].

These results were developed further by T. Oaku, N. Takayama and myself as follows. I gave in [47] an algorithm to compute iterated local cohomology with arbitrary support of holonomic modules, thereby solving the problem of algorithmic determination of local cohomological dimension of algebraic varieties in affine space.

T. Oaku and N. Takayama gave in [37] an algorithm for the restriction of modules to subspaces $x_1 = \ldots = x_d = 0$ in affine n-space. An important application was given in [36], where a method is explained to compute the de Rham cohomology of the complement of the complement of a hypersurface in affine space. In [48] I took up this theme and gave an algorithm that computes the restriction of complexes of D-modules to subspaces $x_1 = \ldots = x_d = 0$ in affine n-space. The complexes in question must have holonomic cohomology. In the same paper I provided as an application an algorithm to compute the de Rham cohomology of the complement of arbitrary subvarieties of affine space.

In [38] all three of us worked together on a method to localize a module at a hypersurface that is holonomic at the complement of the hypersurface, but not necessarily on the hypersurface.

This thesis covers the mentioned results and is based on [32, 34, 36, 37, 38, 47, 48].

We shall now go through a list of objects that will concern us, give definitions and historical background, and explain the interest in computing them.

1.1 Bernstein-Sato Polynomials

Let $f \in R_n = \mathbb{C}[x_1, \ldots, x_n]$ be a polynomial in n variables with complex coefficients. J. Bernstein proved in [2] that if s is a complex parameter then there is a linear differential operator $P(s)$ in the variables $x_1, \ldots, x_n$ which is polynomial in s and satisfies

$$P(s)(f^{s+1}) = b(s)f^s,$$

where $b(s)$ is in $\mathbb{C}[s]$. Here we interpret the action of a partial derivative $\partial_i = \frac{\partial}{\partial x_i}$ on f^s in a purely symbolic way: $\partial_i(f^{s+j}) = (s+j)\partial_i(f) \cdot f^{s+i-1}$.

Bernstein's investigations were spurred by a question of Gelfand's on the existence of holomorphic extensions of the germ f^s in complex affine space.

The important bit is of course that $P(s)$ can be chosen in such a way that $b(s)$ is not identically zero. There is a unique monic nonzero polynomial in $\mathbb{C}[s]$ that divides all other $b(s)$ that occur in a similar equation for the same f. This polynomial is called the *Bernstein-Sato polynomial of* f, and written $b_f(s)$.

M. Kashiwara [23] proved that all roots of $b_f(s)$ are rational and B. Malgrange [29] showed that the roots of $b_f(s)$ provide some information about eigenvalues of certain natural monodromy matrices around the locus described by $f = 0$.

The classical example is $f = x_1^2 + \ldots + x_n^2$, in which case $b_f(s) = (s+1)(s+\frac{n}{2})$ and $P_f(s) = \frac{1}{4}\sum_1^n {\partial_i}^2$.

1.2 Generalized Bernstein-Sato Polynomials of Holonomic Modules

Let D_n stand for the ring of $\mathbb{C}$-linear differential operators on $\mathbb{C}[x_1, \ldots, x_n]$. Then $D_n = \mathbb{C}\langle x_1, \ldots, x_n, \partial_1, \ldots, \partial_n \rangle$, the quotient of the free algebra on the generators $x_1, \ldots, \partial_n$ modulo the commutator relations $[x_i, x_j] = [\partial_i, \partial_j] = 0$, $[\partial_i, x_i] = 1$, $[\partial_i, x_k] = 0$ for all i, j, k with $k \neq i$.

D_n is not commutative, but simple and left and right Noetherian of global homological dimension n. A left D_n-module M is called *holonomic* if and only if $\mathrm{Ext}^i_{D_n}(M, D_n) = 0$ unless $i = n$. In some sense, holonomic mimics Artinian. For example, submodules and quotients of holonomic modules are either zero or holonomic.

If $m \in M$ generates a holonomic left module over D_n and $f \in R_n$, then it is known that there exists a generalized monic nonzero Bernstein-Sato polynomial $b_f^m(s)$ minimal with respect to satisfying an identity

$$P(s)(m \otimes f^{s+1}) = b_f^m(s) m \otimes f^s.$$

Here, the tensor indicates the formal use of a product rule: $\partial_i(m \otimes f^s) = \partial_i(m) \otimes f^s + s\frac{\partial_i(f)}{f} m \otimes f^s$.

Bernstein-Sato polynomials and their generalized versions play a crucial role in representing localizations as finitely generated modules over D_n.

1.3 Localizations

Let $f \in R_n$ and $M = D_n \cdot m$ be a left D_n-module. The localization $M[f^{-1}] = M_f = M \otimes_{R_n} R_n[f^{-1}]$ becomes a D_n-module in a natural way via the product rule. Localizations are of fundamental importance in algebraic geometry since they describe sections of sheaves on special open subsets.

B. Malgrange introduced an action of $D_n\langle t, \partial_t\rangle$ on $M_f[s] \otimes f^s$, see [29] and section 3.1. It turns out that $-\partial_t t$ acts as multiplication by s on $m \otimes f^s$ and $b_f^m(s)$ is the minimal polynomial of $-\partial_t t$ on $(M_f[s] \otimes f^s)/t(M_f[s] \otimes f^s)$. Let a denote the smallest integer root of $b_f^m(s)$. One can deduce that for $a \geq a' \in \mathbb{C}$ the substitution $s \to a'$ turns the element $m \otimes_{R_n} R_n[f^{-1}] \otimes f^s \in M_f[s] \otimes f^s$ into a generator $mf^{a'}$ for M_f over D_n ([23]).

The work in [33] is the first time that Gröbner basis computations led to information about localized objects, usually complicated by the infinite generatedness of such things. We will use methods of [33] as basic building blocks in our thesis.

1.4 Local Cohomology

Let R be a commutative Noetherian ring, I an ideal in R and M an R-module. The i-th *local cohomology functor* with respect to I is the i-th right derived functor of the functor $H^0_I(-)$ which sends M to the I-torsion $\bigcup_{k=1}^{\infty}(0 :_M I^k)$ of M and is denoted by $H^i_I(-)$. Local cohomology was introduced by A. Grothendieck as an algebraic analog of (classical) relative cohomology. An introduction to local cohomology may be found in [5].

The *cohomological dimension* of I in R, denoted by $\mathrm{cd}(R, I)$, is the smallest integer c such that $H^q_I(M) = 0$ for all $q > c$ and all R-modules M. If R is the coordinate ring of an affine variety X and $I \subseteq R$ is the defining ideal of the Zariski closed subset $V \subseteq X$ then the *local cohomological dimension* of V in X is defined as $\mathrm{cd}(R, I)$. It is not hard to show that if X is smooth, then the integer $\dim(X) - \mathrm{cd}(R, I)$ depends only on V but neither on X nor on the embedding $V \hookrightarrow X$.

Knowledge of local cohomology modules provides interesting information, illustrated by the following three situations. Let $I \subseteq R$ and $c = \mathrm{cd}(R, I)$. Then I cannot be generated by fewer than c elements. In fact, no ideal J with the same radical as I will be generated by fewer than c elements.

Let H^i_{dR} stand for the i-th de Rham cohomology group. A second application is a family of results commonly known as Barth theorems which are a generalization of the classical Lefschetz theorem that states that if $Y \subseteq \mathbb{P}^n_{\mathbb{C}}$ is a hypersurface then $H^i_{dR}(\mathbb{P}^n_{\mathbb{C}}) \to H^i_{dR}(Y)$ is an isomorphism for $i < \dim(Y) - 1$ and injective for $i = \dim(Y)$. For example, let $Y \subseteq \mathbb{P}^n_{\mathbb{C}}$ be a closed subset and $I \subseteq R = \mathbb{C}[x_0, \ldots, x_n]$ the defining ideal of Y. Then $H^i_{dR}(\mathbb{P}^n_{\mathbb{C}}) \to H^i_{dR}(Y)$ is an isomorphism for $i < n - \mathrm{cd}(R, I)$ and injective if $i = n - \mathrm{cd}(R, I)$ ([16], theorem III.7.1).

Another consequence of the work of Ogus and Hartshorne ([39], 2.2, 2.3 and [16], IV.3.1) is the following. If $I \subseteq R = \mathbb{C}[x_0, \ldots, x_n]$ is the defining ideal of a complex smooth variety $V \subseteq \mathbb{P}^n_{\mathbb{C}}$ then, for $i < n - \mathrm{codim}(V)$, $\dim_{\mathbb{C}} \mathrm{soc}_R(H^0_{\mathfrak{m}}(H^{n-i}_I(R)))$ equals $\dim_{\mathbb{C}} H^i_x(\tilde{V}, \mathbb{C})$ where $H^i_x(\tilde{V}, \mathbb{C})$ stands for the

i-th singular cohomology group of the affine cone $\tilde{V}$ over V with support in the vertex x of $\tilde{V}$ and with coefficients in $\mathbb{C}$ ($\mathrm{soc}_R(M)$ denotes the socle $(0:_M \mathfrak{m}) \subseteq M$ for any R-module M).

Algorithmic computation of local cohomology modules is the major development in chapter 4

1.5 b-functions and Restriction

Let $D_{n+1} = D_n\langle t, \partial_t\rangle$ and let $m \in M$ generate a holonomic D_{n+1}-module. We define a filtration F^k on D_{n+1} by $F^k =$ the linear span of all monomials in D_{n+1} $t^a\partial_t^b x^\alpha \partial^\beta$ for which $b - a \leq k$. Here we use multi-index notation $x = x_1 \ldots x_n$, $\partial = \partial_1 \ldots \partial_n$, $x^\alpha = x_1^{\alpha_1} \cdot \ldots \cdot x_n^{\alpha_n}$, $\partial^\beta = \partial_1^{\beta_1} \cdot \ldots \cdot x_n^{\beta_n}$. This induces in a natural way a filtration on $D_{n+1} \cdot m$.

We define the *b-function of m relative to the filtration* to be the monic polynomial $b^m(s)$ of smallest degree satisfying

$$b^m(t\partial_t) \cdot F^0 \cdot m \subseteq F^{-1} \cdot m.$$

Holonomicity of $D_{n+1} \cdot m$ guarantees existence of a nonzero $b^m(s)$.

$b^m(s)$ can to some extent be thought of in the following way. Consider the operation $(D_{n+1}/t \cdot D_{n+1}) \otimes_{D_{n+1}} (-)$. This is a right exact functor from the category of left D_{n+1}-modules to that of left D_n-modules. Its left derived functors correspond to the *restriction of a module* on $(n+1)$-dimensional affine space to the hyperplane $t = 0$. Restriction functors are occasionally referred to as *integrals* and written $\int_\pi^i M$ if $\pi : (x,t) \to x$ is the projection $\mathbb{C}^{m+1} \to \mathbb{C}^m$.

It is a priori unclear whether finitely generated modules over D_{n+1} have finitely generated integrals over D_n. Kashiwara proved that if the module in question is holonomic, then this is indeed the case. In order to make this theorem algorithmic, one uses the b-function: the 0-th restriction for example is generated over D_n by the left-multiples of m by $1, \partial_t, \ldots, {\partial_t}^a$, where a is the largest integer root of $b^m(s)$.

One can, up to shift, interpret the Bernstein-Sato polynomial $b_f^m(s)$ as the b-function corresponding to the restriction of the D_{n+1}-module introduced by

Malgrange (see the section on Bernstein-Sato polynomials, 3.1) to the hypersurface $t = f$

We shall discuss these things in detail in the last chapter and use it to compute de Rham cohomology of certain varieties.

1.6 De Rham Cohomology

In his famous paper [16] R. Hartshorne developed the concept of algebraic de Rham cohomology of algebraic varieties as an analog to classical (singular) cohomology. Results of A. Grothendieck and P. Deligne prove that it agrees with classical cohomology if the base field is $\mathbb{C}$. Moreover, Hartshorne also defined the notion of algebraic de Rham cohomology with supports and proved that it fits into certain natural long exact sequences related to inclusion maps (5.11).

In [36], the authors give an algorithm that computes (by Gröbner basis computations in the Weyl algebra) the algebraic de Rham cohomology of the complement U of any given hypersurface Y of $X = \mathbb{C}^n$. Their method is based on the initial definition of Hartshorne, as the hypercohomology of the de Rham complex on U. They show that this complex is in the derived category the same as the tensor product over $\mathcal{O}_X$ of the sheaf of differential n-forms on X with a resolution of $\mathcal{O}_U$, $\mathcal{O}_U$ considered as a module over the sheaf of differential operators on X. The computation of the hypercohomology of the latter complex reduces to computation of usual cohomology of the global sections since U is affine and the sheaves involved are quasi-coherent. An algorithm to compute the cohomology of complexes of the type one gets after taking global sections was given in [37]. The strategy is to use the method of restriction of a D-module to a linear subvariety ([33] and [37], section 5).

Generalizing these ideas we show in chapter 5 how to compute the restriction of a complex of D-modules with holonomic cohomology to a linear subvariety. A corollary is an algorithm to compute de Rham cohomology groups of U with complex coefficients, where $U = \mathbb{C}^n \setminus V(f_1, \ldots, f_r)$ and all $f_i \in R_n$ (see also chapter 5.

1.7 Gröbner Bases

Gröbner bases were used as early as in F. Macaulay's work [28], who was interested in counting the dimension of Artinian modules over polynomial rings. The first formal definition was given by B. Buchberger in his thesis [7] who called them after his advisor W. Gröbner. Buchberger was interested in determining the structure of the residue class ring of an Artinian ideal in a polynomial ring. Nearly at the same time similar ideas were used to prove one of the outstanding results in this century, the resolution of singularities in characteristic zero by H. Hironaka [18].

These theoretical applications of Gröbner bases stood alone until the early eighties when a group of researchers started to investigate properties of Gröbner bases systematically, and then went on to implement them. Most noticeable are the names of D. Eisenbud, D. Spear, F.-O. Schreyer, D. Bayer and M. Stillman, the latter two of which created the first computer package dealing with Gröbner bases: Macaulay [1].

Since that time theory and praxis of Gröbner bases were developed in various directions, here we list only some of them. T. Mora contributed the tangent cone approach for local rings, A. Kandry-Rody and V. Weispfenning investigated the existence of Gröbner basis theories in noncommutative rings over a field, A. Galligo pioneered them in rings of differential operators, and B. Sturmfels wrote a book on the use of Gröbner bases in invariant theory. This list, as mentioned above, is by no means complete or even near completeness.

Practical applications of Gröbner basis computations include computation of Hilbert functions, free resolutions, kernels and preimages, Ext-groups, intersections of modules, module quotients and saturations, dimension and depth of quotient rings, primary decomposition, Bernstein-Sato polynomials, characteristic varieties of holonomic modules and various others to be discussed in this thesis: restriction, local cohomology, de Rham cohomology. Again, this is not a complete list of applications. Also, Gröbner basis theory has been used in other areas of mathematics as for example linear programming [42] and control theory [13].

1.8 Structure of the Thesis

Besides the introduction, this thesis has four chapters which build on each other. In the following chapter we introduce some basic terminology related to D-modules, the Weyl algebra and Gröbner bases in the Weyl algebra.

In chapter three, we make our first steps into the realm of algebraic analysis, the algebraic theory of linear differential operators. We shall study properties of Bernstein-Sato polynomials, b-functions and localizations as well as describe ways to compute some of these algorithmically.

Then we have a chapter on local cohomology. We give basic definitions and spend some time on the motivation for computing local cohomology modules on schemes. Then we present algorithms that compute various local cohomological functors and invariants.

In the last chapter we study de Rham cohomology groups of the complement of an closed subvariety in affine n-space. We generalize a method of T. Oaku and N. Takayama on restriction functors and use the generalization for an algorithm to compute said de Rham cohomology groups.

In most chapters we shall give examples of implementations and discuss performance and natural limits posed by the problem.

1.9 Computability

We will work over a field K of characteristic zero. For the theoretical results, K is arbitrary. However, eventually we will be interested in implementations of our algorithms. With this in mind we will tacitly assume that the coefficients of all quantities we have to work with belong to a subfield K of $\mathbb{C}$ (the complex numbers) that is *computable*. That is to say, elements of K can be represented with a finite set of data, their sum, product and quotient can be calculated in a finite number of steps, and there is a finite procedure that determines whether a given expression of elements of K is zero or not.

2 The Weyl Algebra

2.1 Preliminaries

2.1.1 Notation

Throughout we shall use the following notation: K will denote a field of characteristic zero, $R_n = K[x_1, \ldots, x_n]$ the ring of polynomials over K in n variables, $D_n = K\langle x_1, \partial_1, \ldots, x_n, \partial_n\rangle$ the Weyl algebra over K in n variables, or, equivalently, the ring of K-linear differential operators on R, Δ_n will denote the maximal left ideal $D_n \cdot (\partial_1, \ldots, \partial_n)$ of D_n and I will stand for the ideal $(f_1, \ldots, f_r)$ in R_n. Every D_n-module becomes an R_n-module via the embedding $R_n \hookrightarrow D_n$.

We shall use multi-index notation in R_n and D_n as follows. Usually we will denote a monomial in D_n (written in right normal form, i.e. the derivations come last) by $c_{\alpha\beta}x^\alpha\partial^\beta$ and mean $c_{\alpha\beta}x_1^{\alpha_1}\ldots x_n^{\alpha_n}\partial_1^{\beta_1}\ldots\partial_n^{\beta_n}$.

2.1.2 D-modules

A good introduction to D-modules is the book by Björk, [3]. Here we review chapter 1 of that book.

Let $f \in R_n$. Then the R_n-module $R_n[f^{-1}]$ has a structure as left D_n-module: $x_i(\frac{g}{f^k}) = \frac{x_i g}{f^k}, \partial_i(\frac{g}{f^k}) = \frac{\partial_i(g)f - k\partial_i(f)g}{f^{k+1}}$. This may be thought of as a special case of localizing an D_n-module: if M is a left D_n-module and $f \in R_n$ then $M \otimes_{R_n} R_n[f^{-1}]$ becomes an D_n-module via $\partial_i(m \otimes \frac{g}{f^k}) = m \otimes \partial_i(\frac{g}{f^k}) + \partial_i m \otimes \frac{g}{f^k}$. Localization of D_n-modules lies at the heart of our algorithms.

Of particular interest are the *holonomic* modules which are those finitely generated D_n-modules N for which $\mathrm{Ext}^j_{D_n}(N, D_n)$ vanishes unless $j = n$. Our standard example of a holonomic module is $R = D_n/\Delta_n$, generated by $1 \in R_n$. Holonomic modules are always cyclic and of finite length over D_n. Besides that, if $N = D_n/L$ is holonomic, $f \in R_n$, s is an indeterminate and $\overline{1}$ is the coset of $1 \in D_n$ in N, then there is a nonzero polynomial $b(s)$ in $K[s]$ and an operator $P(s) \in D_n[s]$ such that $P(s)(\overline{1} \otimes f \cdot f^s) = b(s) \cdot \overline{1} \otimes f^s$. The unique monic polynomial that divides all other polynomials satisfying an identity of this type is called the *Bernstein polynomial* of L and f and denoted by $b_f^L(s)$. Any operator $P(s)$ that satisfies $P(s)(\overline{1} \otimes f^{s+1}) = b_f^L(s)\overline{1} \otimes f^s$ we shall call a *Bernstein operator* and refer to the roots of $b_f^L(s)$ as *Bernstein roots* of f on D_n/L.

Any localization of a holonomic module $N = D_n \cdot g$ at a single element (and hence at any finite number of elements) of R_n is holonomic ([3], 1.5.9) and in particular cyclic over D_n, generated by $f^{-a}g$ for sufficiently large $a \in \mathbb{N}$. This facilitates the use of Gröbner bases as computational tool for maps between holonomic modules and their localizations. As a special case we note that localizations of R_n are holonomic, and hence finite, over D_n.

2.2 Gröbner Bases of Modules over the Weyl Algebra

In this section we review some of the concepts and results related to the Buchberger algorithm in modules over Weyl algebras. It turns out that with a little care many of the important constructions from the theory of commutative Gröbner bases carry over to our case. For an introduction into non-commutative monomial orders and related topics, [22] is a good source. Readers interested in proofs and more details are encouraged to look at the book by D. Eisenbud ([10], chapter 15 for the commutative case) or [12], [8], [44], [22] (for the more general situation).

If $m = c_{\alpha\beta}x^\alpha\partial^\beta, c_{\alpha\beta} \in K$, we will say that m has *Bernstein degree* $\deg m = |\alpha+\beta|$ and an operator $P \in D_n$ has Bernstein degree equal to the largest degree of any monomial occurring in P.

Recall that a *monomial order* $<$ in D_n is a total order on the monomials of D_n, subject to $m < m' \Rightarrow mm'' < m'm''$ for all nonzero monomials m, m', m''. Since the product of two monomials in our notation is not likely to be a monomial (as $\partial_i x_i = x_i\partial_i + 1$) it is not obvious that such orderings exist at all. However, the commutator of any two monomials m_1, m_2 will be a polynomial of degree at most $\deg(m_1) + \deg(m_2) - 2$. That means that the degree of an operator and its component of maximal degree is independent of the way it is represented. Thus we may for example introduce a monomial order on D_n by taking any monomial order on $\tilde{D}_n = K[x_1, \dots, x_n, \partial_1, \dots, \partial_n]$ (the polynomial ring in $2n$ variables) that refines the partial order given by total degree, and defining $m_1 > m_2$ in D_n if and only if $m_1 > m_2$ in $\tilde{D}_n$. A wealth of other possibilities is discussed in [41].

Let $<$ be a monomial order on D_n. Let $G = \bigoplus_1^d D_n \cdot \gamma_i$ be the free D_n-module on the symbols $\gamma_1, \dots, \gamma_d$. We define a monomial order on G by $m_i \gamma_i > m_j \gamma_j$ if either $m_i > m_j$ in D_n, or $m_i = m_j$ and $i > j$. The largest monomial $m\gamma$ in an element $g \in G$ will be denoted by $\mathrm{in}(g)$, called the *initial term of g*. Of fundamental importance is

Algorithm 2.1 (Remainder). Let h and $\underline{g} = \{g_i\}_1^s$ be elements of G. Set $h_0 = h, \sigma_0 = 0, j = 0$ and let $\varepsilon_i = \varepsilon(g_i)$ be symbols. Then

`Repeat`

`If` $\mathrm{in}(g_i) | \mathrm{in}(h_j)$ `set`

$$\{h_{j+1} := h_j - \tfrac{\mathrm{in}(h_j)}{\mathrm{in}(g_i)} g_i,$$
$$\sigma_{j+1} := \sigma_j + \tfrac{\mathrm{in}(h_j)}{\mathrm{in}(g_i)} \varepsilon_i,$$
$$j := j + 1\}$$

`Until` `No` $\mathrm{in}(g_i)$ `divides` $\mathrm{in}(h_j)$.

The result is h_a, called a *remainder* $\Re(h; \underline{g})$ *of h under division by $\underline{g}$*, and an expression $\sigma_a = \sum_{i=1}^s a_i \varepsilon_i$ with $a_i \in D_n$ and $\mathrm{in}(a_i g_i) < \mathrm{in}(g)$ for all i. By Dickson's lemma ([22], 1.1) the algorithm terminates. It is worth mentioning that $\Re(h; \underline{g})$ is not uniquely determined, it depends on which g_i we pick amongst those whose initial term divides the initial term of h_j.

Note that if h_a is zero, σ_a tells us how to write h in terms of $\underline{g}$. Such a σ_a is called a *standard expression for h with respect to* $\{g_1, \dots, g_s\}$.

Definition 2.2. If $\mathrm{in}(g_i)$ and $\mathrm{in}(g_j)$ involve the same basis element of G, then we set $s_{ij} = m_{ji} g_i - m_{ij} g_j$ and $\sigma_{ij} = m_{ji} \varepsilon_i - m_{ij} \varepsilon_j$ where $m_{ij} = \frac{\mathrm{lcm}(\mathrm{in}(g_j), \mathrm{in}(g_i))}{\mathrm{in}(g_j)}$. Otherwise, σ_{ij} and s_{ij} are defined to be zero. s_{ij} is called the *S-polynomial* to g_i and g_j.

Suppose $\Re(s_{ij}; \underline{g})$ is zero for all i, j. Then we call $\underline{g}$ a *Gröbner basis* for the module $D_n \cdot (g_1, \ldots, g_s)$.

The following result ([22], Lemma 3.8) indicates the usefulness of Gröbner bases.

Proposition 2.3. *Let $\underline{g}$ be a finite set of elements of G. Then $\underline{g}$ is a Gröbner basis if and only if $h \in D_n \cdot \underline{g}$ implies $\exists i : \mathrm{in}(g_i) | \mathrm{in}(h)$.* □

Computation of Gröbner bases over the Weyl algebra works just as over polynomial rings:

Algorithm 2.4 (Buchberger). Input: $\underline{g} = \{g_1, \ldots, g_s\} \subseteq G$.

Output: a Gröbner basis for $D_n \cdot (g_1, \ldots, g_s)$.

Begin.

`Repeat`

  `If` $h := \Re(s_{ij}; \underline{g}) \neq 0$

    `add` h `to` $\underline{g}$

`Until` $\Re(s_{ij}; \underline{g}) = 0$ `for all` $\{i, j\}$.

`Return` $\underline{g}$.

End.

2.2.1

Now we shall describe the construction of kernels of D_n-linear maps using Gröbner bases. We first consider the case of a map between free D_n-modules.

Let $E = \bigoplus_1^s D_n \varepsilon_i, G = \bigoplus_1^r D_n \gamma_j$ and $\phi : E \to G$ be a D_n-linear map. Assume $\phi(\varepsilon_i) = g_i$. Suppose that in order to make $\underline{g}$ a Gröbner basis we have to add $g_1', \ldots, g_{s'}'$ to $\underline{g}$ which satisfy $g_i' = \sum_{k=1}^s a_{ik} g_k$. We get an induced map

$$\begin{tikzcd} \bigoplus\limits_1^{s+s'} D_n\varepsilon_i \arrow[d, "\pi"] \arrow[dr, "\tilde{\phi}"] & \\ \bigoplus\limits_1^{s} D_n\varepsilon_i \arrow[r, "\phi"'] & \bigoplus\limits_1 D_n\gamma_j \end{tikzcd}$$

where π is the identity on ε_i for $i \leq s$ and sends ε_{i+s}

into $\sum_{k=1}^{s} a_{ik}\varepsilon_k$. Of course, $\tilde{\phi} = \phi\pi$.

The kernel of ϕ is just the image of the kernel of $\tilde{\phi}$ under π. So in order to find kernels of maps between free modules one may assume that the generators of the source are mapped to a Gröbner basis and replace ϕ by $\tilde{\phi}$. Recall from definition 2.2 that $\sigma_{ij} = m_{ji}\varepsilon_i - m_{ij}\varepsilon_j$ or zero, depending on the leading terms of g_i and g_j.

Proposition 2.5. *Assume that $\{g_1, \dots, g_s\}$ is a Gröbner basis. Let $s_{ij} = \sum d_{ijk} g_k$ be standard expressions for the S-polynomials. Then*

$$\{\sigma_{ij} - \sum_k d_{ijk}\varepsilon_k\}_{1 \le i < j \le s}$$

generate the kernel of $\phi : \bigoplus_1^s D_n\varepsilon_i \to \bigoplus_1^r D_n\gamma_j$, sending ε_i to g_i.

The proof proceeds exactly as in the commutative case, see for example [10], section 15.10.8.

2.2.2

We explain now how to find a set of generators for the kernel of an arbitrary D_n-linear map. Let E, G be as in subsection 2.2.1 and suppose $P \subseteq E$, $D_n(q_1, \dots, q_b) = Q \subseteq G$ and $\phi : \bigoplus_1^s D_n\varepsilon_i/P \to \bigoplus_1^r D_n\gamma_j/Q$. It will be sufficient to consider the case $P = 0$ since we may lift ϕ to the free module E surjecting onto E/P.

Let as before $\phi(\varepsilon_i) = g_i$. A kernel element in E is a sum $\sum_i a_i\varepsilon_i, a_i \in D_n$, which if ε_i is replaced by g_i can be written in terms of the generators q_j of Q. Let $\underline{\beta} = \{\beta_1, \dots, \beta_c\}$ be such that $\underline{g} \cup \underline{q} \cup \underline{\beta}$ is a Gröbner basis for $D_n(\underline{g}, \underline{q})$, obtained from $\underline{g} \cup \underline{q}$ by application of algorithm 2.4. Then from algorithm 2.1 we have expressions

$$\beta_i = \sum_j c_{ij} g_j + \sum_k c'_{ik} q_k, \tag{2.1}$$

with $c_{ij}, c'_{ik} \in D_n$. Furthermore, by proposition 2.5, algorithm 2.4 returns a generating set $\underline{\sigma}$ for the syzygies on $\underline{g} \cup \underline{q} \cup \underline{\beta}$. Write

$$\sigma_i = \sum_j a_{ij}\varepsilon_{g_j} + \sum_k a'_{ik}\varepsilon_{q_k} + \sum_l a''_{il}\varepsilon_{\beta_l} \tag{2.2}$$

and eliminate the last sum using the relations (2.1) to obtain syzygies

$$\tilde{\sigma}_i = \sum_j a_{ij}\varepsilon_{g_j} + \sum_k a'_{ik}\varepsilon_{q_k} + \sum_l a''_{il}\left(\sum_v c_{lv}\varepsilon_{g_v} + \sum_w c'_{lw}\varepsilon_{q_w}\right). \tag{2.3}$$

These will then form a generating set for the syzygies on $\underline{g} \cup \underline{q}$. Cutting off the q-part of these syzygies we get a set of forms

$$\left\{\sum_j a_{ij}\varepsilon_{g_j} + \sum_l a''_{il}\left(\sum_v c_{lv}\varepsilon_{g_v}\right)\right\}$$

which generate the kernel of the map $E \to G/Q$.

2.2.3

The comments in this subsection will find their application in algorithm 4.2 which computes the structure of $H^i_{\mathfrak{m}}(H^j_I(R))$ as D_n-module where $\mathfrak{m} = R_n \cdot (x_1, \ldots, x_n)$. Let

$$\begin{tikzcd}
M'_3 \arrow[r, "\alpha"] & M_3 \arrow[r, "\alpha'"] & M''_3 \\
M'_2 \arrow[r, "\beta"] \arrow[u, "\phi'"'] & M_2 \arrow[r, "\beta'"] \arrow[u, "\psi'"'] & M''_2 \arrow[u, "\rho'"'] \\
M'_1 \arrow[r, "\gamma"] \arrow[u, "\phi"'] & M_1 \arrow[r, "\gamma'"] \arrow[u, "\psi"'] & M''_1 \arrow[u, "\rho"']
\end{tikzcd}$$

be a commutative diagram of D_n-modules. Note that the row cohomology of the column cohomology at M_2 is given by

$$\left[\ker(\psi') \cap {\beta'}^{-1}(\operatorname{im}\rho) + \operatorname{im}(\psi)\right] \;/\; \left[\beta(\ker(\phi')) + \operatorname{im}(\psi)\right].$$

In order to compute this we need to be able to find:

- preimages of submodules,
- kernels of maps,
- intersections of submodules.

It is apparent that the second and third calculation is a special case of the first: kernels are preimages of zero, intersections are images of preimages (if ${D_n}^r \overset{\phi}{\to} {D_n}^s/M \overset{\psi}{\leftarrow} {D_n}^t$ is given, then $\mathrm{im}(\phi) \cap \mathrm{im}(\psi) = \psi(\psi^{-1}(\mathrm{im}(\phi)))$).

So suppose in the situation $\phi : {D_n}^r/M \to {D_n}^s/N$, $\psi : {D_n}^t/P \to {D_n}^s/N$ we want to find $\psi^{-1}(\mathrm{im}(\phi))$. We may reduce to the case where M and P are zero and then lift ϕ, ψ to maps into ${D_n}^s$. The elements x in $\psi^{-1}(\mathrm{im}\,\phi) \subseteq {D_n}^t$ are exactly the elements in $\ker({D_n}^t \overset{\psi}{\to} {D_n}^s/N \to {D_n}^s/(N + \mathrm{im}\,\phi))$ and this kernel can be found according to the comments in 2.2.2.

2.3 Gröbner Bases relative to the shifted V_d-filtration

Let D_n be the Weyl algebra in $x_1, \ldots, x_n$ and $0 \leq d \leq n$. In this section we shall introduce a filtration on modules over D_n and a technique to compute Gröbner bases in free modules that respects the filtration. Gröbner bases relative to filtrations are fundamental in all our algorithms.

We need to introduce some terminology from [37] related to the V-filtration.

Definitions 2.6. Fix an integer d with $0 \leq d \leq n$ and set $H = \mathrm{Var}(x_1, \ldots, x_d)$. For $\alpha \in \mathbb{Z}^n$, we set $\alpha_H = (\alpha_1, \ldots, \alpha_d, 0, \ldots, 0)$.

On the ring D_n we define the *V_d-filtration* $F_H^k(D_n)$ as the linear span of all operators $x^\alpha \partial^\beta$ for which $|\alpha_H| + k \geq |\beta_H|$. More generally, on a free D_n-module $A = \oplus_1^m D_n \cdot e_j$ we define $F_H^k(A)[\mathfrak{m}]$, where $\mathfrak{m}$ is an element of $\mathbb{Z}^m$, as $\sum F_H^{k-\mathfrak{m}(j)}(D_n) \cdot e_j$. We shall call $\mathfrak{m}$ the *shift vector.*

We define the *V_d-degree* of an operator $P \in A[\mathfrak{m}]$, $V_d deg(P[\mathfrak{m}])$, to be the smallest k such that $P \in F_H^k(A[\mathfrak{m}])$. For example, $V_d deg(x^\alpha \partial^\beta e_i[\mathfrak{m}]) = |\beta_H - \alpha_H| + \mathfrak{m}(i)$.

If A is a free D_n-module of rank m, the phrase *let a shift vector for A be given* will mean the following. First of all we assume that once and forever a minimal set of generators $\{e_j\}_1^m$ for A has been chosen, and secondly that there is given $\mathfrak{m} \in \mathbb{Z}^m$ defining the V_d-degree on A by the formula of the previous paragraph.

If M is a quotient of the free D_n-module $A = \oplus_1^m D_n \cdot e_j$, $M = A/I$, we define the V_d-filtration on M by $F_H^k(M[\mathfrak{m}]) = F_H^k(A[\mathfrak{m}]) + I$. For submodules N of A we define the V_d-filtration by intersection: $F_H^k(N[\mathfrak{m}]) = F_H^k(A[\mathfrak{m}]) \cap N$. Notice that $F_H^k(M[\mathfrak{m}]) \subseteq F_H^{k+1}(M[\mathfrak{m}])$.

If $A^\bullet$ is a free D_n-resolution of the module M, M being positioned in degree b, we say it is *V_d-strict* if there exist shift vectors $\mathfrak{m}_i$ such that $F_H^k(A^i[\mathfrak{m}_i]) \to F_H^k(A^{i+1}[\mathfrak{m}_{i+1}]) \to F_H^k(A^{i+2}[\mathfrak{m}_{i+2}])$ is exact for all $i < b-1$ and all k, and $F_H^k(A^{b-1}[\mathfrak{m}_{b-1}]) \to F_H^k(A^b[\mathfrak{m}_b]) \to F_H^k(M[\mathfrak{m}_b]) \to 0$ is exact for all k.

If M is a submodule of the free module $A[\mathfrak{m}]$, then a *V_d-strict Gröbner basis* or a *V_d-Gröbner basis* for M is a set of generators $\{m_1, \ldots, m_t\}$ for M which satisfies: for all $m \in M$ we can find $\alpha_i \in D_n$ such that $m = \sum \alpha_i m_i$ and $V_d deg(\alpha_i m_i[\mathfrak{m}]) \leq V_d deg(m[\mathfrak{m}])$ for all i.

In order to compute such V_d-strict Gröbner bases, we shall consider a homogenization technique introduced by Oaku in [33] and worked out in detail in [37] by Oaku and Takayama.

Definition 2.7. If $P \in {D_n}^r[\mathfrak{m}]$ then the V_d-homogenization $h(P)$ is the element in $(D_n[t])^r$

$$h(P[\mathfrak{m}]) = \sum_{i,\alpha,\beta} c_{\alpha\beta} x^\alpha \partial^\beta t^{|\beta_H - \alpha_H| + \mathfrak{m}(i) - k} e_i,$$

where $\{e_i\}_i$ is a basis for $D_n[t]^r$, the sum is taken over all monomials $c_{\alpha\beta} x^\alpha \partial^\beta e_i$ of P and k is the minimal V_d-degree of a nonzero monomial in P.

If $Q = h(P)$ then we say that Q is V_d-homogeneous.

It is easy to see that the product of a V_d-homogeneous operator in $D_n[t]$ and a V_d-homogeneous operator in $D_n[t]^r[\mathfrak{m}]$ is V_d-homogeneous in $D_n[t]^r[\mathfrak{m}]$.

Applying the Buchberger algorithm relative to an arbitrary monomial order on $D_n[t]^r$ to a set of V_d-homogeneous elements in $D_n[t]^r[\mathfrak{m}]$ therefore preserves

V_d-homogeneity. We aim to find a useful monomial order $<_h$ so that a Gröbner basis for $<_h$ transforms into a V_d-strict Gröbner basis if we evaluate t at 1.

We have the following property of V_d-homogenization.

Lemma 2.8 ([37], Lemma 3.6). *For $P_1, \ldots, P_k \in {D_n}^r$, put $P = P_1 + \ldots + P_k$. Then there are $l, l_1, \ldots, l_k \in \mathbb{N}$ such that*

$$t^l h(P) = t^{l_1} h(P_1) + \ldots + t^{l_k} h(P_k).$$

□

Let $\mathbb{L} = \mathbb{N}^n \times \mathbb{N}^n$ parameterize the exponent vectors of monomials in D_n. Then $\mathbb{L} \times \{1, \ldots, r\}$ signifies monomials on ${D_n}^r$.

Let $<$ be a well-order on $\mathbb{L} \times \{1, \ldots, r\}$ satisfying

- $(l, i) < (l', i')$ implies $(l + l'', i) < (l' + l'', i')$, for all $l, l', l'' \in \mathbb{L}$ and all $i, i' \in \{1, \ldots, r\}$.

We then define a new total order $<_F$ on $\mathbb{L} \times \{1, \ldots, r\}$ by $(\alpha, \beta, i) <_F (\alpha', \beta', i')$ if and only if $|\beta_H - \alpha_H| + \mathfrak{m}(i) < |\beta'_H - \alpha'_H| + \mathfrak{m}(i')$ or else $|\beta_H - \alpha_H| + \mathfrak{m}(i) = |\beta'_H - \alpha'_H| + \mathfrak{m}(i')$ and $(\alpha, \beta, i) < (\alpha', \beta', i')$ in the order $<$ on $\mathbb{L} \times \{1, \ldots, r\}$. $<_F$ models order by V_d-degree, but is not a well order. Hence the Buchberger algorithm may fail to terminate.

We define a monomial order $<_h$ on $D_n[t]^r$ by $t^\lambda x^\alpha \partial^\beta e_i <_h t^{\lambda'} x^{\alpha'} \partial^{\beta'} e_{i'}$ if either $\lambda < \lambda'$, or $\lambda = \lambda'$ and $x^\alpha \partial^\beta e_i < x^{\alpha'} \partial^{\beta'} e_{i'}$ in ${D_n}^r$. $<_h$ is then a well order.

The point of this is that if P is a V_d-homogeneous element of $D_n[t]^r[\mathfrak{m}]$ then the leading term of P using the new order $<_h$ maps under evaluation of t at 1 to the leading term of $\tilde{P} \in {D_n}^r[m]$ using the old order $<_F$.

This leads to the computation of V_d-strict Gröbner bases as follows.

Proposition 2.9 ([37], Proposition 3.9). *Let $P_1, \ldots, P_l \in {D_n}^r[\mathfrak{m}]$ with $\mathfrak{m} \in \mathbb{Z}^r$. Using the order $<_h$ compute a Gröbner basis G for the left $D_n[t]$-module generated by $h(P_1), \ldots, h(P_l)$. A V_d-strict Gröbner basis for $D_n \cdot (P_1, \ldots, P_l)$ is obtained by evaluating all elements of G at $t = 1$.* □

A consequence of the general property of Buchberger's algorithm with respect to syzygies yields

Proposition 2.10 ([37], Proposition 3.11). *Suppose the Gröbner basis G of the previous proposition has elements $Q_1(t), \ldots, Q_p(t)$. While executing Buchberger's algorithm we created relations*

$$S_{ji}(t)Q_i(t) - S_{ij}(t)Q_j(t) - \sum_{q=1}^{p} U_{ijq}(t)Q_q(t) = 0.$$

Evaluation at $t = 1$ leads to syzygies on $Q_1(1), \ldots, Q_p(1)$. These syzygies generate the syzygies on $Q_1, \ldots, Q_p$. $\square$

Remark 2.11. V_d-strict Gröbner bases are a special case of Gröbner bases respecting weight vectors: if w is a weight on $\mathbb{N}^n$ (that is, an element of $\mathbb{N}^n$ and $w(\alpha) = w \cdot \alpha$, the scalar product), then we say that $\{P_1, \ldots, P_r\} \subseteq D_n$ is a $(-w, w)$-strict Gröbner basis if for all $P \in D_n \cdot (P_1, \ldots, P_r)$ there are $Q_1, \ldots, Q_r \in D_n$ with $P = \sum Q_i P_i$ and $(-w, w)(Q_i P_i) \leq (-w, w)(P)$ for all i. V_d-strictness corresponds to $w = (\underbrace{1, \ldots, 1}_{d}, \underbrace{0, \ldots, 0}_{n-d})$.

Gröbner bases for $(-w, w)$ can be computed in exactly the same way as V_d-strict Gröbner bases, the only difference being the homogenization procedure and the order $<_h$ on $\mathbb{N} \times \mathbb{L} \times \{1, \ldots, r\}$ which now depend on w.

These things are discussed in detail in [41].

3 Localization Functors

The outline of this chapter is as follows.

In section 3.1 we first present generalizations of some results due to B. Malgrange and M. Kashiwara on D-modules and their localizations. The purpose of that section is to find a representation of $R_n[f^{-1}] \otimes N$ as a cyclic D_n-module if N is a given holonomic D-module (for a definition and some properties of holonomic modules, see subsection 2.1.2 below). Most of the algorithmic ideas in this section appear already in T. Oaku's work [35], [34], [33].

In the following section we consider an alternative way of computing localization via restriction. This is essentially a reprint of the article [38]. In the third section of this chapter we present some computer-based examples obtained by the use of these algorithms.

For this chapter we shall write f_i for the i-th partial derivative $\partial_i(f) = [\partial_i, f]$ of f.

3.1 D-modules after Kashiwara, Malgrange and Oaku

The purpose of this section is as follows. Given $f \in R_n$ and a left ideal $L \subseteq D_n$ such that D_n/L is holonomic and L is f-saturated (i.e. $f \cdot P \in L$ only if $P \in L$), we want to compute the structure of the module $D_n/L \otimes_{R_n} R_n[f^{-1}]$. It turns out that it is useful to know the ideal $J^L(f^s)$ which consists of the operators $P(s) \in D_n[s]$ that annihilate $\overline{1} \otimes f^s \in M := D_n/L \otimes_{R_n} (R_n[f^{-1}][s] \otimes f^s)$ where the bar denotes cosets in D_n/L. In order to find $J^L(f^s)$, we will consider the module M over the ring $D_{n+1} = D_n\langle t, \partial_t\rangle$. It will turn out that one can easily compute the ideal $J^L_{n+1}(f^s) \subseteq D_{n+1}$ consisting of all operators that kill $\overline{1} \otimes 1 \otimes f^s$. In proposition 3.3 we will then explain how to compute $J^L(f^s)$ from $J^L_{n+1}(f^s)$.

Another important result in this section (proposition 3.2) shows how to compute the structure of $D_n/L \otimes_{R_n} R_n[f^{-1}]$ as D_n-module once $J^L(f^s)$ is known.

3.1.1

Consider $D_{n+1} = D_n\langle t, \partial_t\rangle$, the Weyl algebra in $x_1, \ldots, x_n$ and the new variable t. B. Malgrange ([29]) has defined an action of t and ∂_t on $M = D_n/L \otimes_{R_n} R_n[f^{-1}][s] \otimes f^s$ by $t(\overline{P} \otimes g(x,s) \otimes f^s) = \overline{P} \otimes g(x, s+1) f \otimes f^s$ and $\partial_t(\overline{P} \otimes g(x,s) \otimes f^s) = \overline{P} \otimes \frac{-s}{f} g(x, s-1) \otimes f^s$ for $\overline{P} \in D_n/L$. D_n acts on M as expected, the variables by multiplication on the left, the ∂_i by the product rule. One checks that this actually defines a structure of M as a left D_{n+1}-module and that $-\partial_t t$ acts as multiplication by s.

We denote by $J^L_{n+1}(f^s)$ the ideal in D_{n+1} that annihilates the element $\overline{1} \otimes f^s$ in M. Then we have an induced morphism of D_{n+1}-modules $D_{n+1}/J^L_{n+1}(f^s) \to M$ sending $P + J^L_{n+1}(f^s)$ to $P(\overline{1} \otimes 1 \otimes f^s)$.

The following lemma generalizes lemma 4.1 in [29] (as well as part of the proof given there) where the special case $L = (\partial_1, \ldots, \partial_n), D_n/L = R$ is considered. Note that $J^L_{n+1}(f^s)$ makes perfect sense even if L is not holonomic.

Lemma 3.1. *Suppose that $L = D_n \cdot (P_1, \ldots, P_r)$ is f-saturated (i.e., if $f \cdot P \in L$, then $P \in L$). With the above definitions, $J^L_{n+1}(f^s)$ is the ideal generated by $f - t$ together with the images of the P_j under the automorphism ϕ of D_{n+1} induced by $x \to x, t \to t - f$.*

Proof. The automorphism sends ∂_i to $\partial_i + f_i \partial_t$ and ∂_t to ∂_t. So if we write $P_j = P_j(\partial_1, \ldots, \partial_n)$, then $\phi P_j = P_j(\partial_1 + f_1 \partial_t, \ldots, \partial_n + f_n \partial_t)$.

One checks that $(\partial_i + f_i \partial_t)(\overline{Q} \otimes 1 \otimes f^s) = \overline{\partial_i Q} \otimes 1 \otimes f^s$ for all $Q \in D_{n+1}$ so that $\phi(P_j(\partial_1, \ldots, \partial_n))(\overline{1} \otimes 1 \otimes f^s) = \overline{P_j(\partial_1, \ldots, \partial_n)} \otimes 1 \otimes f^s = 0$. By definition, $f \cdot (\overline{1} \otimes 1 \otimes f^s) = t \cdot (\overline{1} \otimes 1 \otimes f^s)$. So $t - f \in J^L_{n+1}(f^s)$ and $\phi(P_j) \in J^L_{n+1}(f^s)$ for $i = 1, \ldots, r$.

Conversely let $P(\overline{1} \otimes 1 \otimes f^s) = 0$. We may assume, that P does not contain any t since we can eliminate t using $f - t$. Now rewrite P in terms of ∂_t and the $\partial_i + f_i \partial_t$. Say, $P = \sum c_{\alpha\beta} \partial_t^\alpha x^\beta Q_{\alpha\beta}(\partial_1 + f_1 \partial_t, \ldots, \partial_n + f_n \partial_t)$, where the $Q_{\alpha\beta}$ are polynomials in n variables and $c_{\alpha\beta} \in K$. Application to $\overline{1} \otimes 1 \otimes f^s$ results in $\sum \partial_t^\alpha (c_{\alpha\beta} x^\beta \overline{Q_{\alpha\beta}(\partial_1, \ldots, \partial_n)} \otimes 1 \otimes f^s)$.

Let $\overline{\alpha}$ be the largest $\alpha \in \mathbb{N}$ for which there is a nonzero $c_{\alpha\beta}$ occuring in $P = \sum c_{\alpha\beta}\partial_t^\alpha x^\beta Q_{\alpha\beta}(\partial_1 + f_1\partial_t, \dots, \partial_n + f_n\partial_t)$. We show that the sum of terms that contain $\partial_t^{\overline{\alpha}}$ is in $D_{n+1} \cdot \phi(L)$ as follows. In order for $P(\overline{1} \otimes 1 \otimes f^s)$ to vanish, the sum of terms with the highest s-power, namely $s^{\overline{\alpha}}$, must vanish, and so $\sum_\beta c_{\overline{\alpha}\beta}(-1/f)^{\overline{\alpha}} x^\beta Q_{\overline{\alpha}\beta}(\partial_1, \dots, \partial_n) \otimes 1 \otimes f^s \in L \otimes R_n[f^{-1}] \otimes f^s$ as $R_n[f^{-1}] \otimes f^s$ is R-flat. It follows, that $\sum_\beta c_{\overline{\alpha}\beta} x^\beta Q_{\overline{\alpha}\beta}(\partial_1, \dots, \partial_n) \in L$ (L is f-saturated!) and hence $\sum_\beta \partial_t^{\overline{\alpha}} c_{\overline{\alpha}\beta} x^\beta Q_{\overline{\alpha}\beta}(\partial_1 + f_1\partial_t, \dots, \partial_n + f_n\partial_t) \in D_{n+1} \cdot \phi(L)$.

So by the first part, $P - \sum_\beta c_{\overline{\alpha}\beta}\partial_t^{\overline{\alpha}} x^\beta Q_{\overline{\alpha}\beta}(\partial_1 + f_1\partial_t, \dots, \partial_n + f_n\partial_t)$ kills $\overline{1} \otimes 1 \otimes f^s$, but is of smaller degree in ∂_t than P was.

The claim follows. □

3.1.2

Let $J^L(f^s)$ stand for the ideal in $D_n[s] \cong D_n[-\partial_t t]$ that kills $\overline{1} \otimes 1 \otimes f^s \in D_n/L \otimes_{R_n} R_n[f^{-1}][s] \otimes f^s$. Note that $J^L(f^s) = J^L_{n+1}(f^s) \cap D_n[-\partial_t t]$. Again, we may talk about $J^L(f^s)$ independently of the holonomicity of L.

Recall that the Bernstein polynomial $b^L_f(s)$ is defined to be the monic generator of the ideal of polynomials $b(s) \in K[s]$ for which there exists an operator $P(s) \in D_n[s]$ such that $P(s)(\overline{1} \otimes f \otimes f^s) = \overline{1} \otimes b(s) \otimes f^s$, and that $b^L_f(s)$ will exist for example if L is holonomic. The following proposition already appears as Proposition 7.3 in [34].

Proposition 3.2. *If L is holonomic and $a \in \mathbb{Z}$ is such that no integer root of $b^L_f(s)$ is smaller than a, then we have isomorphisms*

$$D_n/L \otimes_{R_n} R_n[f^{-1}] \cong D_n[s]/J^L(f^s)|_{s=a} \cong D_n \cdot \overline{1} \otimes f^a \subseteq D_n/L \otimes_{R_n} R_n[f^{-1}]. \tag{3.1}$$

Proof. We mimic the proof given by Kashiwara, who proved the proposition for the case $L = (\partial_1, \dots, \partial_n), D_n/L = R$ ([23], proposition 6.2).

Let us first prove the second equality. Certainly $J^L(f^s)|_{s=a}$ kills $\overline{1} \otimes f^a$. So we have to show that if $P(\overline{1} \otimes f^a) = 0$ then $P \in J^L(f^s) + D_n[s] \cdot (s - a)$. To that end note that st acts as $t(s-1)$ which means that $t \cdot (D_n[s]/J^L(f^s))$ is a left $D_n[s]$-module. Identify $D_n[s]/J^L(f^s)$ with $\mathcal{N}^L_f := D_n[s] \cdot (\overline{1} \otimes 1 \otimes f^s)$.

By definition, $b_f^L(s)$ is the minimal polynomial for which there is $P(s)$ with $b_f^L(s)(\overline{1}\otimes 1\otimes f^s) = P(s)\overline{1}\otimes f\otimes f^s = t\cdot P(s-1)(\overline{1}\otimes 1\otimes f^s)$. So $b_f^L(s)$ multiplies $D_n[s]\cdot(\overline{1}\otimes 1\otimes f^s)$ into $t\cdot D_n[s](\overline{1}\otimes 1\otimes f^s)$ and whenever the polynomial $b(s)\in K[s]$ is relatively prime to $b_f^L(s)$ its action on $\mathcal{N}_f^L/t\cdot\mathcal{N}_f^L$ is injective. $b_f^L(s)$ is therefore the characteristic polynomial of the endomorphism s on $\mathcal{N}_f^L/t\cdot\mathcal{N}_f^L$.

Since by hypothesis $s-a+j$ is not a divisor of $b_f^L(s)$ for $0<j\in\mathbb{N}$,

$$(s-a+j)\mathcal{N}_f^L\cap t\cdot\mathcal{N}_f^L\subseteq(s-a+j)t\cdot\mathcal{N}_f^L. \tag{3.2}$$

So $(s-a+m)\mathcal{N}_f^L\cap t^m\mathcal{N}_f^L\subseteq(s-a+m)t\mathcal{N}_f^L\cap t^m\mathcal{N}_f^L = t[(s-a+m-1)\mathcal{N}_f^L\cap t^{m-1}\mathcal{N}_f^L]$ whenever $m\geq 1$.

We show now by induction on m that $(s-a+m)\mathcal{N}_f^L\cap t^m\mathcal{N}_f^L\subseteq(s-a+m)t^m\mathcal{N}_f^L$ for $m\geq 1$. The claim is clear for $m=1$ from equation (3.2). So let $m>1$. The inductive hypothesis states that $(s-a+m-1)\mathcal{N}_f^L\cap t^{m-1}\mathcal{N}_f^L\subseteq(s-a+m-1)t^{m-1}\mathcal{N}_f^L$. The previous paragraph shows that $(s-a+m)\mathcal{N}_f^L\cap t^m\mathcal{N}_f^L\subseteq t\left[(s-a+m-1)\mathcal{N}_f^L\cap t^{m-1}\mathcal{N}_f^L\right]$. Combining these two facts we get

$$\begin{aligned}(s-a+m)\mathcal{N}_f^L\cap t^m\mathcal{N}_f^L &\subseteq t(s-a+m-1)t^{m-1}\mathcal{N}_f^L\\ &= (s-a+m)t^m\mathcal{N}_f^L.\end{aligned}$$

Now if $P(s)\in D_n[s]$ is of degree m in the ∂_i and $P(a)(\overline{1}\otimes f^a)=0$, then $P(s+m)\cdot f^m+J^L(f^s)\in(s-a+m)\cdot\mathcal{N}_f^L$ because we can interpret $P(s+m)(\overline{1}\otimes f^m\otimes f^s)$ as a polynomial in $s+m$ with root a. But then $P(s+m)(\overline{1}\otimes f^m\otimes f^s) = P(s+m)(f^m\overline{1}\otimes 1\otimes f^s)$ is in

$$(s-a+m)\mathcal{N}_f^L\cap t^m\mathcal{N}_f^L\subseteq(s-a+m)t^m\mathcal{N}_f^L,$$

implying $P(s+m)(f^m\overline{1}\otimes 1\otimes f^s)=(s-a+m)Q(s)(f^m\overline{1}\otimes 1\otimes f^s)$ for some $Q(s)\in D_n[s]$. In other words, $P(s)-(s-a)Q(s-m)\in J^L(f^s)$.

For the first isomorphism we have to show that $D_n\cdot(\overline{1}\otimes f^a)=D_n/L\otimes_{R_n}R_n[f^{-1}]$. It suffices to show that every term of the form $\overline{Q}\otimes f^m f^a$ is in the module generated by $(\overline{1}\otimes f^a)$ for all $m\in\mathbb{Z}$. Furthermore, we may assume that Q is a monomial in $\partial_1,\ldots,\partial_n$.

Existence and definition of $b_f^L(s)$ provides an operator $P(s)$ with $[b_f^L(s-1)]^{-1}P(s-1)(\overline{1}\otimes 1\otimes f^s) = \overline{1}\otimes f^{-1}\otimes f^s$. As $b_f^L(a-m)\neq 0$ for all $0 < m \in \mathbb{N}$ we have $\overline{1}\otimes f^m f^a \in D_n\cdot(\overline{1}\otimes f^a)$ for all m. Now let Q be a monomial in $\partial_1, \dots, \partial_n$ of ∂-degree $j > 0$ and assume that $\overline{Q'}\otimes f^m f^a \in D_n\cdot(\overline{1}\otimes f^a)$ for all m and all operators Q' of ∂-degree lower than j. Then $Q = \partial_i Q'$ for some $1 \leq i \leq n$. Fix $m \in \mathbb{Z}$. By assumption on j, for some P' we have $P'(\overline{1}\otimes f^a) = \overline{Q'}\otimes f^m f^a$. So

$$\overline{Q}\otimes f^m f^a = \partial_i P'(\overline{1}\otimes f^a) - f_i\cdot(a+m)\overline{Q'}\otimes f^{m-1}f^a \in D_n\cdot(\overline{1}\otimes f^a). \quad (3.3)$$

The claim follows by induction. This completes the proof of the proposition. □

We remark that if any $a \in \mathbb{Z}$ satisfies the conditions of the proposition, then so does every integer smaller than a.

3.1.3

The purpose of this subsection is to review some algorithms due to T. Oaku. In [33] Theorem 19, Oaku showed how to construct a generating set for $J^L(f^s)$ in the case where $L = (\partial_1, \dots, \partial_n)$. According to 3.1.2, $J^L(f^s)$ is the intersection of $J^L_{n+1}(f^s)$ with $D_n[-\partial_t t]$. We shall explain how one may calculate $J \cap D_n[-\partial_t t]$ whenever $J \subseteq D_{n+1}$ is any given ideal and as a corollary develop an algorithm that for f-saturated D_n/L computes $J^L(f^s)$.

On $D_{n+1}[y_1, y_2]$ define weights $w(t) = w(y_1) = -w(\partial_t) = -w(y_2) = 1$, $w(x_i) = w(\partial_i) = 0$. If $P = \sum_i P_i \in D_{n+1}[y_1, y_2]$ and all P_i are monomials, then we will write $(P)^h$ for the operator $\sum_i P_i \cdot y_1^{d_i}$ where $d_i = \max_j(w(P_j)) - w(P_i)$ and call it the y_1-*homogenization* of P. w-homogeneity is then a special case of V-homogeneity.

Recall that the Buchberger algorithm preserves homogeneity in the following sense: if a set of generators for an ideal is given and these generators are homogeneous with respect to the weights above, then any new generator for the ideal constructed with the classical Buchberger algorithm will also be homogeneous. (This is a consequence of the facts that the y_i commute with all other variables and that $\partial_t t = t\partial_t + 1$ is homogeneous of weight zero.)

Proposition 3.3 ([32], Theorem 18). *Let $J = D_{n+1} \cdot (Q_1, \ldots, Q_r)$. Let I be the left ideal in $D_{n+1}[y_1]$ generated by the y_1-homogenizations $(Q_i)^h$ of the Q_i, relative to the weight w above, and set $\tilde{I} = D_{n+1}[y_1, y_2] \cdot (I, 1 - y_1 y_2)$. Let G be a Gröbner basis for $\tilde{I}$ under a monomial order that eliminates y_1, y_2. For each $P \in G$ set $P' = t^{-w(P)} P$ if $w(P) < 0$ and $P' = \partial_t^{w(P)} P$ if $w(P) \geq 0$ and let $G' = \{P' : P \in G\}$. Then $G_0 = G' \cap D_n[-\partial_t t]$ generates $J \cap D_n[-\partial_t t]$.*

Proof. Note first that G consists of w-homogeneous operators and so $w(P)$ is well defined for $P \in G$.

Suppose $P \in G_0$. Hence $P \in \tilde{I}$. So $P = Q_{-1} \cdot (1 - y_1 y_2) + \sum a_i \cdot (Q_i)^h$ where the a_i are in $D_{n+1}[y_1, y_2]$. Since $P \in D_n[-\partial_t t]$, the substitution $y_i \to 1$ shows that $P = \sum a_i(1,1) \cdot (Q_i)^h(1,1) = \sum a_i(1,1) \cdot Q_i \in J$. Therefore $G_0 \subseteq J \cap D_n[-\partial_t t]$.

Now assume that $P \in J \cap D_n[-\partial_t t]$. So P is w-homogeneous of weight 0. Also, $P \in J$ and J is contained in $I(1)$, the ideal of operators $Q(1) \subseteq D_{n+1}$ for which $Q(y_1) \in I$. By lemma 3.4 below (taken from [35]), $y_1^a P \in I$ for some $a \in \mathbb{N}$. Therefore $P = (1 - (y_1 y_2)^a) P + (y_1 y_2)^a P \in \tilde{I}$.

Let $G = \{P_1, \ldots, P_b, P_{b+1}, \ldots, P_c\}$ and assume that $P_i \in D_{n+1}$ if and only if $i \leq b$. Buchberger algorithm gives a standard expression $P = \sum a_i P_i$ with all $\mathrm{in}(a_i P_i) \leq \mathrm{in}(P)$. That implies that a_{b+i} is zero for positive i and a_i does not contain y_1, y_2 for any i.

Since P, P_i are w-homogeneous, the same is true for all a_i, from Buchberger algorithm. In fact, $w(P) = w(a_i) + w(P_i)$ for all i. As $w(P) = 0$ (and t, ∂_t are the only variables with nonzero weight that may appear in a_i) we find $a_i' \in D_{n+1}$ of weight zero with $a_i = a_i' \cdot t^{-w(P_i)}$ or $a_i = a_i' \cdot \partial_t^{w(P_i)}$, depending on whether $w(P_i)$ is negative or positive.

It follows that $P = \sum_1^b a_i P_i = \sum_1^b a_i' P_i' \in D_n[-\partial_t t] \cdot G_0$. □

Lemma 3.4 ([32], Lemma 15). *Let I be a w-homogeneous ideal in $D_{n+1}[y_1]$ with respect to the weights introduced before the proposition and $I(1)$ defined as in the proof of the proposition. Assume $P \in D_{n+1}$ is a w-homogeneous operator. Then $P \in I(1)$ implies $y_1^a P \in I$ for some a.*

Proof. Note first that $y_1 \to 1$ will not lead to cancellation of terms in any homogeneous operator as $w(y_1) \neq 0$.

If $P \in I(1)$, $P = \sum Q_i(1)$, with all Q_i w-homogeneous in I. Then the y_1-homogenization of $Q_i(1)$ will be a divisor of Q_i and the quotient will be some power of y_1, say $y_1^{\eta_i}$. Homogenization of the equation $P = \sum Q_i(1)$ results in $y_1^{\eta} P = \sum Q_i(1)^h$ (since P is homogeneous) so that

$$y_1^{\eta+\max(\eta_i)} P = \sum y_1^{\max(\eta_i)-\eta_i} Q_i \in I.$$

$\square$

So we have

Algorithm 3.5. Input: $f \in R_n, L \subseteq D_n$ such that L is f-saturated.

Output: Generators for $J^L(f^s)$.

Begin

1. For each generator Q_i of $D_{n+1} \cdot (L, t)$ compute the image $\phi(Q_i)$ under $x_i \to x_i, t \to t - f, \partial_i \to \partial_i + f_i \partial_t, \partial_t \to \partial_t$.
2. Homogenize all $\phi(Q_i)$ with respect to the new variable y_1 relative to the weight w introduced before proposition 3.3.
3. Using an order that eliminates y_1, y_2 compute a Gröbner basis for the ideal generated by $(\phi(Q_1))^h, \ldots, (\phi(Q_r))^h$, $1 - y_1 y_2$ in $D_{n+1}[y_1, y_2]$.
4. Select the operators $\{P_j\}_1^b$ in this basis which do not contain y_1, y_2.
5. For each P_j, $1 \leq j \leq b$, if $w(P_j) > 0$ replace P_j by $P_j' = \partial_t^{w(P_j)} P_j$. Otherwise replace P_j by $P_j' = t^{-w(P_j)} P_j$.
6. Return the new operators $\{P_j'\}_1^b$.

End.

This algorithm was already stated in Proposition 7.1 of [34].

In order to guarantee existence of the Bernstein polynomial $b_f^L(s)$ we assume for our next result that L is holonomic. Let $|s|$ denote the complex absolute value of s.

Corollary 3.6. *Suppose L is a holonomic ideal in D_n. If $J^L(f^s)$ is known or it is known that L is f-saturated, then the Bernstein polynomial $b_f^L(s)$ of $R_n[f^{-1}] \otimes_{R_n} D_n/L$ can be found from $(b_f^L(s)) = D_n[s] \cdot (J^L(f^s), f) \cap K[s]$.*

Moreover, let $b_f^L(s) = s^d + b_{d-1}s^{d-1} + \ldots + b_0$ and set $B = \max_i\{|b_i|^{1/(d-i)}\}$. In order to find the smallest integer root of $b_f^L(s)$, one only needs to check the integers between $-2B$ and $2B$.

If in particular $L = (\partial_1, \ldots, \partial_n)$, it suffices to check the integers between $-b_{d-1}$ and -1.

Proof. If L is f-saturated, propositions 3.1 and 3.3 enable us to find $J^L(f^s)$. The first part follows then easily from the definition of $b_f^L(s)$: as $(b_f^L(s) - P_f^L \cdot f)(\overline{1} \otimes 1 \otimes f^s) = 0$ it is clear that $b_f^L(s)$ is in $K[s]$ and in $D_n[s](J^L(f^s), f)$. Using an elimination order on $D_n[s]$, $b_f^L(s)$ will be (up to a scalar factor) the unique element in the reduced Gröbner basis for $D_n[s] \cdot (J^L(f^s), f)$ that contains no x_i nor ∂_i.

Now suppose $|s_0| = 2B\rho$ where B is as defined above and $\rho > 1$. Assume also that s_0 is a root of $b_f^L(s)$. We find

$$\begin{aligned}(2B\rho)^d = |s_0|^d &= |-\sum_0^{d-1} b_i {s_0}^i| \leq \sum_0^{d-1} B^{d-i}|s_0|^i \\ &= B^d \sum_0^{d-1} (2\rho)^i \leq B^d((2\rho)^d - 1),\end{aligned}$$

using $\rho \geq 1$. By contradiction, s_0 is not a root.

The final claim is a consequence of Kashiwara's work [23] where it is proved that if $L = (\partial_1, \ldots, \partial_n)$ then all roots of $b_f^L(s)$ are negative and hence $-b_{n-1}$ is a lower bound for each single root. ☐

For purposes of reference we also list algorithms that compute the Bernstein polynomial and the localization of a holonomic module.

Algorithm 3.7. Input: $f \in R_n, L \subseteq D_n$ such that D_n/L is holonomic and f-torsionfree.

Output: The Bernstein polynomial $b_f^L(s)$.

Begin

1. Determine $J^L(f^s)$ following algorithm 3.5.

2. Find a reduced Gröbner basis for the ideal $J^L(f^s) + D_n[s] \cdot f$ using an elimination order for x and ∂.

3. Pick the unique element $b(s) \in K[s]$ contained in that basis and return it.

End.

Algorithm 3.8. Input: $f \in R_n, L \subseteq D_n$ such that D_n/L is holonomic and f-torsionfree.

Output: Generators for an ideal J such that $D_n/L \otimes_{R_n} R_n[f^{-1}] \cong D_n/J$.

Begin

1. Determine $J^L(f^s)$ following algorithm 3.5.

2. Find the Bernstein polynomial $b^L_f(s)$ using algorithm 3.7.

3. Find the smallest integer root a of $b^L_f(s)$.

4. Replace s by a in all generators for $J^L(f^s)$ and return these generators.

End.

Algorithms 3.7 and 3.8 appear already in [34] as Theorem 6.14 and Proposition 7.3. Correctness of step 4 in algorithm 3.8 follows from proposition 3.2.

3.2 Localization by Restriction

In this section we introduce the notion of integration functors, the D-module categorical direct image functor. We will later have occasion to study a close relative in more detail (see chapter 5). For now we only use it to develop an alternative method to compute localizations that applies to a wider class of modules.

As before, let $R_n = K[x_1, \dots, x_n]$ be the ring of polynomials in n variables over the field K of characteristic zero and $D_n = R_n\langle\partial_1, \dots, \partial_n\rangle$ the n-th Weyl algebra over K. Suppose $f \in R_n$, let J be a left ideal of D_n and set $M = D_n/J$. Let $X = K^n, Y = \mathrm{Var}(f) \subseteq X, U = X \setminus Y$ and $j : U \hookrightarrow X$ the natural inclusion.

Recall that if $\phi : T \to S$ is a differentiable map of differentiable manifolds then there are associated direct image functors $\int_\phi^i$ from the category of sheaves of $\mathcal{D}_T$-modules to the category of $\mathcal{D}_S$-modules (cf. [3]). Similarly, there is a notion of inverse image functor ϕ^{-1} from $\mathcal{D}_S$-modules to $\mathcal{D}_T$-modules.

In the landmark paper [24] M. Kashiwara proved that $\int_j j^{-1}M$ is holonomic provided that M is holonomic on U, but not necessarily on all of X. In this section we make this important result algorithmic in the sense that we provide an algorithm that computes the module structure of $\int_j j^{-1}(M)$ over D_n.

We remark that U is affine and $X = K^n$, and so $\int_j j^{-1}(M) = M \otimes_{R_n} R_n[f^{-1}]$ as D_n-module. An algorithm to compute $M \otimes_{R_n} R_n[f^{-1}]$ as a D_n-module has been given in [34] under the assumption that M is holonomic on K^n and f-torsion free, and in [36] under the assumption that M is holonomic on K^n. The advantage of the present algorithm, besides removing the requirement of holonomicity on $\mathrm{Var}(f)$, is that the natural map $M \to M \otimes_{R_n} R_n[f^{-1}]$ can be traced, which is an important property for other computations as local cohomology ([34], [47]) and de Rham cohomology ([37], [48]).

3.2.1 Description of the Algorithm

We shall start with a motivating construction which will also provide a skeleton of the proof of the correctness of our algorithm.

Assume that K is in fact the field $\mathbb{C}$ of complex numbers. Let s, v be two new variables and let $D_v = D_n\langle v, \partial_v\rangle$.

Notation 3.9. We will denote $\partial f/\partial x_i$ by f_i. If $P \in D_v$ is an operator interpreted as a polynomial in $\partial_1, \ldots, \partial_n$, we will simply write $P(\partial_x)$. Similarly we will abbreviate $P(\partial_1 - v^2 f_1 \partial_v, \ldots, \partial_n - v^2 f_n \partial_v)$ by $P(\partial_x - v^2 f_x \partial_v)$.

Consider the ring D_v and the $\mathbb{C}$-vectorspace $R_n[f^{-1}][s] \otimes f^{-s}$. Here, f^{-s} should be thought of as a symbol that behaves like the complex valued function f^{-s} under differentiation and the tensor is over $\mathbb{C}$, similar to the construction in section 3.1. We shall turn this vectorspace into a module over D_v by means of the action $\bullet$ defined as follows. We require that all x_i act as multiplication on the left factor, while ∂_i acts by the "product rule" $\partial_i \bullet (\frac{g(x,s)}{f^m} \otimes f^{-s}) = \partial_i(\frac{g(x,s)}{f^m}) \otimes f^{-s} + \frac{g(x,s)}{f^{m+1}}(-s) f_i \otimes f^{-s}$. Thus, we only need to define the action of v and ∂_v. We set

$$\begin{aligned} v \bullet \left(\frac{g(x,s)}{f^m} \otimes f^{-s} \right) &= \frac{g(x,s+1)}{f^{m+1}} \otimes f^{-s}, \\ \partial_v \bullet \left(\frac{g(x,s)}{f^m} \otimes f^{-s} \right) &= -(s-2) \frac{f \cdot g(x,s-1)}{f^m} \otimes f^{-s}. \end{aligned}$$

We observe that v is simply shifting s up by one (in particular, the extra f in the denominator comes from shifting the exponent in f^{-s}) and ∂_v is shifting s down by one, together with the "differential" $2-s$ of the shift operator.

Remark 3.10. The construction in section 3.1 and the present one are two from an infinite family corresponding to the introduction of a new variable t (or v) satisfying $t^a f = 1$ with $a \in \mathbb{Z}$. 3.1 corresponds to $a = -1$, our present one to $a = 1$.

Armed with this action on $R_n[f^{-1}][s] \otimes f^{-s}$ we can define an action of D_v on $D_n/J \otimes_{R_n} (R_n[f^{-1}][s] \otimes f^{-s})$ as follows. v and ∂_v commute with the first factor in the tensor product, x_i acts by left multiplication on the first factor and ∂_i acts by the product rule.

We detect a certain set of trivial relations for this action. Namely, let us ask which operators on D_v are candidates for annihilating the element $\overline{1} \otimes 1 \otimes f^{-s}$, the bar denoting cosets in D_n/J. Independently of J, $1 - vf$ is such an annihilator. Assume that J is generated by $\{P_j(\partial_x)\}_1^r$ where we think of P_j as a polynomial in ∂_x with left coefficients in R_n. Let us define a ring map ϕ from D_n to D_v by mapping $P \in D_n$ to $\phi(P_j(\partial_x)) = Q_j(\partial_x) := P_j(\partial_x - v^2 f_x \partial_v) \in D_v$ (recall $f_i = \partial_i(f)$). It is easy to see that this assignment preserves commutators and hence is an actual ring map. We remark that ϕ represents the map $U \xrightarrow{\cong} \mathrm{Var}(1-vf) \subseteq \mathbb{C}^{n+1}$ given by $x \to (1/f(x), x)$ on the level of differential operators, while the inclusion $\phi(D) \hookrightarrow D_v$ corresponds to $\mathrm{Var}(1-vf) \hookrightarrow \mathbb{C}^{n+1}$.

We claim that $Q_j(\partial_x)$ annihilates $\overline{1} \otimes 1 \otimes f^{-s}$. To prove this, we observe that $(\partial_i - v^2 f_i \partial_v) \bullet (\overline{P} \otimes 1 \otimes f^{-s})$ equals $\overline{\partial_i P} \otimes 1 \otimes f^{-s}$ for all $P \in D_n$. Thus $Q_i(\partial_x) \bullet (\overline{1} \otimes 1 \otimes f^{-s}) = \overline{P_i(\partial_x)} \otimes 1 \otimes f^{-s} = 0$. We point out that $-v\partial_v + 1$ acts as multiplication by s and that this construction is (formally) independent of the base field.

Our main statement is the following

Theorem 3.11. *Let $M = D_n/J$. Consider the left ideal $D_v \cdot \{1 - xv, \phi(J)\}$ and the right ideal $\partial_v \cdot D_v$ in D_v. The quotient $D_v/(D_v \cdot \{1 - xv, \phi(J)\} + \partial_v \cdot D_v)$ is isomorphic to $M \otimes R_n[f^{-1}]$ as D_n-module. Moreover, the natural map $M \to M \otimes_{R_n} R_n[f^{-1}]$ induced by ϕ sends $\overline{1} \in M$ to $\overline{1} \otimes \frac{1}{f^2}$.*

It is worth noticing that this quotient is in fact a left D_n-module due to the fact that D_n and ∂_v commute.

In general, the D_n-module $D_v/(D_v \cdot \{1 - xv, \phi(J)\} + \partial_v \cdot D_v)$ will not be a finitely generated D_n-module. If however M is holonomic on $U = K^n \setminus \mathrm{Var}(f)$, then $M \otimes_{R_n} R_n[f^{-1}] = D_v/(D_v \cdot \{1 - xv, \phi(J)\} + \partial_v \cdot D_v)$ will be holonomic, and in particular finitely generated. In order to compute the structure of the localized module, we observe that it is the integration of the left D_v-module $D_v/D_v \cdot \{1 - xv, \phi(J)\}$ with respect to ∂_v (cf. [34, Theorem 5.7], [37, Section 6]). Methods for the algorithmic computation of the 0-th integral worked out there (i.e., the restriction to $v = 0$ of the Fourier transform of the module,) together with our theorem give the following algorithm. Let N denote $D_v/(\partial_v \cdot D_v + D_v \cdot \{1 - fv, \phi(J)\})$.

Algorithm 3.12. Input: $f \in R_n$; $\{P_1, \ldots, P_r\} \subseteq D_n$ generating J, D_n/J holonomic on $U = K^n \setminus \mathrm{Var}(f)$.
Output: $k \in \mathbb{N}$; $\{Q_1, \ldots, Q_t\} \subseteq D_n$ such that $D_n/(Q_1, \ldots, Q_t)$ is isomorphic to $D_n/J \otimes_R R_n[f^{-1}]$ generated by $\overline{1} \otimes \frac{1}{f^{k+2}}$.

Begin

1. For $P_i, i = 1, \ldots, r$ compute $\phi(P_i) \in D_v$ defined by replacing ∂_x by $\partial_x - v^2 f_x \partial_v$.

2. Compute the b-function $b(s)$ for integration of $D_v/D_v \cdot \{1-fv, \phi(J)\}$ with respect to ∂_v. That is, find $K[v\partial_v] \cap \mathrm{in}_w(D_v \cdot \{1-fv, \phi(J)\})$ where w is the weight assigning 1 to v, -1 to ∂_v and 0 to all other variables. Replace $v\partial_v$ by $-s-1$.

3. Let a be the largest non-negative integer root of $b(s)$. If there is no such root, then output $I = D_n$, else continue. The integral N is generated by the coset of v^a in N.

4. Compute the annihilator I over D_n of v^a in N.

5. $D_n/J \otimes_{R_n} R_n[f^{-1}]$ is generated by $\overline{1} \otimes f^{-(a+2)}$ and isomorphic to D_n/I.

6. Return a and I.

End.

An algorithm to perform step 2 of the algorithm is given in [33, Algorithm 4.5].

The steps 3 and 4 are nothing but an algorithm to get the 0-th integral of D-modules. Here is a more precise description of steps 3 and 4.

1. Let $G = \{g_1, \dots, g_m\}$ be a Gröbner basis of $D_v \cdot \{1-vf, \phi(J)\}$ with respect to w. w-Gröbner bases are a special type of V-strict Gröbner bases, see section 2.3.

2. Put
$$\mathrm{ord}_w(g) := \max\left\{ w \cdot (\alpha, \beta) \mid g = \sum a_{\alpha\beta} v^\alpha \partial_v^\beta \right\}$$
following the notation of [41].

3. Let G_a be the set
$$\{\, \Re(v^i g_j; \partial_v) \mid j = 1, \dots, m, 0 \le i \le a - \mathrm{ord}_w(g_j)\}.$$
where we compute *right* remainders. (Recall that $\Re(g;\partial_v)$ means to compute the remainder of g with respect to ∂_v, here generating a right ideal

in D_n. For example,

$$\Re(v^2\partial_v^2;\partial_v) = \Re(\partial_v^2 v^2 - 4\partial_v v + 2;\partial_v) = 2.$$

)

4. Regard $D_n \cdot \{G_a\}$ as a left submodule of the free module

$$D_n\, v^0 + D_n\, v^1 + \cdots + D_n\, v^a = {D_n}^{a+1}$$

and find generators of $D_n \cdot \{G_a\} \cap D_n\, v^a$. The intersection can be computed by using an order to eliminate $v^0, \ldots, v^{a-1}$, i.e, by using an order $\succ$ such that $P_i v^i \succ P_a v^a$ for all $i = 0, \ldots, a-1$ and $P_i, P_a \neq 0$ in D_n.

3.2.2 Proof of correctness of the algorithm

Let us first provide the

Proof of the Theorem: Given an operator $P \in D_v$ we shall define its *normal form* in D_v. The goal is a presentation of P where $\partial_i - v^2 f_i \partial_v$ take the position of ∂_i. In other words, we are aiming for a sum of the form $P = \sum \partial_v^a v^b p_{ab}(x) q_{ab}(\partial_x - v^2 f_x \partial_v)$ where p_{ab} and q_{ab} are polynomials in n variables over K. To this end, write first P as $\sum \partial_v^a v^b p_{ab}(x) q_{ab}(\partial_x)$. The operator $P^1 := \sum \partial_v^a v^b p_{ab}(x) q_{ab}(\partial_x - v^2 f_x \partial_v)$ has the property that $P - P^1$ will have lower degree in ∂_x than P. If $P - P^1 = 0$, quit. Otherwise write $P - P^1$ as $\sum \partial_v^a v^b p_{ab}^1(x) q_{ab}^1(\partial_x)$ and set $P^2 := \partial_v^a v^b p_{ab}^1(x) q_{ab}^1(\partial_x - v^2 f_x \partial_v)$. Repeat this procedure until we arrive at $P^l = 0$. Then $P = P^1 + \cdots + P^l$ and this sum is the desired normal form in D_v. We shall write $\tilde{P}$ for the normal form of P in D_v.

The normal form in D_v of an operator induces a normal form in $D_v/\partial_v \cdot D_v$ by removing all terms contained in $\partial_v \cdot D_v$ from the normal form of P in D_v. We denote the normal form of $P \in D_v$ in $D_v/\{\partial_v\} \cdot D_v$ by $\overline{\tilde{P}}$. Of course, $P + \partial_v \cdot D_v = \tilde{P} + \partial_v \cdot D_v$ as cosets. We notice that both normal forms are unique. As an example, consider the normal form of ∂_i in $D_v/\partial_v \cdot D_v$: $P = \partial_i$, $P^1 = \partial_i - v^2 f_i \partial_v$, thus $P^2 = v^2 f_i \partial_v = \partial_v v^2 f_i - 2 f_i v$, $P^3 = 0$ and hence $\overline{\tilde{P}} = (\partial_i - v^2 f_i \partial_v) - 2 f_i v + \{\partial_v\} \cdot D_v$.

As a D_n-module, the quotient $N = D_v/(\partial_v \cdot D_v + D_v \cdot \{1 - vf, \phi(J)\})$ is spanned by the powers of v, which are in normal form. Let us try to understand the relations among these powers. To this end note that $(1-fv)(\partial_i - v^2 f_i \partial_v) = (\partial_i - v^2 f_i \partial_v + f_i v)(1-fv)$ which implies that $\overline{(1-fv)\tilde{P}} = 0$ in N for all $P \in D_v$. This proves that $\overline{f(v\tilde{P})} = \overline{\tilde{P}} = \overline{P}$ in N for all $P \in D_v$ and we conclude that every element in N is divisible by f.

Moreover, the division by f is unique since if $f\overline{\tilde{Q}} = f\overline{\tilde{Q}'}$ in N then of course $\overline{vf(\tilde{Q} - \tilde{Q}')} = 0$ in N and that implies $\overline{\tilde{Q}} = \overline{\tilde{Q}'}$ in N. Conversely, for all $\overline{\tilde{P}} \in N$ there is a unique $\overline{\tilde{Q}} \in N$ for which $\overline{v\tilde{Q}} = \overline{\tilde{P}}$ because then necessarily $\overline{\tilde{Q}} = f\overline{\tilde{P}}$. So f is an invertible operator on N and the inverse is given by v acting on the normal form representative of a coset.

We shall now make use of the action we defined in the previous section.

The map $D_v \to (D_n/J) \otimes_{R_n} (R_n[f^{-1}][s] \otimes f^{-s})$ sending 1 to $\overline{1} \otimes 1 \otimes f^{-s}$ defined via the action $\bullet$ is surjective. Consider the combined map

$$D_v \to D_n/J \otimes_{R_n} (R_n[f^{-1}, s] \otimes f^{-s}) \to \frac{\big((D/J) \otimes_{R_n} (R_n[f^{-1}, s] \otimes f^{-s})\big)}{(s-2)(D/J) \otimes_{R_n} (R_n[f^{-1}, s] \otimes f^{-s})}.$$

This is a surjective map of left D_n-modules and the kernel contains $\partial_v \cdot D_v + D_v \cdot \{1 - vf, \phi(J)\}$. If an operator in D_v is in the kernel, we can assume it to be in normal form in $D_v/\partial_v \cdot D_v$ since $\partial_v \cdot D_v$ is in the kernel. Any such operator may be written as $v^k P(\partial_x - v^2 f_x \partial_v)$ where $P(\partial_x) \in D_n$. We conclude that $P(\partial_x - v^2 f_x \partial_v)$ must already be in the kernel since $v^k P(\partial_x - v^2 f_x \partial_v) \bullet (\overline{1} \otimes 1 \otimes f^{-s}) = v^k \bullet \overline{P(\partial_x)} \otimes 1 \otimes f^{-s} = \overline{P(\partial_x)} \otimes f^{-k} \otimes f^{-s}$. Thus $P(\partial_x - v^2 f_x \partial_v) \bullet (\overline{1} \otimes 1 \otimes f^{-s}) \in (s-2)(D/J) \otimes_{R_n} (R_n[f^{-1}][s] \otimes f^{-s})$ and therefore $f^j P(\partial_x) \in D[s] \cdot (J, (s-2))$ for some $j \in \mathbb{N}$. Since neither P nor J does contain s in the generators, setting $s = 2$ we see that $P(\partial_x)$ is in the f-saturation of J, $f^j P(\partial_i) \in J$.

$f^j P(\partial_x) \in J$ implies that $f^j P(\partial_x - v^2 f_x \partial_v) \in \phi(J)$. Modulo $D_v \cdot (1 - vf)$ this is equivalent to $P(\partial_x - v^2 f_x \partial_v) \in \phi(J)$ since $f^j P(\partial_x - v^2 f_x \partial_v)$ is in normal form. This simply means that P is in the kernel of $D_v \to (D_n/J) \otimes_{R_n} (R_n[f^{-1}][s] \otimes f^{-s}) \to \frac{(D_n/J) \otimes_{R_n} (R_n[f^{-1}][s] \otimes f^{-s})}{(s-2)(D_n/J) \otimes_{R_n} (R_n[f^{-1}][s] \otimes f^{-s})}$ if and only if it is in $\partial_v \cdot D_v + D_v \cdot \{1 - vf, \phi(J)\}$.

Finally we observe that

$$\left((D_n/J)\otimes_{R_n}(R_n[f^{-1}][s]\otimes f^{-s})\right)/(s-2)\left((D_n/J)\otimes_{R_n}(R_n[f^{-1}][s]\otimes f^{-s})\right)$$

is isomorphic to $(D_n/J)\otimes_{R_n} R_n[f^{-1}]$ by means of the map $\overline{P}\otimes\frac{g(x,s)}{f^m}\otimes f^{-s}\to \overline{P}\otimes\frac{g(x,2)}{f^{m+2}}$. $\square$

Remark 3.13. The theorem and its proof generalize nearly verbatim to the situation where M is any finitely generated D_n-module.

Now let us consider the situation in which $M = D_n/J$ is holonomic on U. Then $D_v/(\partial_v\cdot D_v + D_v\cdot\{1-fv,\phi(J)\})$ is a finite D_n-module as it is isomorphic to the module $M\otimes_{R_n} R_n[f^{-1}]$ which is holonomic by theorem 1.3 of [24]. We can find a generator by computing the b-function for integration along ∂_v for $D_v/D_v\cdot\{1-vf,\phi(J)\}$ as in step 2 of algorithm 3.12. If a is the largest root of the b-function then v^a is a generator for $D_v/(\partial_v\cdot D_v + D_v\cdot\{1-fv,\phi(J)\}) = M\otimes_{R_n} R_n[f^{-1}]$ (compare [36], algorithm 5.4). Thus in order to represent the localization $M\otimes_{R_n} R_n[f^{-1}]$ as a quotient of D_n all one needs to do is to find the annihilator of v^a over D_n. This shows the correctness of our algorithm.

Remark 3.14. Again, the algorithm generalizes to the non-cyclic situation. Let $M = {D_n}^m/J$, ${D_n}^m = \oplus_1^m D_n e_j$. The modifications are as follows. Compute m separate b-functions $b_j(s)$ to the integration of $D_v(e_j+\phi(J))/(D_v\cdot\{\phi(J),(1-vf)e_j\})$ along ∂_v. $M\otimes_{R_n} R_n[f^{-1}]$ is generated by the cosets of $v^{a_j}e_j$ in N where a_j is the largest integer root of $b_j(s)$.

3.2.3 Categorical explanation of the algorithm

Let us now give a more categorical explanation of the validity of our algorithm. Recall that $X = K^n$, $Y = \mathrm{Var}(F)$, $U = X\setminus Y$.

Decompose $j : K^n\setminus V(f)\hookrightarrow K^n$ as $j = p\circ\iota\circ\phi$;

$$\begin{array}{ccc} W & \xrightarrow{\iota} & K^{n+1} \\ \uparrow{\scriptstyle\phi} & & \downarrow{\scriptstyle p} \\ K^n\setminus V(f) & \xrightarrow{j} & K^n \end{array}$$

where $\phi : K^n \setminus V(f) \to W = \{(v,x) \mid v \cdot f(x) = 1\}$ is defined by $x \mapsto (1/f(x), x)$; $\iota : W \hookrightarrow K^{n+1}$ is the closed embedding, and $p : K^{n+1} \to K^n$ is the natural projection $v \to 0$. Then we have

$$M \otimes_{R_n} R_n[f^{-1}] = \int_j j^{-1}M = \int_p \int_\iota \int_\phi j^{-1}M$$

by the chain rule of integration functors (see, e.g., [3, p.251, 6.4 Proposition], [19, 1.5.1]). Note that the chain rule holds in the derived category in general, but, in this case, $\int_j$ is an exact functor since $\mathcal{D}_{X \overset{j}{\leftarrow} U} = \mathcal{D}_X[1/f]$ is flat over $\mathcal{D}_U$ and j is an affine morphism. $\int_\iota \int_\phi$ is also an exact functor as a combination of an isomorphism and Kashiwara equivalence. Hence, we have only to compute 0-th integrals in each step of the computation of the integral functors. $\int_\phi j^{-1}M$ is obtained by the coordinate transformation represented by our ring map ϕ; consider the map

$$y_i = x_i, \ (i = 1, \ldots, n), \ y_{n+1} = 1/f(x).$$

Then, $\partial_{x_i} = \partial_{y_i} - \frac{f_i}{f^2}\partial_{y_i} = \partial_{y_i} - f_i v^2 \partial_{y_i}$ modulo $1 - fv = 0$. These relations commute with each other and define our ring map ϕ. $\int_\iota$ is nothing but Kashiwara equivalence corresponding to $\phi(D) \hookrightarrow D_v$ induced by $W \hookrightarrow K^{n+1}$. It follows that $\int_\iota \int_\phi j^{-1}M = D_v/(D_v \cdot \{\phi(J), 1 - fv\})$, compare also Proposition A.1 (p. 596) of [35]. Integration under the projection p corresponds then to our last step accomplished by forming the quotient $\left(\int_\iota \int_\phi j^{-1}M\right)/\partial_v \cdot \left(\int_\iota \int_\phi j^{-1}M\right)$.

Since $j^{-1}M$ is holonomic, so is $\int_\iota \int_\phi j^{-1}M$ and hence the b-function for integration is nonzero, thus guaranteeing termination of the search for generators of $\int_j j^{-1}(M)$.

3.3 Examples

Example 3.15. For our first example we take $n = 1, x = x_1, K = \mathbb{C}, J = x\partial_x + \lambda, \lambda \in \mathbb{C}, f(x) = x$.

In this scenario, we have to compute the integral of the module $D_v/D_v \cdot (1 - xv, x(\partial_x - v^2\partial_v) + \lambda)$ along ∂_v. $D_v \cdot (1 - xv, x(\partial_x - v^2\partial_v) + \lambda)$ agrees with the left ideal $D_v \cdot \{1 - xv, \partial_x x - \partial_v v + 1 + \lambda\}$ from which one sees that the b-function is $s(s+1+\lambda)$. Thus the largest integer root is either 0 or $-1-\lambda$, depending on whether λ is a negative integer or not. So $M \otimes_{R_1} R_1[x^{-1}]$ is generated by $\overline{1} \otimes x^{-(-1-\lambda)-2}$ in the former and $\overline{1} \otimes x^{-2}$ in the latter case.

If for example $\lambda = -7$ we compute a b-function of $s(s-6)$ indicating that $M \otimes_{R_1} R_1[x^{-1}]$ is generated by $\overline{1} \otimes x^{-8}$. Since in this case $M = D_1 \cdot x^7 \cong R_1$, we conclude that $M \otimes_{R_1} R_1[f^{-1}]$ is in fact generated by $1/x$, as it should.

If on the other hand $\lambda = 1/2$ then $b(s) = s(s+3/2)$ and hence the largest integer root is 0. Thus, $M \otimes_{R_1} R_1[x^{-1}]$ is generated by v^0 corresponding to $1/x^2$. This is simply stating that M is already isomorphic to $M \otimes_{R_1} R_1[x^{-1}]$.

Example 3.16. In this example we consider the left ideal J generated by

$$\partial_x(x^2-y^3) \text{ and } \partial_y(x^2-y^3).$$

These are annihilators of the function $1/f$, where $f = x^2 - y^3$, but they do not generate the annihilating ideal (see, e.g., [33]). The left ideal J is not holonomic since the characteristic variety of J is $\mathrm{Var}(x^3-y^2) \cup \mathrm{Var}(\xi_x, \xi_y)$, the first component of which has dimension 3 in $\mathbb{C}^4 = \{(x, y; \xi_x, \xi_y)\}$. As to an algorithmic method to get the characteristic variety, see [32]. We give below the output of a computer session using the computer algebra system Kan/sm1 ([44]) interspersed with comments. We remark that in this case M restricted to U is an $\mathcal{O}_U$-coherent free module of rank one where $U = \mathbb{C}^2 \setminus \mathrm{Var}(f)$.

```
/ff
  [ ((x^2-y^3)*(Dx - v^2*2*x*Dt) + 2*x)
    ((x^2-y^3)*(Dy + v^2*3*y^2*Dt) -3*y^2)
    (v*(x^2-y^3)-1)
  ]
def
```

(this is $\phi(J)$)

```
sm1>ff [(v)] intbfm ::
[    $216*s^4+1296*s^3+2586*s^2+1716*s$ ]
[[6,1],[s+2,1],[6*s+11,1],[6*s+13,1],[s,1]]
```

(these are the factors of the b-function)

```
sm1>ff [(v)] -2 0 1 intall_s ;
Completed.
```

(computing the integration)

```
 0-th cohomology:  [    1 ,
   [    -3*x*Dx-2*y*Dy-18 ,
         3*y^2*Dx+2*x*Dy ,
         -2*y^3*Dy+2*x^2*Dy-18*y^2 ]  ]
-1-th cohomology:  [    0 , [   ]  ]
```

The integration $\int_j j^{-1}(M)$ is not $\mathcal{O}_X$-coherent, although of course it is still coherent over the sheaf of differential operators $\mathcal{D}_X$ on $X = \mathbb{C}^2$. Since localization is an exact functor, the first cohomology group (corresponding to the first higher order integration) was known to be zero. The 0-th cohomology above coincides with the annihilating ideal of the function f^{-3}.

Example 3.17. Let $n = 3$, and consider the left ideal J generated by the system

$$\begin{array}{rcl} (x^3 - y^2z^2)\partial_x & + & 3x^2, \\ (x^3 - y^2z^2)\partial_y & - & 2yz^2, \\ (x^3 - y^2z^2)\partial_z & - & 2yz^2. \end{array}$$

These operators annihilate the exponential function $e^{1/f}$ where $f(x, y, z) = x^3 - y^2z^2$. The characteristic variety of $M = D_3/J$ has six components, defined by the prime ideals (y, x), (z, x), (ξ_x, ξ_y, ξ_z), (ξ_y, z, x), (ξ_z, y, x) and $(y\xi_y - z\xi_z, 2x\xi_x + 3z\xi_z, 8z\xi_x^3 + 27\xi_y^2\xi_z, 8y\xi_x^3 + 27\xi_y\xi_z^2, -4z^2\xi_x^2 + 9x\xi_y^2, -4yz\xi_x^2 + 9x\xi_y\xi_z, -4y^2\xi_x^2 + 9x\xi_z^2, 2z^3\xi_x\xi_z + 3x^2\xi_y^2, 2yz^2\xi_x + 3x^2\xi_y, 2y^2z\xi_x + 3x^2\xi_z, -yz^3\xi_z + x^3\xi_y, -y^2z^2 + x^3, x^3\xi_y^2 - z^4\xi_z^2)$. All but the first two are of dimension three. This implies that the non-holonomic locus of M is contained in the hypersurface $x = 0$. Here, we used [43] to obtain the primary ideal decomposition.

Hence we may apply our algorithm to compute $M \otimes_{R_3} R_3[x^{-1}]$. We remark that contrary to the previous example in this case $j^{-1}M$ is holonomic but not coherent as $\mathcal{O}_U$-module on $U = X \setminus \mathrm{Var}(x)$. Using Kan/sm1 again one obtains the eight operators

$$
\begin{aligned}
&-3y\partial_y + 3z\partial_z,\\
&-2xyz^2\partial_x - 3x^3\partial_y - 4yz^2,\\
&-2xy^2z\partial_x - 3x^3\partial_z - 4y^2z,\\
&6xz^3\partial_x\partial_z + 9x^3\partial_y^2 + 6xz^2\partial_x + 6yz^2\partial_y + 6z^3\partial_z + 12z^2,\\
&-6y^2z^3\partial_z + 4x^4\partial_x + 12x^3z\partial_z + 8x^3 + 12,\\
&6yz^4\partial_z^2 - 4x^4\partial_x\partial_y - 12x^3z\partial_y\partial_z + 18yz^3\partial_z - 8x^3\partial_y - 12\partial_y,\\
&4x^5\partial_x^2 + 12x^4z\partial_x\partial_z + 9x^3z^2\partial_z^2 + 32x^4\partial_x + 51x^3z\partial_z + 40x^3 + 12x\partial_x + 24,\\
&-6z^5\partial_z^3 + 4x^4\partial_x\partial_y^2 + 12x^3z\partial_y^2\partial_z - 36z^4\partial_z^2 + 8x^3\partial_y^2 - 36z^3\partial_z + 12\partial_y^2
\end{aligned}
$$

which annihilate the function $x^{-2}e^{1/f}$. The characteristic variety of this holonomic left ideal of D has the same last four components as the characteristic variety of M while the first two components (of dimension 4) are replaced by (ξ_z, ξ_y, x) and (z, y, x) (of dimension 3).

4 Local Cohomology

As before, let $R_n = K[x_1, \dots, x_n]$ with K a computable field of characteristic zero, and set $D_n = D(R_n, K)$, the n-th Weyl algebra over K. In this chapter we set $\mathfrak{m} = R_n \cdot (x_1, \dots, x_n)$. f_i will not denote the i-th derivative of f anymore, but rather one of r functions $f_1, \dots, f_r$ generating the ideal I in R_n.

We will combine the results from the previous chapters to obtain algorithms that compute for given $i, j, k \in \mathbb{N}, I \subseteq R_n$ the local cohomology modules $H^k_I(R_n), H^i_{\mathfrak{m}}(H^j_I(R_n))$ and the invariants $\lambda_{i,j}(R_n/I)$ associated to I.

In the following section we wrap up the reasons that give local cohomology modules an action over D_n in our situation.

In section 4.2 we describe our main results, namely algorithms that for arbitrary i, j, k, I determine the structure of $H^k_I(R_n), H^i_{\mathfrak{m}}(H^j_I(R_n))$ and find $\lambda_{i,j}(R_n/I)$. Some of these algorithms have been implemented in the programming language C.

In the third section we display some results on the Lyubeznik numbers in a local ring, covering the zero-, one- and two-dimensional cases. We also exhibit an algorithm to compute these invariants if the local ring in question is the localization of a polynomial ring of characteristic zero at a maximal prime.

In the following section we discuss computation of local cohomology and local cohomological dimension in spaces that are not affine n-space. We also make some remarks on ambient singular spaces.

The last section is devoted to computational examples.

4.1 Local Cohomology as D-module

4.1.1 General Facts

Local cohomology and the cohomological dimension have been studied by many authors, for example R. Hartshorne [15], A. Ogus [39], R. Hartshorne and R. Speiser [17], C. Peskine and L. Szpiro [40], G. Faltings [11], C. Huneke and G. Lyubeznik [20], C. Huneke and R. Y. Sharp [21]. Yet despite this extensive effort, the problem of finding an algorithm for the computation of cohomological dimension remained open in general. For the determination of $\mathrm{cd}(R,I)$ it is in fact enough to find an algorithm to decide whether or not the local cohomology module $H^i_I(R)=0$ for given i,R,I. This is because $H^q_I(R)=0$ for all $q>c$ implies $\mathrm{cd}(R,I)\le c$ (see [15], section 1). In [27], G. Lyubeznik gave an algorithm for deciding whether or not $H^i_I(R)=0$ for all $I\subseteq R=K[x_1,\dots,x_n]$ where K is a field of positive characteristic. The main purposes of this chapter is to produce such an algorithm in the case where K is a computable field containing the rational numbers and $R=R_n=K[x_1,\dots,x_n]$. (By a computable field we mean a subfield K of $\mathbb{C}$ such that K is described by a finite set of data and for which addition, subtraction, multiplication and division as well as the test whether the result of any of these operations is zero in the field can be executed by the Turing machine.)

Since in such a situation the local cohomology modules $H^i_I(R_n)$ have a natural structure of finitely generated left D_n-modules ([24], and in the algebraic context [26]), explicit computations may be performed. Using this finiteness we employ the theory of Gröbner bases to develop algorithms that give a representation of $H^i_I(R_n)$ and $H^i_{\mathfrak{m}}(H^j_I(R_n))$ for all triples $i,j\in\mathbb{N}, I\subseteq R$ in terms of generators and relations over D_n. This also leads to an algorithm for the computation of the invariants $\lambda_{i,j}(R_n/I)=\dim_K \mathrm{soc}_R(H^i_{\mathfrak{m}}(H^{n-j}_I(R)))$ introduced in [26]. Our algorithms are in part modelled after algorithms due to T. Oaku. In [35], [34], [33] he develops a method that computes explicitly $H^i_I(R_n)$ if I is a complete intersection and $i=\mathrm{depth}(I)$.

We remark that if R is an arbitrary finitely generated K-algebra and I is an ideal in R then, if R is regular, our algorithms can be used to determine $\mathrm{cd}(R,I)$ for all ideals I of R, but if R is not regular, then the problem of algorithmic determination of $\mathrm{cd}(R,I)$ remains open (see subsection 4.4).

It turns out that $H^k_I(M)$ may be computed as follows for any R_n-module M. Let $\check{C}^\bullet(f_i)$ be the complex $0\to R_n \overset{1\to\frac{1}{1}}{\longrightarrow} R_n[f_i{}^{-1}]\to 0$. Then $H^k_I(M)$ is the k-th cohomology group of the *Čech complex* defined by $\check{C}^\bullet(M;f_1,\dots,f_r)=\bigotimes_1^r \check{C}^\bullet(f_i)\otimes M$. Unfortunately, explicit calculations are complicated by the fact that $H_I^k(M)$ is rarely finitely generated as R-module. This difficulty disappears for $H_I^k(R_n)$ if we enlarge the ring to D_n, in essence because $R_n[f^{-1}]$ is finitely generated over D_n for all $f \in R_n$.

4.1.2 The Čech complex

Our local cohomology modules are D_n-modules and in fact holonomic: we know already that the modules in the Čech complex are holonomic, it suffices to show that the maps are D_n-linear. All maps in the Čech complex are direct sums of localization maps. Suppose $R_n[f^{-1}]$ is generated by f^s and $R_n[(fg)^{-1}]$ by $(fg)^t$. We may replace s, t by their minimum u and then we see that the inclusion $R_n[f^{-1}] \to R_n[(fg)^{-1}]$ is nothing but the map $D_n/\operatorname{ann}(f^u) \to D_n/\operatorname{ann}((fg)^u)$ sending $P + \operatorname{ann}(f^u)$ to $P \cdot g^{-u} + \operatorname{ann}((fg)^u)$. We conclude that $\check{C}^i(N; f_1, \ldots, f_r) \to \check{C}^{i+1}(N; f_1, \ldots, f_r)$ is an D_n-linear map between holonomic modules for every holonomic N. One can prove that kernels and cokernels of D_n-linear maps between holonomic modules are holonomic. Holonomicity of $H_I^k(R_n)$ follows.

Similarly, $H_{\mathfrak{m}}^i(H_I^j(R_n))$ is holonomic for $i, j \in \mathbb{N}$ (since $H_I^j(R_n)$ is holonomic).

4.2 Computing local Cohomology

4.2.1 Computation of $H_I^k(R_n)$

Here we will describe an algorithm that takes in a finite set of polynomials $\underline{f} = \{f_1, \ldots, f_r\} \subset R_n$ and returns a presentation of $H_I^k(R_n)$ where $I = (f_1, \ldots, f_r)$. In particular, if $H_I^k(R_n)$ is zero, then the algorithm will return the zero presentation.

Consider the Čech complex associated to $f_1, \ldots, f_r$ in R_n,

$$0 \to R_n \to \bigoplus_1^r R_n[f_i^{-1}] \to \bigoplus_{1 \le i < j \le r} R_n[(f_i f_j)^{-1}] \to \cdots$$
$$\cdots \to R_n[(f_1 \cdot \ldots \cdot f_r)^{-1}] \to 0. \quad (4.1)$$

Its k-th cohomology group is $H_I^k(R_n)$. The map

$$\check{C}^k = \bigoplus_{1\le i_1<\cdots<i_k\le r} R_n[(f_{i_1}\cdot\ldots\cdot f_{i_k})^{-1}] \to \bigoplus_{1\le j_1<\cdots<j_{k+1}\le r} R_n[(f_{j_1}\cdot\ldots\cdot f_{j_{k+1}})^{-1}] = \check{C}^{k+1} \quad (4.2)$$

is the sum of maps

$$R_n[(f_{i_1}\cdot\ldots\cdot f_{i_k})^{-1}] \to R_n[(f_{j_1}\cdot\ldots\cdot f_{j_{k+1}})^{-1}] \quad (4.3)$$

which are either zero (if $\{i_1,\ldots,i_k\} \not\subseteq \{j_1,\ldots,j_{k+1}\}$) or send $\frac{1}{1}$ to $\frac{1}{1}$, up to sign. Recall that $D_n/\Delta_n \cong R_n$ and identify $R_n[(f_{i_1}\cdot\ldots\cdot f_{i_k})^{-1}]$ with $D_n/J^{\Delta_n}((f_{i_1}\cdot\ldots\cdot f_{i_k})^s)|_{s=a}$ and $R_n[(f_{j_1}\cdot\ldots\cdot f_{j_{k+1}})^{-1}]$ with $D_n/J^{\Delta_n}((f_{j_1}\cdot\ldots\cdot f_{j_{k+1}})^s)|_{s=b}$ where a, b are sufficiently small integers. By proposition 3.2 we may assume that $a = b \le 0$. Then the map (4.2) is in the nonzero case multiplication from the right by $(f_l)^{-a}$ where $l = \{j_1,\ldots,j_{k+1}\}\backslash\{i_1,\ldots,i_k\}$, again up to sign.

It follows that the matrix representing the map $\check{C}^k \to \check{C}^{k+1}$ in terms of D_n-modules is very easy to write down once the annihilator ideals and Bernstein polynomials for all k- and $(k+1)$-fold products of the f_i are known: the entries are 0 or $\pm f_l^{-a}$ where f_l is the new factor.

Let Θ_k^r be the set of k-element subsets of $1,\ldots,r$ and for $\theta\in\Theta_k^r$ write F_θ for the product $\prod_{i\in\theta} f_i$. We have demonstrated the correctness and finiteness of the following algorithm.

Algorithm 4.1. Input: $f_1,\ldots,f_r \in R_n; k\in\mathbb{N}$.

Output: $H_I^k(R_n)$ in terms of generators and relations as finitely generated D_n-module.

Begin

1. Compute the annihilator ideal $J^{\Delta_n}((F_\theta)^s)$ and the Bernstein polynomial $b_{F_\theta}^{\Delta_n}(s)$ for all $(k-1)$-, k- and $(k+1)$-fold products F_θ of $f_1,\ldots,f_r$ as in 3.5 and 3.7 (so θ runs through $\Theta_{k-1}^r \cup \Theta_k^r \cup \Theta_{k+1}^r$).

2. Compute the smallest integer root a_θ for each $b_{F_\theta}^{\Delta_n}(s)$, let a be the minimum of all a_θ and replace s by a in all the annihilator ideals.

3. Compute the two matrices M_{k-1}, M_k representing the D_n-linear maps $\check{C}^{k-1} \to \check{C}^k$ and $\check{C}^k \to \check{C}^{k+1}$ as explained in subsection 4.2.1.

4. Compute a Gröbner basis G for the kernel of the map

$$\bigoplus_{\theta \in \Theta_k^r} D_n \to \bigoplus_{\theta \in \Theta_k^r} D_n/J^{\Delta_n}((F_\theta)^s)|_{s=a} \xrightarrow{M_k} \bigoplus_{\theta \in \Theta_{k+1}^r} D_n/J^{\Delta_n}((F_\theta)^s)|_{s=a}$$

   as in 2.2.2.

5. Compute a Gröbner basis G_0 for a lift to $\bigoplus_{\theta \in \Theta_k^r} D_n$ of the module

$$\operatorname{im}(M_{k-1}) \subseteq \bigoplus_{\theta \in \Theta_k^r} D_n/J^{\Delta_n}((F_\theta)^s)|_{s=a}.$$

6. Compute the remainders of all elements of G with respect to G_0.

7. Return these remainders and G_0.

End.

The nonzero elements of G generate the quotient $G/G_0 \cong H_I^k(R_n)$ so that $H_I^k(R_n) = 0$ if and only if all returned remainders are zero.

4.2.2 Computation of $H^i_{\mathfrak{m}}(H^j_I(R_n))$

As a second application of Gröbner basis computations over the Weyl algebra we show now how to compute $H^i_{\mathfrak{m}}(H^j_I(R_n))$. Note that we cannot apply lemma 3.1 to $D_n/L = H^j_I(R_n)$ since $H^j_I(R_n)$ may well contain some torsion.

As in the previous sections, $\check{C}^j(R_n; f_1, \dots, f_r)$ denotes the j-th module in the Čech complex to R_n and $\{f_1, \dots, f_r\}$. Let $\check{C}^{\bullet\bullet}$ be the double complex with $\check{C}^{i,j} = \check{C}^i(R_n; x_1, \dots, x_n) \otimes_{R_n} \check{C}^j(R_n; f_1, \dots, f_r)$, the vertical maps $\phi^{\bullet\bullet}$ induced by the identity on the first factor and the usual Čech maps on the second, whereas the horizontal maps $\xi^{\bullet\bullet}$ are induced by the Čech maps on the first factor and the identity on the second. Since $\check{C}^i(R; x_1, \dots, x_n)$ is R_n-projective, the column cohomology of $\check{C}^{\bullet\bullet}$ at (i, j) is $\check{C}^i(R_n; x_1, \dots, x_n) \otimes_{R_n} H^j_I(R_n)$ and the induced horizontal maps in the j-th row are simply the maps in the Čech complex $\check{C}^\bullet(H^j_I(R_n); x_1, \ldots, x_n)$. It follows that the row cohomology of the column cohomology at (i_0, j_0) is $H^{i_0}_{\mathfrak{m}}(H^{j_0}_I(R_n))$.

Now $\check{C}^{i,j}$ is a direct sum of modules $R_n[g^{-1}]$ where $g = x_{\alpha_1} \cdot \ldots \cdot x_{\alpha_i} \cdot f_{\beta_1} \cdot \ldots \cdot f_{\beta_j}$. So the whole double complex can be rewritten in terms of D_n-modules and D_n-linear maps using 3.8:

$$\begin{tikzcd}
\check{C}^{i-1,j+1} \arrow[r, "\xi^{i-1,j+1}"] & \check{C}^{i,j+1} \arrow[r, "\xi^{i,j+1}"] & \check{C}^{i+1,j+1} \\
\check{C}^{i-1,j} \arrow[u, "\phi^{i-1,j}"'] \arrow[r, "\xi^{i-1,j}"] & \check{C}^{i,j} \arrow[u, "\phi^{i,j}"'] \arrow[r, "\xi^{i,j}"] & \check{C}^{i+1,j} \arrow[u, "\phi^{i+1,j}"'] \\
\check{C}^{i-1,j-1} \arrow[u, "\phi^{i-1,j-1}"'] \arrow[r, "\xi^{i-1,j-1}"] & \check{C}^{i,j-1} \arrow[u, "\phi^{i,j-1}"'] \arrow[r, "\xi^{i,j-1}"] & \check{C}^{i+1,j-1} \arrow[u, "\phi^{i+1,j-1}"']
\end{tikzcd}$$

Using the comments in subsection 2.2.3, we may now compute the modules $H^i_{\mathfrak{m}}(H^j_I(R_n))$. More concisely we have, denoting by $X_{\theta'}$ in analogy to F_θ the product $\prod_{\alpha\in\theta'} x_\alpha$, the following

Algorithm 4.2. Input: $f_1, \ldots, f_r \in R_n; i_0, j_0 \in \mathbb{N}$.

Output: $H^{i_0}_{\mathfrak{m}}(H^{j_0}_I(R_n))$ in terms of generators and relations as finitely generated D_n-module.

Begin.

1. For $i = i_0 - 1, i_0, i_0 + 1$ and $j = j_0 - 1, j_0, j_0 + 1$ compute the annihilators $J^{\Delta_n}((F_\theta \cdot X_{\theta'})^s)$ and Bernstein polynomials $b^{\Delta_n}_{F_\theta \cdot X_{\theta'}}(s)$ of $F_\theta \cdot X_{\theta'}$ where $\theta \in \Theta^r_j, \theta' \in \Theta^n_i$.

2. Let a be the minimum integer root of the product of all these Bernstein polynomials and replace s by a in all the annihilators computed in the previous step.

3. Compute the matrices to the D_n-linear maps $\phi^{i,j} : \check{C}^{i,j} \to \check{C}^{i,j+1}$ and $\xi^{i,j} : \check{C}^{i,j} \to \check{C}^{i+1,j}$, again for $i = i_0 - 1, i_0, i_0 + 1$ and $j = j_0 - 1, j_0, j_0 + 1$.

4. Compute Gröbner bases for the modules

$$G = \ker(\phi^{i_0,j_0}) \cap \left[(\xi^{i_0,j_0})^{-1}(\mathrm{im}(\phi^{i_0+1,j_0-1}))\right] + \mathrm{im}(\phi^{i_0,j_0-1})$$

and $G_0 = \xi^{i_0-1,j_0}(\ker(\phi^{i_0-1,j_0})) + \mathrm{im}(\phi^{i_0,j_0-1})$.

5. Compute the remainders of all elements of G with respect to G_0 and return these remainders together with G_0.

End.

The elements of G will be generators for $H^{i_0}_{\mathfrak{m}}(H^{j_0}_I(R_n))$ and the elements of G_0 generate the relations that are not syzygies.

4.3 Some Remarks on Lyubeznik Numbers

4.3.1 Introduction

Throughout this section let k be a field and A be a local k-algebra. It is shown in [26], that if A is the quotient of a regular local ring $(R, \mathfrak{m}, k)$ of dimension n containing k, $\phi : R \twoheadrightarrow A$, $\ker\phi = I$, then the Bass number $\lambda_{p,i}(A) = \mu_p(\mathfrak{m}, H^{n-i}_I(R)) = \dim_k \mathrm{Ext}^p_R(k, H^{n-i}_I(R))$ is finite and a function of A, i, p alone but not of R or ϕ.

Only little is known about the $\lambda_{p,i}$ so far, but they carry interesting information. For example, if $R = \mathbb{C}[x_0, \ldots, x_n]$, $\hat{R} = \mathbb{C}[[x_1, \ldots, x_n]]$ and $I \subseteq R$ is the defining ideal of a smooth variety $V \subseteq \mathbb{P}^n_{\mathbb{C}}$ then, for $i < n - \mathrm{codim}(V)$, $\lambda_{0,i}(\hat{R}/I \cdot \hat{R}) = \dim_{\mathbb{C}}\left(H^i_x(\tilde{V}, \mathbb{C})\right)$ where $H^i_x(\tilde{V}, \mathbb{C})$ stands for the i-th singular cohomology group of the affine cone $\tilde{V}$ over V with support in the vertex x of $\tilde{V}$ and with coefficients in $\mathbb{C}$.

Since completion does not change $\lambda_{p.i}(A)$ ([26], Lemma 4.2) one may assume that $R = k[[x_1, \ldots, x_n]]$. As $\Gamma_I(-) = \Gamma_{\sqrt{I}}(-)$, $\lambda_{p,i}(A) = \lambda_{p,i}(A_{red})$. Hence we assume that I is radical. One has $H^{n-i}_I(R) = 0$ for $i > \dim(I)$ and $\lambda_{p,i}(A) = 0$ for $p > i$ by [26], (4.4i) and (4.4ii).

We define the type of the ring $A = R/I$ to be the matrix $\Lambda(A)$ where $\Lambda(A)_{i,j} = \lambda_{i,j}(A)$ for $0 \le i, j \le n$.

Recall the Hartshorne-Lichtenbaum vanishing theorem ([15], Theorem 3.1) which we denote by HLVT and in essence states that in a complete local domain $(R, \mathfrak{m}, k)$ $H_I^n(R) = 0$ if and only if I is not $\mathfrak{m}$-primary. As is well known, $H_{\mathfrak{m}}^n(R) = E_R(k)$, the R-injective hull of k.

Note that by virtue of the spectral sequence

$$E_2^{pq} = H_{\mathfrak{m}}^p(H_I^q(R)) \Rightarrow E_\infty^{pq} = H_{\mathfrak{m}}^{p+q}(R) \tag{4.4}$$

and HLVT we have $\Lambda(A) = (1)$ if A is Artinian, and $\Lambda(A) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ if $\dim(A) = 1$.

G. Lyubeznik asked in [26] whether $\lambda_{d,d}(A) = 1$ for any A and proved it to be true for A normal. We shall show that this is not the case in general.

4.3.2 Equidimension two

We shall assume that k is separably closed. This means that in R we can use the second vanishing theorem, due to Ogus, Hartshorne-Speiser and Huneke-Lyubeznik (see [20], Theorem 2.4): for $\sqrt{I} \subsetneq \mathfrak{m}$, we have $H_I^{n-1}(R) = 0$ if and only if the punctured spectrum of R/I is connected.

Assume first that I is prime of dimension 2. Then the second vanishing theorem together with the spectral sequence (4.4) shows that R/I is of type $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

Now we consider the puredimensional case.

Lemma 4.3. *Let $I = \bigcap_1^s P_i$ such that $V(I)\backslash\{\mathfrak{m}\}$ is connected and all P_i are prime ideals of dimension 2. Then $H_I^{n-1}(R) = 0$ and I is of type* $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

Proof. If $s = 1$, the claim follows from the previous paragraph. Assume that the claim has been proved for $s - 1$, we show it for s. It is easy to see that we can arrange the P_i in such a way that $J = \bigcap_1^{s-1} P_i$ has a connected punctured spectrum. (Namely, create a graph G with vertices labelled by the P_i and with edges (P_i, P_j) if and only if P_i meets P_j in a curve. By hypothesis the graph is connected. It suffices to find a vertex which if removed does not disconnect the graph. Such a point is given by any vertex of degree one in any spanning tree of G.) By induction, $H_J^{n-1}(R) = 0$. Since $\mathrm{Var}(I) \setminus \{\mathfrak{m}\}$ is connected, $J + P_s$ is of dimension one. The Mayer-Vietoris sequence contains a piece

$$H_J^{n-1}(R) \oplus H_{P_s}^{n-1}(R) \to H_{JP_s}^{n-1}(R) \to H_{J+P_s}^{n}(R) = 0,$$

where the zero on the right follows from HLVT.

So $H_{JP_s}^{n-1}(R)$ is a quotient of $H_J^{n-1}(R) \oplus H_{P_s}^{n-1}(R)$. The second vanishing theorem for P_s shows that $H_I^{n-1}(R) = 0$. The lemma follows as in the single prime case from the spectral sequence (4.4). $\square$

Proposition 4.4. *Let I be radical of pure dimension 2. Let a be the number of connected components of the punctured spectrum of R/I. Then I is of type* $\begin{pmatrix} 0 & a-1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & a \end{pmatrix}$ *and $H_I^{n-1}(R) = E_R(k)^{a-1}$.*

Proof. If $a = 1$, the claim follows from the previous lemma. If $a > 1$, write $I = \bigcap_1^a J_k$ where each J_k defines a connected component of $\mathrm{Spec}(R/I) \setminus \{\mathfrak{m}\}$. Set $J = \bigcap_1^{a-1} J_k$. By induction, $\lambda_{2,2}(R/J) = a - 1$ and $\lambda_{2,2}(R/J_a) = 1$. Since $\mathfrak{m}$ is minimal to $J + J_a$, $H_{J+J_a}^{n-1}(R) = H_{J+J_a}^{n-2}(R) = 0$ by HLVT. Hence $H_{JJ_a}^{n-2}(R) = H_J^{n-2}(R) \oplus H_{J_a}^{n-2}(R)$ so that $\lambda_{2,2}(R/I) = a - 1 + 1$. Moreover, the Mayer-Vietoris sequence to J and J_a also contains a piece

$$0 \to H_J^{n-1}(R) \oplus H_{J_a}^{n-1}(R) \to H_{JJ_a}^{n-1}(R) \to H_{J+J_a}^{n}(R) \to 0$$

where the last zero comes from HLVT. By induction, the term on the left is isomorphic to $E_R(k)^{a-1}$ and in particular injective. The sequence splits and the proposition follows. $\square$

4.3.3 The mixed case

Recall the Brodmann exact sequence ([4], Lemma 3.9) associated to I and $x \in \mathfrak{m}$:

$$\cdots \to H^i_{I,x}(R) \to H^i_I(R) \to \left(H^i_I(R)\right)_x \to H^{i+1}_{I,x}(R) \to \cdots$$

where the map $H^i_{I,x}(R) \to H^i_I(R)$ is induced by the inclusion $I \hookrightarrow I + R \cdot x$. Note that if M is an R-module and $x \in \mathfrak{m}$ then $\operatorname{Hom}_R(k, M_x) = 0$.

Write $I = J_1 \cap J_2$ where J_i is of pure dimension i and let a be the number of connected components of $\operatorname{Spec}(R/J_2) \setminus \{\mathfrak{m}\}$. Let $x \in J_2 \setminus \bigcup\{P|P \in \operatorname{ass}(I), \dim(P) = 1\}$. Observe that $I + R \cdot x = J_2$. Consider the associated Brodmann sequence

$$\begin{aligned} 0 \quad \to \quad & H^{n-2}_{J_2}(R) \to H^{n-2}_I(R) \to \left(H^{n-2}_I(R)\right)_x \to \\ & H^{n-1}_{J_2}(R) \to H^{n-1}_I(R) \to \left(H^{n-1}_I(R)\right)_x \to H^n_{J_2}(R) = 0 \end{aligned}$$

where the zero on the right comes from HLVT. By [26] (4.4iii), the inclusion $H^{n-2}_{J_2}(R) \to H^{n-2}_I(R)$ is an isomorphism. Hence $\lambda_{2,2}(R/I) = \lambda_{2,2}(J_2)$ and we have a four piece exact sequence

$$0 \to \left(H^{n-2}_I(R)\right)_x \to H^{n-1}_{J_2}(R) \to H^{n-1}_I(R) \to \left(H^{n-1}_I(R)\right)_x \to 0.$$

Let F be the kernel of the map $H^{n-1}_I(R) \to (H^{n-1}_I(R))_x$ and split the sequence into two short exact sequences. Then application of $\operatorname{Ext}^\bullet_R(k, -)$ to the first sequence yields

$$\begin{aligned} 0 \quad \to \quad & 0 \to k^{a-1} \to \operatorname{Ext}^0_R(k, F) \to \\ & 0 \to 0 \to \operatorname{Ext}^1_R(k, F) \to \cdots \end{aligned}$$

according to proposition 4.4. Hence $\operatorname{Ext}^0_R(k, F) = k^{a-1}$ and $\operatorname{Ext}^i_R(k, F) = 0$ for $i > 0$. Application of $\operatorname{Ext}^\bullet_R(k, -)$ to the second sequence yields then

$$\begin{aligned} 0 \quad \to \quad & k^{a-1} \to k^{\lambda_{0,1}(R/I)} \to 0 \to \\ & 0 \to k^{\lambda_{1,1}(R/I)} \to 0 \to \cdots . \end{aligned}$$

This proves that $\lambda_{1,1}(R/I) = 0$, $\lambda_{0,1}(R/I) = a - 1$ and the type of I equals the type of J_2.

We present our conclusions in form of the following

Proposition 4.5. *Let I be a radical two-dimensional ideal of the complete regular ring $(R, \mathfrak{m}, k)$ where k is separably closed. Write $I = J_1 \cap J_2$ where each J_i is of pure dimension i. Let a be the number of connected components of* $\operatorname{Spec}(R/J_2) \setminus \{\mathfrak{m}\}$. *Then I is of type* $\begin{pmatrix} 0 & a & 0 \\ 0 & 0 & 0 \\ 0 & 0 & a+1 \end{pmatrix}$. *In particular, the type is independent of the one-dimensional components of I.* □

Remark 4.6. Since these results were obtained in 1995 some new facts have been discovered. Known to us are the following.

In [30], the author gives a combinatorial algorithm to calculate the $\lambda_{p,i}(A)$ from a primary decomposition of I assuming that I is a monomial ideal. He observed that there are examples of monomial ideals in any dimension such that $\lambda_{d,d}(R_n/I)$ is a given positive integer.

In [49] the $\lambda_{p,i}(A)$ for monomial I are investigated in relation to certain Ext-modules. Related results have been obtained in [31], where certain combinatorial properties of $H^i_I(R)$ are studied in the monomial case.

C. Sabbah and R. Garcia (1997) investigated isolated complex singularities and proved that for such cases the Lyubeznik numbers relate to the topology of the real link of the singularity. Finally, the pure-dimensional results of this section were also obtained by Kawasaki (1998). As far as I know these results are all unpublished.

4.3.4 Algorithmic Computation of $\lambda_{i,n-j}(R/I)$

Now consider the case $R = R_n$.

$H^i_{\mathfrak{m}}(H^j_I(R_n))$ is an injective $\mathfrak{m}$-torsion R_n-module of finite socle dimension $\lambda_{i,n-j}$ (which depends only on i, j and the quotient R_n/I) and so isomorphic to $(E_{R_n}(K))^{\lambda_{i,n-j}}$ where $E_{R_n}(K)$ is the injective hull of K over R_n. We are now in a position that allows computation of these invariants of R_n/I.

Algorithm 4.7. Input: $f_1, \ldots, f_r \in R_n; i, j \in \mathbb{N}$.

Output: $\lambda_{i,n-j}(R_n/(f_1, \ldots, f_r))$.

Begin.

1. Using Algorithm 4.2 find $g_1, \ldots, g_l \in {D_n}^d$ and $h_1, \ldots, h_e \in {D_n}^d$ such that $H^i_{\mathfrak{m}}(H^j_I(R_n))$ is isomorphic to $D_n(g_1, \ldots, g_l)$ modulo $H = D_n(h_1, \ldots, h_e)$.

2. Assume that g_1 is not in H. If such a g_1 cannot be chosen, quit.

3. Find a monomial $m \in R$ such that $m \cdot g_1 \notin H$ but $x_i m g_1 \in H$ for all x_i.

4. Replace H by $D_n m g_1 + H$ and reenter at step 2.

5. $\lambda_{i,n-j}(R_n/I)$ equals the number of times step 3 was executed.

End.

The justification of the correctness of the algorithm is as follows. We know that $(D_n \cdot g_1 + H)/H$ is $\mathfrak{m}$-torsion (as $H^i_{\mathfrak{m}}(H^j_I(R_n))$ is) and so it is possible (with trial and error) to find the monomial m in step 3. Then the element $mg_1 + H/H$ has annihilator equal to $\mathfrak{m}$ and therefore generates an D_n-module isomorphic to $D_n/D_n \cdot \mathfrak{m} \cong E_{R_n}(K)$. The injection $(D_n \cdot mg_1 + H)/H \hookrightarrow (D_n \cdot (g_1, \ldots, g_l) + H)/H$ splits as map of R_n-modules because of injectivity and so the cokernel $D_n(g_1, \ldots, g_l)$ modulo $D_n(mg_1, h_1, \ldots, h_e)$ is isomorphic to $(E_{R_n}(K))^{\lambda_{i,n-j}-1}$.

Reduction of the g_i with respect to a Gröbner basis of the new relation module and repetition will lead to the determination of $\lambda_{i,n-j}$.

4.4 Local Cohomology in ambient Spaces other than $\mathbb{A}^n_K$

If $R_n = K[x_1, \ldots, x_n]$, $I \subseteq R_n$, $X = \mathrm{Spec}(R_n)$ and $V = \mathrm{Spec}(R_n/I)$, then knowledge of $H^i_I(R_n)$ for all $i \in \mathbb{N}$ answers of course the question about the local cohomological dimension of V in X. It is worth mentioning, that if $W \subseteq X$ is any smooth variety containing V then our algorithm 4.1 for the computation of $H^i_I(R_n)$ also leads to a determination of the local cohomological dimension of V in W. Namely, if J stands for the defining ideal of W in X so that $A = R_n/J$ is the affine coordinate ring of W and if we set $c = \mathrm{ht}(J)$, then it can be shown that $H_I^{i-c}(A) = \mathrm{Hom}_{R_n}(A, H_I^i(R_n))$ for all $i \in \mathbb{N}$. As $H_I^i(R_n)$ is I-torsion (and hence J-torsion), $\mathrm{Hom}_{R_n}(A, H_I^i(R_n))$ is zero if and only if $H_I^i(R_n) = 0$. It follows that the local cohomological dimension of V in W equals $\mathrm{cd}(R_n, I) - c$ and in fact $\{q \in \mathbb{N} : H_I^q(R_n) \neq 0\} = \{q \in \mathbb{N} : H_I^{q-c}(A) \neq 0\}$.

If however W is not smooth, no algorithms for the computation of either $H_I^i(A)$ or $\mathrm{cd}(A, I)$ are known, irrespective of the characteristic of the base field.

4.5 Implementation and Examples

Some of the algorithms described above have been implemented as C-scripts and tested on some examples.

4.5.1

The algorithm 3.5 with $L = \Delta_n$ has been implemented by Oaku using the package Kan (see [45]) which is a postscript language for computations in the Weyl algebra and in polynomial rings. [46] is theoretically able to compute $H_I^i(R_n)$ for arbitrary $i, R_n = \mathbb{Q}[x_1, \ldots, x_n], I \subseteq R_n$ in the above described terms of generators and relations, using algorithm 4.1. An implementation of the algorithms in this chapter in Macaulay2 is pending. It is expected that in the future also a Kan-based implementation will work for $R_n = K[x_1, \ldots, x_n]$ where K is any computable field of characteristic zero and also algorithms for computation of $H_{\mathfrak{m}}^i(H_I^j(R_n))$ and $\lambda_{i,j}(R_n)$ will be implemented, but see the comments in 4.5.2 below.

Example 4.8. Let I be the ideal in $R_6 = K[x_1, \ldots, x_6]$ that is generated by the 2×2 minors f, g, h of the matrix $\begin{pmatrix} x_1 & x_2 & x_3 \\ x_4 & x_5 & x_6 \end{pmatrix}$. Then $H_I^i(R_6) = 0$ for $i < 2$ and $H_I^2(R_6) \neq 0$ because I is a height 2 prime and $H_I^i(R_6) = 0$ for $i > 3$ because I is 3-generated, so the only remaining case is $H_I^3(R_6)$. This module in fact does not vanish, but until the discovery of our algorithm, its non-vanishing was a rather non-trivial fact. Our algorithm provides the first known proof of this fact by direct calculation.

Previously, Hochster pointed out that $H_I^3(R_6)$ is nonzero, using the fact that the map $K[f,g,h] \to R_6$ splits (compare [20], Remark 3.13) and Bruns and Schwänzl ([6], the corollary to Lemma 2) provided a topological proof of the nonvanishing of $H_I^3(R_6)$ via étale cohomology. Both of these proofs are quite ingenious and work only in very special situations.

Using the program [46], one finds that $H_I^3(R_6)$ is isomorphic to a cyclic D_6-module generated by $1 \in D_6$ subject to relations $x_1 = \ldots = x_6 = 0$. This is a straightforward computational proof of the non-vanishing of $H_I^3(R_6)$. Of course this proof gives more than simply the non-vanishing. Since $D_6/D_6(x_1,\ldots,x_6)$ is isomorphic to $E_{R_6}(R_6/(x_1,\ldots,x_6))$, the injective hull of $R_6/(x_1,\ldots,x_6) = K$ in the category of R_6-modules, our proof implies that $H_I^3(R_6) \cong E_{R_6}(K)$.

4.5.2

Computation of Gröbner bases in many variables is in general a time- and space consuming enterprise. Already in (commutative) polynomial rings the worst case performance for the number of elements in reduced Gröbner bases is doubly exponential in the number of variables and the degrees of the generators. In the (relatively small) example above R_6 is of dimension 6, so that the intermediate ring $D_{n+1}[y_1,y_2]$ contains 16 variables. In view of these facts the following idea has proved useful.

The general context in which lemma 3.1 and proposition 3.2 were stated allows successive localization of R_{fg} in the following way. First one computes $R_n[f^{-1}]$ according to algorithm 3.8 as quotient of D_n by a certain holonomic ideal $L = J^{\Delta_n}(f^s)|_{s=a}, a \ll 0$. Then $R_n[(fg)^{-1}]$ may be computed using 3.8 again since $R_n[(fg)^{-1}] \cong R_n[g^{-1}] \otimes D_n/L$. (Note that all localizations of R_n are automatically f-torsion free for $f \in R_n$ as R_n is a domain.) This process may be iterated for products with any finite number of factors. Note also that the exponents for the various factors might be different. This requires some care as the following situations illustrate. Assume first that -1 is the smallest integer root of the Bernstein polynomials of f and g (both in R_n)

with respect to the holonomic module R_n. Assume further that $R_n[(fg)^{-1}] \cong D_n \cdot f^{-2}g^{-1} \supsetneq D_n \cdot (fg)^{-1}$. Then $R_n[f^{-1}] \to R_n[(fg)^{-1}]$ can be written as $D_n/\operatorname{ann}(f^{-1}) \to D_n/\operatorname{ann}(f^{-2} \cdot g^{-1})$ sending $P \in D_n$ to $P \cdot f \cdot g$.

Suppose on the other hand that we are interested in $H^2_I(R_n)$ where $I = (f, g, h)$ and we know that $R_n[f^{-1}] = D_n \cdot f^{-2} \supsetneq D_n \cdot f^{-1}, R_n[g^{-1}] = D_n \cdot g^{-2}$ and $R_n[(fg)^{-1}] = D_n \cdot f^{-1}g^{-2}$. (In fact, the degree 2 part of the Čech complex of example 4.8 consists of such localizations.) It is tempting to write the embedding $R_n[f^{-1}] \to R_n[(fg)^{-1}]$ with the use of a Bernstein operator (if $P_f(s)f^{s+1} = b_f^{\Delta_n}(s)f^s$ then take $s = -2$) but as f^{-1} is not a generator for $R_n[f^{-1}]$, $b_f^{\Delta_n}(-2)$ will be zero. In other words, we must write $R_n[(fg)^{-1}]$ as $D_n/\operatorname{ann}((fg)^{-2})$ and then send $P \in \operatorname{ann}(f^{-2})$ to $P \cdot g^2$.

The two examples indicate how to write the Čech complex in terms of generators and relations over D_n while making sure that the maps $\check{C}^k \to \check{C}^{k+1}$ can be made explicit using the f_i - the exponents used in $\check{C}^k$ have to be at least as big as those in $\check{C}^{k-1}$ (for the same f_i).

Remark 4.9. One might hope that for all holonomic fg-torsionfree modules $M = D_n/L$ we have (with $R_n[g^{-1}] \otimes M \cong D_n/L'$):

$$\min\{s \in \mathbb{Z} : b_f^L(s) = 0\} \leq \min\{s \in \mathbb{Z} : b_f^{L'}(s) = 0\}.$$

This would guarantee, that successive localization at the factors of a product does not lead to matrices in the Čech complex with entries of higher degree than localization at the product at once.

However, as was pointed out by an unknown referee, the following example shows that this hope is unfounded. Let $R_5 = \mathbb{C}[x_1, \ldots, x_5]$, $f = x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2$. One may check that then $b_f^{\Delta_n}(s) = (s+1)(s+5/2)$. Hence $R_n[f^{-1}] = D_5 \cdot f^{-1}$, let $L = \ker(D_5 \to D_5 \cdot f^{-1})$. Set $g = x_1$. Then $b_g^{\Delta_n}(s) = s + 1$, let $L' = \ker(D_5 \to D_5 \cdot g^{-1})$.

Computations with [46] or [45] show that $b_f^{L'}(s) = (s+1)(s+2)(s+5/2)$ and $b_g^L(s) = (s+1)(s+3)$. This shows that $R_5[(fg)^{-1}]$ is generated by $f^{-2}g^{-1}$ or $f^{-1}g^{-3}$ but not by $f^{-1}g^{-2}$ and in particular not by $f^{-1}g^{-1}$. Notice that this example not only disproves the above inequality but also shows the inequality to be wrong if $\mathbb{Z}$ is replaced by $\mathbb{R}$ (as $-3 < \min(-5/2, -1)$).

Nonetheless, localizing $R_n[(fg)^{-1}]$ successively as $(R_n[f^{-1}])[g^{-1}]$ is advantageous, heuristically. For one, it allows the exponents of the various factors to be distinct which is useful for the subsequent cohomology computation: it helps to keep the degrees of the maps small. (So for example $R_n[(x \cdot (x^2 + y^2))^{-1}]$ can be written as $D_n \cdot x^{-1}(x^2 + y^2)^{-2}$ instead of $D_n \cdot (x^{-2} \cdot (x^2 + y^2)^{-2})$. On the other hand, since the computation of Gröbner bases is doubly exponential it seems to be advantageous to break a big problem (localization at a product) into many "easy" problems (successive localization).

An interesting case of this behaviour is our example 4.8. If we compute $R_6[(fgh)^{-1}]$ as $((R_n[f^{-1}])[g^{-1}])[h^{-1}]$, the calculation uses approximately 2.5 kB and lasts 10 seconds on a Sun workstation using [46]. If one tries to localize R at the product of the three generators at once, [46] needs about 30 minutes.

So, one should proceed as follows for the computation of the Čech complex $\check{C}^\bullet(R_n; f_1, \ldots, f_r)$. First compute $J^{\Delta_n}((f_i)^s)$ for all i, find all minimal integer Bernstein roots β_i of f_i on R_n and substitute them into the appropriate annihilator ideals. Iteratively use algorithm 3.8 in order to compute $R_n[(f_{i_1} \cdot \ldots \cdot f_{i_k} \cdot f_{i_{k+1}})^{-1}]$ from $R_n[(f_{i_1} \cdot \ldots \cdot f_{i_k})^{-1}]$ If all Bernstein roots for $f_{i_{k+1}}$ on $R_n[(f_1 \cdot \ldots \cdot f_k)^{-1}]$ are greater or equal to $\beta_{i_{k+1}}$ we are fine. As soon as one of them is smaller than the corresponding β, we need to replace β_i by that root and start from the beginning.

5 de Rham Cohomology

In this chapter we shall prove

Theorem 5.1. *(see Theorem 5.30) The de Rham cohomology groups of the complement of an affine complex variety are effectively computable by means of Gröbner basis computations in Weyl algebras.*

In fact, we shall first generalize the restriction process to the restriction of a complex to a linear subvariety. Then, as applications, we obtain an algorithm that computes de Rham cohomology of arbitrary Zariski-open U, and an algorithm that computes de Rham cohomology of Zariski open sets with supports in a Zariski closed subset Z of U.

Now we shall give a detailed overview of the structure of this chapter. Let $D_n = \mathbb{C}[x_1, \ldots, x_n]\langle\partial_1, \ldots, \partial_n\rangle$, the n-th Weyl algebra over $\mathbb{C}$.

First of all, in section 5.1, we show that if U is the complement of any Zariski closed set Y defined by $F = \{f_1, \ldots, f_r\}$ in X then computation of the de Rham cohomology of U can be performed by computing the cohomology of the tensor product over D_n of a D_n-free resolution of $D_n/(\partial_1, \ldots, \partial_n) \cdot D_n$ with the Mayer-Vietoris complex $MV^\bullet(F)$ associated to $f_1, \ldots, f_r$ (cf. subsection 5.1.4). If $r = 1$, $MV^\bullet(F) = (0 \to R_n[{f_1}^{-1}] \to 0)$.

In the following section we compute a certain D_n-free complex that is quasi-isomorphic to $\mathcal{F}(MV^\bullet(F))$, the Fourier transform of the Mayer-Vietoris complex. In fact, let $C^\bullet$ be an arbitrary complex of finitely generated D_n-modules such that the cohomology of $C^\bullet$ is specializable to the subspace $x_1 = \ldots = x_d = 0$ for fixed $0 \le d \le n$ (cf. definition 5.11). We present a method that computes a D_n-free complex $A^\bullet$ that is quasi-isomorphic to $C^\bullet$ and has certain properties related to the V_d-filtration (for facts about the V_d-filtration, see section 2.3 and also [37]).

Section 5.3 is devoted to the explicit computation of the derived tensor product $D_n/(x_1, \ldots, x_d) \cdot D_n \otimes^L_{D_n} C^\bullet$, $0 \le d \le n$, where $C^\bullet$ is required to have cohomology that is specializable to the subspace $x_1 = \ldots = x_d = 0$ but otherwise is arbitrary. As a corollary of this computation we give in section 5.4 an algorithm that computes $H^{\bullet}_{dR}(U,\mathbb{C})$, algorithm (5.30).

In section 5.5 we review the definition of de Rham cohomology with supports and derive an algorithm that computes $H^i_{dR,Z}(X\backslash Y,\mathbb{C})$ for arbitrary closed subvarieties Y, Z of X. The idea here is similar to the original argument, twisted with the Čech complex associated to Z.

5.1 Algebraic de Rham Cohomology

5.1.1 Notation

Throughout this chapter, we shall use the following notation. $\mathbb{C}$ will stand for the field of complex numbers, X denotes affine n-dimensional $\mathbb{C}$-space $\mathbb{C}^n$ and Y will be a closed subvariety of X cut out by polynomials $\{f_1,\ldots,f_r\}\subseteq R$ where $R=\mathbb{C}[x_1,\ldots,x_n]$. Set $U=X\backslash Y$.

D_n will be the ring of differential operators on X generated by the multiplications by the x_i (which we will call also x_i) and the partial derivatives $\partial_i=\partial/\partial x_i$. Set $\mathcal{O}_X$ to be the structure sheaf on X. $\mathcal{D}_X$ will be the sheaf version of D_n, $\mathcal{D}_X=\mathcal{O}_X\otimes_{R_n}D_n$. Let $\Omega_d=D_n/D_n\cdot(\partial_1,\ldots,\partial_d)$, $\tilde{\Omega}_d=D_n/D_n\cdot(x_1,\ldots,x_d)$ for $1\le d\le n$. In the special case $n=d$ we write Ω for Ω_n and define $\Omega(\mathcal{D}_X)=\Omega\otimes_{D_n}\mathcal{D}_X$.

If $\mathcal{M}$ is a $\mathcal{D}_X$- or D_n-module, $\Omega^\bullet(\mathcal{M})$ will throughout stand for the de Rham complex of $\mathcal{M}$. In other words, $\Omega^k(\mathcal{M})=\mathcal{M}\otimes_{\mathbb{Z}}\bigwedge^k(\mathbb{Z}^n)$ where $\bigwedge^k(\mathbb{Z}^n)$ is spanned by the symbols $dx_{i_1}\wedge\ldots\wedge dx_{i_k}$ and the differential δ is defined in the usual way: $\delta(u\otimes dx_{i_1}\wedge\ldots\wedge dx_{i_k})=\sum_{j=1}^n(\partial_j\cdot u)\otimes dx_j\wedge dx_{i_1}\wedge\ldots\wedge dx_{i_k}$. If V is a variety, $\Omega^\bullet(V)$ will denote the de Rham complex on V. Furthermore, set $\Omega^\bullet=\Gamma(X,\Omega^\bullet(X))$.

If I is a finite set, $|I|$ will denote its cardinality.

5.1.2 Definition of de Rham Cohomology

Recall the notion of completion $\hat{\mathcal{G}}$ of a quasi-coherent sheaf $\mathcal{G}$ on the scheme A with respect to the closed subset B: by definition, if V is open in A, $\hat{\mathcal{G}}(V\cap B)=$

$\lim\limits_{\longleftarrow k}(\mathcal{G}(V)/\mathcal{I}^k(V)\cdot\mathcal{G}(V))$ where $\mathcal{I}$ is the sheaf of ideals defining B.

Algebraic de Rham cohomology of an arbitrary closed subset B of an arbitrary smooth scheme A over any field K of characteristic zero is defined as the hypercohomology of the complex $\hat{\Omega}^\bullet(A)$ where the hat denotes completion of $\Omega^\bullet(A)$ with respect to the system of ideals $\mathcal{I}$ which defines B in A. (For a precise definition of the maps in $\hat{\Omega}^\bullet(A)$ see [16], page 22.) It is shown in [16] that this definition does not depend on the embedding of B in A nor in fact on A itself.

In the special case where B is smooth, we may take $B = A$ and then the sheaf of ideals $\mathcal{I}$ is the zero sheaf. In particular, for open subsets of X, $H^i_{dR}(U,\mathbb{C})$ is the hypercohomology of the complex $\Omega^\bullet(U)$.

5.1.3 The Idea of Oaku and Takayama

For this subsection, assume that $r = 1$ and $F = \{f\}$ so that $Y = \mathrm{Var}(f)$ is a hypersurface. Let $j : U \hookrightarrow X$ be the embedding.

The basic observation is the following

Lemma 5.2. *The complex $\Omega^\bullet(D_n) = \Omega^\bullet$ is (a complex in the category of right D_n-modules and in that category) quasi-isomorphic to the complex that is zero except in degree n and whose n-th entry is the right D_n-module $D_n/(\partial_1,\ldots,\partial_n)\cdot D_n = \Omega$. A corresponding statement holds for $\mathcal{D}_X$.* □

The hypercohomology of $\Omega^\bullet(\mathcal{O}_U)$ equals the cohomology of the global sections because U is affine and $\Omega^\bullet(\mathcal{O}_U)$ consists of quasi-coherent sheaves. Also, $\Omega^\bullet(\mathcal{O}_U)$ and $j_*(\Omega^\bullet(\mathcal{O}_U))$ have the same cohomology and hence $H^\bullet_{dR}(U,\mathbb{C})$ is the cohomology of $j_*(\Omega^\bullet(\mathcal{O}_U)) = \Omega^\bullet(\mathcal{D}_X)\otimes_{\mathcal{D}_X} j_*(\mathcal{O}_U)$.

Since $\Omega^\bullet(\mathcal{D}_X)$ is a complex of free $\mathcal{D}_X$-modules, the lemma shows that the cohomology of $j_*(\Omega^\bullet(\mathcal{O}_U))$ are the torsion sheaves $\mathcal{T}or^{\mathcal{D}_X}_{n-\bullet}(\Omega(\mathcal{D}_X), j_*(\mathcal{O}_U))$. The cohomology of the global sections of this complex is simply $\mathrm{Tor}^{D_n}_{n-\bullet}(\Omega, R_n[f^{-1}])$. A crucial point is to use the fact that Tor is balanced.

Let $A^\bullet$ be a truncated resolution of $R_n[f^{-1}]$ by finitely generated free D_n-modules in the category of left D_n-modules of length greater than n. That this is possible follows for example from the fact that D_n is left-Noetherian and that $R_n[f^{-1}]$ is D_n-cyclic ([3]).

Then the cohomology of $\Omega \otimes_{D_n} A^\bullet$ is the de Rham cohomology of U with coefficients in $\mathbb{C}$ shifted by n, since $H^{-i}(\Omega \otimes_{D_n} A^\bullet) = \mathrm{Tor}_i^{D_n}(\Omega, R_n[f^{-1}])$ and $\mathrm{Tor}_i^{D_n}(\Omega, R_n[f^{-1}]) = 0$ for $i < 0$ and $i > n$.

T. Oaku and N. Takayama gave an algorithm in [37] for the computation of the cohomology groups of this kind of complex. It is in fact explained how one can find the cohomology groups of the complex $D_n/(x_1, \ldots, x_n) \cdot D_n \otimes^L_{D_n} M$ where M is an arbitrary holonomic D_n-module and the tensor product is considered as an element in the derived category. The present problem can be reduced to that case by applying the Fourier automorphism:

Definition 5.3. The d-th *Fourier transform* $\mathcal{F}_d$ of a free D_n-module is a $\mathbb{C}$-linear map and defined as follows. $\mathcal{F}_d(x_i) = \partial_i, \mathcal{F}_d(\partial_i) = -x_i$ for $1 \leq i \leq d$. $\mathcal{F}_d$ leaves $\mathbb{C}\langle x_{d+1}, \partial_{d+1}, \ldots, x_n, \partial_n\rangle$ invariant.

We write simply $\mathcal{F}$ for $\mathcal{F}_n$ as before.

So computation of $H^i_{dR}(U, \mathbb{C})$ for a hypersurface can be summarized as follows ([36], algorithm 2.1):

Algorithm 5.4. Input: $f \in R$, $i \in \mathbb{N}$.

Output: $H^i_{dR}(U, \mathbb{C})$ where $U = \mathbb{C}^n \setminus V(f)$.

Begin

1. Find a suitable finite free resolution $A^\bullet$ of the D_n-module $\mathcal{F}(R_n[f^{-1}])$, $\mathcal{F}(R_n[f^{-1}])$ positioned in degree n.

2. Replace each D_n by the right D_n-module $\Omega \cong \mathbb{C}[\partial_1, \ldots, \partial_n]$ in that resolution and shift the complex n steps to the right.

3. Truncate the resolution using the method of [37] to a complex of finite dimensional $\mathbb{C}$-vectorspaces.

4. Take the i-th cohomology.

End.

5.1.4 Computing de Rham Cohomology for arbitrary Y

Let Y now be cut out by the r polynomials $f_1, \ldots, f_r$. The central problem arises from the fact that computation of the hypercohomology of $\Omega^\bullet(\mathcal{D}_U) \otimes_{\mathcal{D}_U} \mathcal{O}_U$ is not just $\operatorname{Tor}_{n-\bullet}^{D_n}(\Omega, \Gamma(U, \mathcal{O}_U))$ anymore, due to the existence of higher cohomology of quasi-coherent sheaves on U. The strategy is to find an open covering of U such that each of the open sets in the covering is acyclic for cohomology of quasi-coherent sheaves.

Definition 5.5. Let $\Re :=$ the nonempty subsets of $\{1, \ldots, r\}$. We shall picture them as increasing sequences. Set $U_i = X \setminus \operatorname{Var}(f_i)$. More generally for $I \in \Re$, we define $U_I = \bigcap_{i \in I} U_i$.

Similarly, set $f_I = \prod_{i \in I} f_i$ with the special cases $f_I = f_i$ if $I = \{i\}$. Write $\mathcal{O}_{U_I}$ as $\mathcal{O}_I$.

To get started, notice that $U_I = X \setminus \operatorname{Var}(f_I)$. This means in particular, that by Oaku–Takayama the de Rham cohomology groups of U_I with coefficients in $\mathbb{C}$ are computable as the cohomology of $\Omega^\bullet(D_n) \otimes_{D_n} R_n[{f_I}^{-1}]$. Notice also that $U = X \setminus Y$ is just the union of all the U_I.

In [16], page 28, Hartshorne defines how de Rham cohomology of schemes may be recovered from the de Rham complexes on the open sets in a finite covering. For our U that works as follows.

For each I let $X_I = \prod_{i \in I} U_i$. Then U_I embeds in X_I as the diagonal. As X_I is smooth, $\hat{\Omega}^\bullet(X_I)$ computes de Rham cohomology of U_I, the hat denoting completion at the closed subscheme $U_I \subseteq X_I$.

Consider the direct image $\mathcal{M}_I^\bullet$ of $\hat{\Omega}^\bullet(X_I)$ in U, induced by the inclusion $j_I : U_I \hookrightarrow U$.

Since U_I is smooth, $\hat{\Omega}^\bullet(X_I)$ is naturally quasi-isomorphic to $\Omega^\bullet(\mathcal{O}_I)$, cf. [16], Proposition II.1.1. Hence $\mathcal{M}_I^\bullet$ is naturally quasi-isomorphic to $j_{I*}(\Omega^\bullet(\mathcal{O}_I))$, the direct image of $\Omega^\bullet(\mathcal{O}_I) = \Omega^\bullet(D_n) \otimes_{D_n} \mathcal{O}_I$.

For $j \notin I$, the natural maps $X_{I \cup j} \twoheadrightarrow X_I$ and $U_{I \cup j} \hookrightarrow U_I$ give a natural map $\hat{\Omega}^\bullet(X_I) \to \hat{\Omega}^\bullet(X_{I \cup j})$. Similarly, we get chain maps $\phi_{I,j}^\bullet : \Omega^\bullet(\mathcal{O}_I) \to \Omega^\bullet(\mathcal{O}_{I \cup j})$

induced from the inclusion $U_{I\cup j} \hookrightarrow U_I$. It is not hard to see that the natural quasi-isomorphisms from $\hat{\Omega}^\bullet(X_I)$ to $\Omega^\bullet(\mathcal{O}_I)$ transform the map $\hat{\Omega}^\bullet(X_I) \to \hat{\Omega}^\bullet(X_{I\cup j})$ into $\phi^\bullet_{I,j}$. So the same is true for the direct images in U, the induced maps between which we also denote by $\phi^\bullet_{I,j}$.

Multiply $\phi^\bullet_{I,j}$ by $(-1)^{\mathrm{sgn}(I,j)}$, $\mathrm{sgn}(I,j)$ being the number of elementary permutations that are needed to make (I,j) an actual element of $\Re$, that is, an increasing sequence.

Let us write $\mathcal{J}^\bullet_I := j_{I*}(\Omega^\bullet(D_n) \otimes_{D_n} \mathcal{O}_I)$, a complex of sheaves on U. We will now construct a double complex $\mathcal{MV}(\mathcal{J})$ out of all the $\mathcal{J}^\bullet_I$. Let $\mathcal{MV}(\mathcal{J})^{k,l} = \bigoplus_{|I|=l} \mathcal{J}^k_I$. The maps in horizontal (k-) direction are simply the direct sums of the differentials of the $\mathcal{J}^\bullet_I$ involved, while the vertical (l-) maps are defined to be the sums of all maps which are composed as follows: $\bigoplus_{|I|=l} \mathcal{J}^k_I \xrightarrow{nat} \mathcal{J}^k_I \xrightarrow{\phi^k_{I,j}} \mathcal{J}^k_{I\cup j} \hookrightarrow \bigoplus_{|I'|=l+1} \mathcal{J}^k_{I'}$.

Notice that this is in fact a double complex (and in particular anticommutative) due to the sign rule that applies to the $\phi^k_{I,j}$.

Then, according to Hartshorne, the de Rham cohomology of U is the hypercohomology of the associated total complex $Tot^\bullet(\mathcal{MV}(\mathcal{J}))$ since $\mathcal{J}^\bullet_I$ and $\mathcal{M}^\bullet_I$ are naturally quasi-isomorphic. Of course, $\mathcal{MV}(\mathcal{J})$ is just the origin of the usual Mayer-Vietoris spectral sequence of de Rham cohomology and sometimes called the Čech-de Rham complex .

In the hypersurface case U is affine, so hypercohomology on U was cohomology of the global sections. Now we claim

Lemma 5.6. *The complex $j_{I*}(\Omega^\bullet(\mathcal{O}_I))$ consists entirely of sheaves that have no higher cohomology on U.*

Proof. In order to see this observe that it is sufficient to show that $j_{I*}(\mathcal{O}_I)$ has this property, because Ω^i is D_n-free. If $\mathcal{E}^\bullet_I$ is an $\mathcal{O}_I$-injective resolution of $\mathcal{O}_I$ on U_I, then $j_{I*}(\mathcal{E}^\bullet_I)$ is a complex of flasque sheaves on U as direct images of flasque sheaves are flasque. Moreover, as U_I is affine, j_{I*} is an exact functor on quasi-coherent sheaves (and $\mathcal{O}_{U_I}$-morphisms), because $R^i j_{I*}(-)$ is the sheaf associated to the presheaf $V \to H^i(V \cap U_I, -)$ for open subsets V of U. Hence we actually get a flasque resolution of $j_{I*}(\mathcal{O}_I)$. Taking global sections we see that $j_{I*}(\mathcal{O}_I)$ has no higher cohomology on U, as $\Gamma(U, j_{I*}(\mathcal{E}_I^i)) = \Gamma(U_I, \mathcal{E}_I^i)$. $\square$

Remark 5.7. We note in passing that we actually showed that $H^i(j_{I*}(\mathcal{G}), U)$ vanishes for positive i and all quasi-coherent $\mathcal{G}$ on U_I.

So the complex $Tot^\bullet(\mathcal{MV}(\mathcal{J}))$ consists of $\Gamma(U, -)$-acyclic sheaves. Thus, in order to compute its hypercohomology it suffices to compute the cohomology of the global sections of that complex. We arrive at

Proposition 5.8. *The de Rham cohomology of U with coefficients in $\mathbb{C}$, which may be computed as the hypercohomology of the complex $Tot^\bullet(\mathcal{MV}(\mathcal{J}))$, agrees with the cohomology of the global sections of $Tot^\bullet(\mathcal{MV}(\mathcal{J}))$ and can be computed as $H^\bullet(\Omega^\bullet \otimes_{D_n} MV^\bullet)$, where*

$$MV^\bullet : 0 \to \oplus_{|I|=1} R_n[{f_I}^{-1}] \to \ldots \to \oplus_{|I|=r} R_n[{f_I}^{-1}] \to 0. \tag{5.1}$$

Proof. This follows from the discussion before the proposition, since the global sections on U of $j_{I*}(\Omega^\bullet(\mathcal{O}_I))$ are exactly $\Omega^\bullet(D_n) \otimes_{D_n} R_n[{f_I}^{-1}]$ and hence $\Gamma(U, Tot^\bullet(\mathcal{MV}(\mathcal{J}))) = \Omega^\bullet(D_n) \otimes_{D_n} MV^\bullet$. $\square$

Remark 5.9. Recall that for any set of polynomials $\{p_i\}_1^m$, the Čech complex $\check{C}^\bullet(R_n; p_1, \ldots, p_m) := \bigotimes_1^m \check{C}^\bullet(p_i)$ is defined by $\check{C}^\bullet(p_i) = (0 \to R_n \xrightarrow{1 \to \frac{1}{1}} R_n[{p_i}^{-1}] \to 0)$. Thus, $MV^i = \check{C}^{i+1}(R_n; f_1, \ldots, f_r)$ if $i \geq 0$ and zero otherwise.

In the special case where $r = 1$, so $I = (f)$, one sees that the complex $MV^\bullet$ degenerates to $(0 \to R_n[f^{-1}] \to 0)$ reducing to the case from [36].

In chapter 4 we gave (besides other things) an algorithm that explicitly computes the Čech complex to a finite set of polynomials as a complex of finitely generated left D_n-modules by means of Gröbner basis computations (algorithm 4.1). Using this algorithm we may explicitly compute $MV^\bullet$ for given $f_1, \ldots, f_r$. For the convenience of the reader we reproduce that algorithm here:

Algorithm 5.10. Input: $f_1, \ldots, f_r \in R_n$.

Output: The Čech complex $\check{C}^\bullet(R_n; f_1, \ldots, f_r)$ in terms of generators and relations as finitely generated D_n-modules.

Begin

1. Compute the annihilator ideal $J^{\Delta_n}((F_I)^s)$ and the Bernstein polynomial $b_{F_I}^{\Delta_n}(s)$ for all k-fold products F_I of $f_1, \ldots, f_r$, k running through $1, \ldots, r$ (so I runs through $\Re$).

2. Compute the smallest integer root a_I for each $b_{F_I}^{\Delta_n}(s)$, let a be the minimum of all a_I and replace s by a in all the annihilator ideals.

3. Set $\check{C}^k = \oplus_{|I|=k} D_n / J^{\Delta_n}((F_I)^s)|_{s=a}$.

4. Compute the matrices M_k representing the D_n-linear maps $\check{C}^k \to \check{C}^{k+1}$.

5. Return these matrices and the presentations for the modules.

End.

It will now be our task to develop an algorithm that computes the cohomology of $\Omega^\bullet \otimes_{D_n} MV^\bullet = \Omega \otimes_{D_n}^L MV^\bullet$.

5.2 Computing V_d-strict D_n-free Resolutions of Complexes

Unless stated otherwise, all modules in this and the next section are understood to be left D_n-modules. In these two sections we shall be concerned with finding the cohomology of the complex $\Omega^\bullet \otimes_{D_n} MV^\bullet$ where $MV^\bullet$ is the Mayer-Vietoris complex (5.1) to $f_1, \ldots, f_r$. More generally we will develop algorithms that find the cohomology of $\Omega \otimes_{D_n}^L C^\bullet$ where $C^\bullet$ is an arbitrary complex of finitely generated D_n-modules with specializable cohomology (cf. Definitions 2.6, 5.11). In particular, in this section we find a free D_n-complex quasi-isomorphic to a given complex $C^\bullet$ with special properties related to the so-called V-filtration, while in the next section we develop methods to evaluate the cohomology of the derived tensor product.

It has been shown by T. Oaku and N. Takayama in [37] (Proposition 3.8 and following remarks) how to compute V_d-strict Gröbner bases, and for any D_n-module M positioned in degree b a free V_d-strict resolution $(A^\bullet[\mathfrak{m}_\bullet], \phi^\bullet)$ of

$M[\mathfrak{m}_b]$, $A^i = \oplus_1^{r_i} D_n, r_i = 0$ if $i > b$. The construction given in [37] allows for arbitrary $\mathfrak{m}_b$ as we saw in chapter 2.

The method employed is to construct a free resolution with the usual technique of finding a Gröbner basis for $\ker(A^i \to A^{i+1})$ and calculating the syzygies on this basis. The trick is to impose an order that refines the partial ordering given by V_d-degree, together with a homogenization technique.

The vectors $\mathfrak{m}_i$ are obtained for each A^i with falling i: if A^i maps its generators on a Gröbner basis of $\ker(A^{i+1} \to A^{i+2})$ then the shift component $\mathfrak{m}_i(j)$ corresponding to the j-th generator e_j of A^i is defined as $V_d deg(\phi^i(e_j)[\mathfrak{m}_{i+1}])$.

We need to generalize the definitions of [37] to the case where the complex $A^\bullet$ is not a resolution and M is a complex rather than a module.

Definitions 5.11. A complex of free D_n-modules $\cdots \to A^{i-1} \stackrel{\phi^{i-1}}{\to} A^i \stackrel{\phi^i}{\to} A^{i+1} \to \cdots$ is called *V_d-adapted at A^i* with respect to certain shift vectors $\mathfrak{m}_{i-1}$, $\mathfrak{m}_i$, $\mathfrak{m}_{i+1}$ if $\phi^i F_H^k(A^i[\mathfrak{m}_i]) \subseteq F_H^k(A^{i+1}[\mathfrak{m}_{i+1}])$ and also $\phi^{i-1} F_H^k(A^{i-1}[\mathfrak{m}_{i-1}]) \subseteq F_H^k(A^i[\mathfrak{m}_i])$ for all k.

We shall say that the complex is *V_d-strict at A^i* if it is V_d-adapted at A^i and moreover $\operatorname{im}(\phi^{i-1}) \cap F_H^k(A^i[\mathfrak{m}_i]) = \operatorname{im}(\phi^{i-1}|_{F_H^k(A^{i-1}[\mathfrak{m}_{i-1}])})$ for all k.

Suppose P^1 and $P^0 = \oplus_1^m D_n \cdot e_j$ are free D_n-modules, $\phi : P^0 \to P^1$ a D_n-linear map and assume that on P^1 a shift vector $\mathfrak{m}_1$ is given. We define *the obvious shift* on P^0 by setting $\mathfrak{m}_0(j) = V_d deg(\phi(e_j[\mathfrak{m}_1]))$.

For $1 \le d \le n$ we set $\theta_d = x_1\partial_1 + \ldots + x_d\partial_d$ and $\theta_0 = 0$. Recall that a D_n-module $M = A[\mathfrak{m}]/I$ is called *specializable to H* if there is a polynomial $b(s)$ in a single variable such that $b(\theta_d + k)F_H^k(M[\mathfrak{m}]) \subseteq F_H^{k-1}(M[\mathfrak{m}])$ for all k (cf. [37]). Introducing

$$\operatorname{gr}_H^k(M[\mathfrak{m}]) = (F_H^k(M[\mathfrak{m}]))/(F_H^{k-1}(M[\mathfrak{m}])),$$

this can be written as $b(\theta_d + k)\operatorname{gr}_H^k(M[\mathfrak{m}]) = 0$. The polynomial b may depend on $\mathfrak{m}$, while its existence does not ([25]).

Notice that independently of d, $\operatorname{gr}_H^\bullet(D_n[0]) \cong D_n$, as a ring.

The main purpose of this section is to construct for a given D_n-finite complex

$0 \to C^1 \to \ldots \to C^r \to 0$ with cohomology specializable to $H = V(x_1, \ldots, x_d)$ a quasi-isomorphic free V_d-strict complex $A^\bullet[\mathfrak{m}_\bullet]$.

Remark 5.12. If $A^\bullet[\mathfrak{m}_\bullet]$ is a free resolution of M and V_d-strict in our sense it is also V_d-strict in the sense of [37]. In fact our definition is a natural generalization to complexes that are not resolutions.

Let $\cdots \to A^{i-1}[\mathfrak{m}_{i-1}] \xrightarrow{\phi^{i-1}} A^i[\mathfrak{m}_i] \xrightarrow{\phi^i} A^{i+1}[\mathfrak{m}_{i+1}] \to \cdots$ be a free V_d-strict complex. Then the V_d-filtration on A^i induces a filtration on the i-cycles $Z^i = \ker \phi^i$ and since the complex is V_d-strict this gives a natural filtration on the cohomology module H^i,

$$F_H^k(H^i[\mathfrak{m}_i]) = F_H^k(Z^i[\mathfrak{m}_i]) / \operatorname{im} F_H^k(A^{i-1}[\mathfrak{m}_{i-1}]).$$

That this is a filtration follows from

$$F_H^k(Z^i[\mathfrak{m}_i]) \cap \operatorname{im}(F_H^{k+1}(A^{i-1}[\mathfrak{m}_{i-1}])) = \operatorname{im}(F_H^k(Z^{i-1}[\mathfrak{m}_{i-1}])).$$

Let B^i be $\operatorname{im}(\phi^{i-1})$, the i-boundaries. The short exact sequences $0 \to Z^i \to A^i \to B^{i+1} \to 0$ give rise to short exact sequences of groups $0 \to F_H^k(Z^i[\mathfrak{m}_i]) \to F_H^k(A^i[\mathfrak{m}_i]) \to F_H^k(B^{i+1}[\mathfrak{m}_{i+1}]) \to 0$ since the complex is V_d-strict. Similarly, the short exact sequences $0 \to B^i \to Z^i \to H^i \to 0$ induce short exact sequences $0 \to F_H^k(B^i[\mathfrak{m}_i]) \to F_H^k(Z^i[\mathfrak{m}_i]) \to F_H^k(H^i[\mathfrak{m}_i]) \to 0$.

This in turn induces short exact sequences of the graded objects,

$$0 \to \operatorname{gr}_H^k(B^i[\mathfrak{m}_i]) \to \operatorname{gr}_H^k(Z^i[\mathfrak{m}_i]) \to \operatorname{gr}_H^k(H^i[\mathfrak{m}_i]) \to 0 \tag{5.2}$$

and

$$0 \to \operatorname{gr}_H^k(Z^i[\mathfrak{m}_i]) \to \operatorname{gr}_H^k(A^i[\mathfrak{m}_i]) \to \operatorname{gr}_H^k(B^{i+1}[\mathfrak{m}_{i+1}]) \to 0. \tag{5.3}$$

These sequences are the main feature of V_d-strict complexes. They say that forming graded objects and taking cohomology commutes for V_d-strict complexes.

Now let $0 \to C^1 \to \cdots \to C^r \to 0$ be a given complex. We shall break the construction of $A^\bullet[\mathfrak{m}_\bullet]$ into several steps. First we find representations

$C^i = A^i[\mathfrak{m}_i]/I^i$ with A^i free such that the complex $A^\bullet[\mathfrak{m}_\bullet]/I^\bullet$ is V_d-strict. For this we shall utilize the general fact that a bounded complex may be broken up into a finite number of short exact sequences. The second step will be to turn a V_d-strict complex into a V_d-strict free complex. For the rest of the chapter, if X is a module, P_X will denote a free module projecting onto X and I_X the kernel of that projection.

Lemma 5.13. *Let $0 \to P_A/I_A \to P_B/I_B \to P_C/I_C \to 0$ be exact and assume that on P_C there is given a shift vector $\mathfrak{m}_C$.*

Then P_B/I_B can be replaced by a certain other quotient of a free module isomorphic to P_B/I_B, such that there exist shift vectors $\mathfrak{m}_A, \mathfrak{m}_B$ making the sequence V_d-strict.

Proof. We remark that making the sequence V_d-adapted is trivial (but not good enough).

Set $Q_B = P_A \oplus P_C$. Pick a D_n-morphism ψ from P_C to P_B/I_B that lifts $P_B/I_B \to P_C/I_C$. We then define $Q_B \to P_B/I_B$ as $(Q_B \twoheadrightarrow P_A \twoheadrightarrow P_A/I_A \hookrightarrow P_B/I_B) + (Q_B \twoheadrightarrow P_C \xrightarrow{\psi} P_B/I_B)$.

Then Q_B projects onto P_B/I_B. Let $I_{A,C} = \ker(Q_B \twoheadrightarrow P_B/I_B)$. $I_{A,C}$ contains $I_A \oplus 0$, corresponding to the natural inclusion $P_A/I_A \hookrightarrow P_B/I_B = Q_B/I_{A,C}$.

Now define the shift on Q_B by taking the given shift from P_C on the second component, and for the generators of P_A take an arbitrary shift for the moment.

It is clear that the resulting short exact sequence

$$0 \to P_A[\mathfrak{m}_A]/I_A \to Q_B[\mathfrak{m}_A, \mathfrak{m}_C]/I_{A,C} \to P_C[\mathfrak{m}_C]/I_C \to 0$$

is V_d-adapted. It is just as clear that it is strict at P_C/I_C and P_A/I_A. Let $b = \sum \alpha_i a_i + \sum \gamma_j c_j$ be an element of Q_B that is sent to zero in P_C/I_C. Here, $\{a_i\}$ and $\{c_j\}$ represent a basis for P_A and P_C respectively, while α_i and γ_j are in D_n. That means that $\sum \gamma_j c_j \in I_C$. Since $I_{A,C}$ contains for all elements $c \in I_C$ an element (a_c, c) (after all, modulo the first component of Q_B, elements of $0 \oplus I_C$ must be zero in $Q_B/I_{A,C}$!) we can transform our element b modulo

$I_{A,C}$ into an element of $P_A \oplus 0$. We would like this element to have V_d-degree at most $V_d deg(b)$.

Remember that we can still choose the shifts for P_A. Take a V_d-strict Gröbner basis $\{n_i\}$ for I_C relative to the given shift $\mathfrak{m}_C$. For each n_i find a m_i in P_A such that $(m_i, n_i) \in I_{A,C}$. Now define the shift on P_A in such a way that m_i has V_d-degree at most equal to the V_d-degree of n_i for all i. We claim that with this shift the sequence is V_d-strict. To see that, return to $b = \sum \alpha_i a_i + \sum \gamma_i c_i$. The V_d-degree of b is the maximum of the degrees of the two sums. Since $\sum \gamma_i c_i$ is in I_C, we can write it as a sum $\sum \delta_i n_i$, where the V_d-degree of the sum, let's call it e, is the V_d-degree of the largest summand in the sum, because the n_i form a V_d-strict Gröbner basis. Modulo $I_{A,C}$, this is the same as the sum $-\sum \delta_i m_i$, which has lower or equal V_d-degree, by construction of the shift on P_A. Then $\sum \alpha_i a_i - \sum \delta_i m_i \in P_A$ is an operator that maps onto b, modulo $I_{A,C}$, and has V_d-degree at most equal to e. Strictness at $Q_B/I_{A,C}$ follows.

This ends the proof. $\square$

Notice that this creates a commutative diagram with exact and V_d-strict rows and columns

$$\begin{tikzcd}
 & 0 & 0 & 0 & \\
0 \arrow[r] & P_A/I_A \arrow[u] \arrow[r] & P_B/I_B \arrow[u] \arrow[r] & P_C/I_C \arrow[u] \arrow[r] & 0 \\
0 \arrow[r] & P_A \arrow[u] \arrow[r] & Q_B \arrow[u] \arrow[r] & P_C \arrow[u] \arrow[r] \arrow[ul, "\psi"'] & 0 \\
0 \arrow[r] & I_A \arrow[u] \arrow[r] & I_{A,C} \arrow[u] \arrow[r] & I_C \arrow[u] \arrow[r] & 0 \\
 & 0 \arrow[u] & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}$$

and ψ is V_d-adapted. We will have need of a slight improvement of lemma 5.13:

Lemma 5.14. *Suppose we have two short exact sequences*

$$0 \to P_A/I_A \to P_B/I_B \to P_C/I_C \to 0$$

and

$$0 \to P_D/I_D \to P_A/I_A \to P_F/I_F \to 0$$

and assume that on P_C is given a shift vector $\mathfrak{m}_C$. Then one can rewrite P_A/I_A as $Q_A/I_{D,F}$ and P_B/I_B as $Q_B/I_{D,F,C}$ and find shift vectors $\mathfrak{m}_A, \mathfrak{m}_B, \mathfrak{m}_D, \mathfrak{m}_F$ such that the resulting two sequences are exact and V_d-strict.

Proof. First use the first half of the proof of the previous lemma to write P_A/I_A as $Q_A/I_{D,F}$ and then with that presentation of P_A/I_A rewrite P_B/I_B as $Q_B/I_{D,F,C}$. So in particular, $Q_B = Q_A \oplus P_C = P_D \oplus P_F \oplus P_C$.

In order to find the proper shift vectors, proceed as follows:

1. Take a V_d-strict Gröbner basis $\{c_i\}$ for I_C relative to the given shift $\mathfrak{m}_C$. For all i find $a_i = (d_i, f_i)$ such that $(d_i, f_i, c_i) \in I_{D,F,C}$.

2. Pick a shift $\mathfrak{m}_F$ on F such that $V_d deg(f_i[\mathfrak{m}_F]) \leq V_d deg(c_i[\mathfrak{m}_C])$ for all i.

3. Compute a V_d-strict Gröbner basis $\{f_{i'}\}$ for I_F, using the shift $\mathfrak{m}_F$. For all i' find $d'_{i'}$ with $(d'_{i'}, f'_{i'}) \in I_{D,F}$.

4. Pick a shift $\mathfrak{m}_D$ on P_D such that $V_d deg(d_i[\mathfrak{m}_D]) \leq V_d deg(c_i[\mathfrak{m}_C])$ for all i and $V_d deg(d'_{i'}[\mathfrak{m}_D]) \leq V_d deg(f'_{i'}[\mathfrak{m}_F])$ for all i.

By arguments similar to those that prove lemma 5.13, the sequences are V_d-strict. □

Lemma 5.13 and 5.14 providing the basis for the construction, the following result is the inductive step:

Lemma 5.15. *Let A, B, C be three submodules of free modules F_A, F_B, F_C. Assume that $0 \to A \xrightarrow{\phi_A} B \xrightarrow{\phi_B} C \to 0$ is exact and V_d-strict, relative to some shift vectors on F_A, F_B, F_C.*

Then one can construct a diagram

$$
\begin{tikzcd}
 & 0 & 0 & 0 & \\
0 \arrow[r] & A \arrow[u] \arrow[r, "\phi_A"] & B \arrow[u] \arrow[r, "\phi_B"] & C \arrow[u] \arrow[r] & 0 \\
0 \arrow[r] & P_A \arrow[u] \arrow[r] & P_B \arrow[u] \arrow[r] & P_C \arrow[u] \arrow[r] & 0 \\
0 \arrow[r] & I_A \arrow[u] \arrow[r] & I_B \arrow[u] \arrow[r] & I_C \arrow[u] \arrow[r] & 0 \\
 & 0 \arrow[u] & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}
\tag{5.4}
$$

such that

- *all P_X are free,*
- *all rows and columns are exact,*
- *there are shift vectors $\mathfrak{m}_A, \mathfrak{m}_B, \mathfrak{m}_C$ such that if P_A, P_B, P_C are shifted accordingly, all rows and columns become V_d-strict.*

In fact, we may prescribe $0 \to I_A \to P_A[\mathfrak{m}_A] \to A \to 0$, provided it is V_d-strict.

Proof. Let $\{a_i\}$ and $\{c_j\}$ be V_d-strict Gröbner bases for A and C inside F_A and F_C respectively.

Let P_A be the free module on the symbols $\{e_{a_i}\}$, define the projection from P_A to A by $e_{a_i} \to a_i$. Let P_C be a free module on the symbols $\{e_{c_j}\}$, define the projection $P_C \to C$ by $e_{c_j} \to c_j$. Define degree shifts on P_A, P_C in the obvious way.

Set $P_B = P_A \oplus P_C$. For the generators of P_B corresponding to $\{c_i\}$ use a lift $\psi : P_C \to B$ for ϕ_B (which exists as P_C is free) that satisfies: $V_d deg(\psi(e_{c_i})) \leq V_d deg(c_i)$ (which exists because $0 \to A \to B \to C \to 0$ is V_d-strict). Then define $P_B \twoheadrightarrow B$ by $(P_B \twoheadrightarrow P_A \twoheadrightarrow A \hookrightarrow B) + (P_B \twoheadrightarrow P_C \xrightarrow{\psi} B)$. Set I_A, I_B, I_C to be the kernels of the corresponding natural projections. This gives the required diagram with exact rows and columns.

It is clear that $0 \to P_A \to P_B$ and $P_B \to P_C \to 0$ are V_d-strict. If an element of V_d-degree e is in the kernel of $P_B \to P_C$, then its second component (the one in P_C) is zero, so the V_d-degree came from the P_A-component. Hence the second row is V_d-strict. Then automatically the third row is too.

By [37], the remarks after proposition 3.11, the outer columns are V_d-strict.

Let $b \in B$ be of V_d-degree e. Then $\phi_B(b) = \sum \alpha_j c_j$ with $V_d deg(\alpha_j c_j) \leq V_d deg(\phi_B(b)) \leq V_d deg(b)$ for all j as $\{c_j\}$ is a V_d-strict Gröbner basis. Therefore $b - \sum \alpha_j \psi(e_{c_j}) \in \ker(\phi_B) \cap F_H^{V_d deg(b[\mathfrak{m}_A,\mathfrak{m}_C])}(B[\mathfrak{m}_A, \mathfrak{m}_C])$. By exactness and V_d-strictness of the sequence, $b - \sum \alpha_j \psi(e_{c_j}) = \sum \alpha'_i a_i$ with $V_d deg(\alpha'_i a_i) \leq V_d deg(b)$. This proves that $\{\phi_A(a_i)\} \cup \{\psi(e_{c_j})\}$ is a V_d-strict Gröbner basis for $B[\mathfrak{m}_A, \mathfrak{m}_C]$. It follows that $Q_B \twoheadrightarrow B$ is V_d-strict, and hence the entire column.
□

We have now assembled enough machinery to find a D_n-free V_d-strict complex that is quasi-isomorphic to the given complex $C^\bullet$.

As a first step, write down all the short exact sequences $0 \to B^i \to Z^i \to H^i \to 0$ and $0 \to Z^i \to C^i \to B^{i+1} \to 0$. That is to say, find presentations of these modules in terms of finitely generated free modules modulo a finite number of relations. Observe that $B^{i+1}, C^{i+1}, Z^{i+1}, H^{i+1}$ are all zero for $i \geq r$.

Invoke lemma 5.14 with $P_C/I_C = B^{r+1} = 0$ to find a presentation for Z^r, H^r, B^r and C^r together with shift vectors $\mathfrak{m}_{Z,r}, \mathfrak{m}_{C,r}, \mathfrak{m}_{H,r}$ and $\mathfrak{m}_{B,r}$ such that there are commutative diagrams

$$\begin{tikzcd}
 & 0 & 0 & 0 & \\
0 \arrow[r] & Z^r \arrow[r] \arrow[u] & C^r \arrow[r] \arrow[u] & B^{r+1} = 0 \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & P_{Z,r}[\mathfrak{m}_{Z,r}] \arrow[r] \arrow[u] & P_{C,r}[\mathfrak{m}_{C,r}] \arrow[r] \arrow[u] & P_{B,r+1}[\mathfrak{m}_{B,r+1}] = 0 \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & I_{Z,r}[\mathfrak{m}_{Z,r}] \arrow[r] \arrow[u] & I_{C,r}[\mathfrak{m}_{C,r}] \arrow[r] \arrow[u] & I_{B,r+1}[\mathfrak{m}_{B,r+1}] = 0 \arrow[r] \arrow[u] & 0 \\
 & 0 \arrow[u] & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}$$

and

$$\begin{tikzcd}
 & 0 & 0 & 0 & \\
0 \arrow[r] & B^r \arrow[u] \arrow[r] & Z^r \arrow[u] \arrow[r] & H^r \arrow[u] \arrow[r] & 0 \\
0 \arrow[r] & P_{B,r}[\mathfrak{m}_{B,r}] \arrow[u] \arrow[r] & P_{Z,r}[\mathfrak{m}_{Z,r}] \arrow[u] \arrow[r] & P_{H,r}[\mathfrak{m}_{H,r}] \arrow[u] \arrow[r] & 0 \\
0 \arrow[r] & I_{B,r}[\mathfrak{m}_{B,r}] \arrow[u] \arrow[r] & I_{Z,r}[\mathfrak{m}_{Z,r}] \arrow[u] \arrow[r] & I_{H,r}[\mathfrak{m}_{H,r}] \arrow[u] \arrow[r] & 0 \\
 & 0 \arrow[u] & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}$$

where all P_X are free, that has exact and V_d-strict rows and columns. Then invoke the lemma again, this time starting with the shift just obtained on B^r and construct representations for $Z^{r-1}, C^{r-1}, H^{r-1}, B^{r-1}$ and shifts on $P_{Z,r-1}, P_{C,r-1}, P_{H,r-1}, P_{B,r-1}$. r-fold repetition leads to V_d-strict commutative diagrams with exact rows and columns

$$\begin{tikzcd}
 & 0 & 0 & 0 & \\
0 \arrow[r] & Z^i \arrow[u] \arrow[r] & C^i \arrow[u] \arrow[r] & B^{i+1} \arrow[u] \arrow[r] & 0 \\
0 \arrow[r] & P_{Z,i}[\mathfrak{m}_{Z,i}] \arrow[u] \arrow[r] & P_{C,i}[\mathfrak{m}_{C,i}] \arrow[u] \arrow[r] & P_{B,i+1}[\mathfrak{m}_{B,i+1}] \arrow[u] \arrow[r] & 0 \\
0 \arrow[r] & I_{Z,i}[\mathfrak{m}_{Z,i}] \arrow[u] \arrow[r] & I_{C,i}[\mathfrak{m}_{C,i}] \arrow[u] \arrow[r] & I_{B,i+1}[\mathfrak{m}_{B,i+1}] \arrow[u] \arrow[r] & 0 \\
 & 0 \arrow[u] & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}$$

and

$$\begin{tikzcd}
 & 0 & 0 & 0 & \\
0 \arrow[r] & B^i \arrow[u] \arrow[r] & Z^i \arrow[u] \arrow[r] & H^i \arrow[u] \arrow[r] & 0 \\
0 \arrow[r] & P_{B,i}[\mathfrak{m}_{B,i}] \arrow[u] \arrow[r] & P_{Z,i}[\mathfrak{m}_{Z,i}] \arrow[u] \arrow[r] & P_{H,i}[\mathfrak{m}_{H,i}] \arrow[u] \arrow[r] & 0 \\
0 \arrow[r] & I_{B,i}[\mathfrak{m}_{B,i}] \arrow[u] \arrow[r] & I_{Z,i}[\mathfrak{m}_{Z,i}] \arrow[u] \arrow[r] & I_{H,i}[\mathfrak{m}_{H,i}] \arrow[u] \arrow[r] & 0 \\
 & 0 \arrow[u] & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}$$

for $1 \leq i \leq r$. The point of this procedure is the creation of a presentation of C^i as $P_{C,i}/I_{C,i}$ with V_d-strict maps between these modules.

Now we assemble a resolution for $C^\bullet$ as follows. First find a V_d-strict resolution for B^0 using the method of [37]. With lemma 5.15 find step by step a resolution for Z^0 and H^0 over $0 \to B^0 \to Z^0 \to H^0 \to 0$. Then, using the resolution for Z^0, construct resolutions for C^0, B^1 over $0 \to Z^0 \to C^0 \to B^1 \to 0$. Now use the one obtained for B^1 to find one for Z^1, H^1 over $0 \to B^1 \to Z^1 \to H^1 \to 0$. In this way construct resolutions for Z^i, H^i, C^i and B^{i+1} for $1 \le i \le r$ which fit into appropriate V_d-strict and exact commutative diagrams.

Let us denote the l-th module of the resolution for X^i (with X being Z, C, B or H) by $P^l_{X,i}$, and the l-th map $P^l_{X,i} \to P^{l-1}_{X,i}$ by $\delta^l_{X,i}$. Then by construction $P^l_{Z,i} = P^l_{B,i} \oplus P^l_{H,i}$ and $P^l_{C,i} = P^l_{Z,i} \oplus P^l_{B,i+1}$. We define a map $\varepsilon^l_{C,i}$ from $P^l_{C,i}$ to $P^l_{C,i+1}$ as the combined map $P^l_{C,i} \twoheadrightarrow P^l_{B,i+1} \hookrightarrow P^l_{Z,i+1} \hookrightarrow P^l_{C,i+1}$, multiplied by $(-1)^l$. So up to sign $\varepsilon^l_{C,i}$ is $P^l_{C,i} = P^l_{B,i} \oplus P^l_{H,i} \oplus P^l_{B,i+1} \twoheadrightarrow P^l_{B,i+1} \hookrightarrow P^l_{B,i+1} \oplus P^l_{H,i+1} \oplus P^l_{B,i+2}$. Clearly $P^l_{C,i} \to P^l_{C,i+1} \to P^l_{C,i+2}$ is the zero map.

We have created a free double complex $\{P^\bullet_{C,\bullet}, \delta^\bullet_{C,\bullet}, \varepsilon^\bullet_{C,\bullet}\}$ over D_n, $\{P^\bullet_{C,i}, \delta^\bullet_{C,i}\}$ is V_d-strict for fixed i and $\{P^l_{C,\bullet}, \varepsilon^l_{C,\bullet}\}$ is V_d-strict for fixed l. Moreover, the associated total complex $Tot^\bullet(P^\bullet_{C,\bullet})$ is quasi-isomorphic to $C^\bullet$ and by construction V_d-adapted. We shall write $\delta_T(-)$ for the differential of $Tot^\bullet(P^\bullet_{C,\bullet})$.

Proposition 5.16. *$Tot^\bullet(P^\bullet_{C,\bullet})$ is in fact V_d-strict.*

Proof. To that end assume that the element $p = p^0_i \oplus p^1_{i+1} \oplus \cdots \oplus p^{r-i}_r \in Tot^i(P^\bullet_{C,\bullet}) = P^0_{C,i} \oplus P^1_{C,i+1} \oplus \cdots \oplus P^{r-i}_{C,r}$ is in the image of the total differential δ_T, and that the V_d-degree of p under the shift vectors is e. We have to show that p is the image of an element of V_d-degree at most e. We need to take a closer look at the maps and modules in front of us.

$P^l_{C,i}$ is by construction $P^l_{B,i} \oplus P^l_{H,i} \oplus P^l_{B,i+1}$. The map $\delta^{l+1}_{C,i} : P^{l+1}_{C,i} = P^{l+1}_{B,i} \oplus P^{l+1}_{H,i} \oplus P^{l+1}_{B,i+1} \to P^l_{C,i} = P^l_{B,i} \oplus P^l_{H,i} \oplus P^l_{B,i+1}$ is on the first component the differential from the resolution $P^\bullet_{B,i}$ while the map from $P^{l+1}_{H,i} \oplus P^{l+1}_{B,i+1} \to P^l_{C,i}$ is defined using certain lifts, obtained while using lemma 5.15. Inspection shows that the matrix which represents $P^{l+1}_{C,i} \to P^{l}_{C,i}$ looks like this:

$$\begin{pmatrix} \delta^{l+1}_{B,i} & \phi^{l+1}_{1,i} & \psi^{l+1}_{1,i} \\ 0 & \delta^{l+1}_{H,i} & \psi^{l+1}_{2,i} \\ 0 & 0 & \delta^{l+1}_{B,i+1} \end{pmatrix}$$

where $\phi^{l+1}_{1,i} : P^{l+1}_{H,i} \to P^{l}_{B,i}, \psi^{l+1}_{1,i} : P^{l+1}_{B,i+1} \to P^{l}_{B,i}, \psi^{l+1}_{2,i} : P^{l+1}_{B,i+1} \to P^{l}_{H,i}$ are the maps that are used to produce the mentioned lifts. $\phi^{l+1}_{1,i}$, $\psi^{l+1}_{1,i}$,$\psi^{l+1}_{2,i}$ are all V_d-adapted by construction.

We shall argue by decreasing induction on the variable s, starting with $s = r+1$, that the components p^{s-i}_{s} of p may be assumed to be zero modulo images of degree no greater than e under the total differential. We will at the same time show that we may assume that the third component of p^{s-i-1}_{s-1} is zero. For $s \le r$ this will follow from the induction. For $s = r+1$ it follows from the fact that $B^{r+1} = 0$.

So assume that $s \le r$, that p has only zero components beyond the s-th component and that the third piece (to $P^{s-i}_{B,s+1}$) of the s-th component of p is zero.

The following lemma will essentially show that our p is then in fact the image of an element in $Tot^{i-1}(P^{\bullet}_{C,\bullet})$ with zero component in $P^{s-i+1}_{B,s+1}$ and only zeros in all columns beyond the s-th.

Lemma 5.17. *Let $(a,b,0) \in P^{s-i}_{C,s}$ and assume $(a,b,0) = \delta^{s-i+1}_{C,s}(\alpha,\beta,\gamma)$ with $(\alpha,\beta,\gamma) \in P^{s-i+1}_{C,s}$. Then $(a,b,0) = \delta^{s-i+1}_{C,s}(\alpha',\beta',0)$ for some $(\alpha',\beta',0)$ in $P^{s-i+1}_{C,s}$ where $V_d deg(\alpha',\beta',0) \le V_d deg(a,b,0)$.*

Proof. By construction, $(\psi^{s-i+1}_{1,s}(\gamma), \psi^{s-i+1}_{2,s}(\gamma)) \in \ker(P^{s-i}_{B,s} \oplus P^{s-i}_{H,s} \to P^{s-i-1}_{B,s} \oplus P^{s-i-1}_{H,s})$. Since this kernel is exactly $\delta^{s-i+1}_{Z,s}(P^{s-i+1}_{B,s} \oplus P^{s-i+1}_{H,s})$ we must have $(\psi^{s-i+1}_{1,s}(\gamma), \psi^{s-i+1}_{2,s}(\gamma)) = \delta^{s-i+1}_{Z,s}(\alpha'',\beta'')$ where we can pick α'' and β'' to be of V_d-degree at most $V_d deg(\gamma)$. Thus $\delta^{s-i+1}_{C,s}(\alpha,\beta,\gamma) = (\delta^{s-i+1}_{B,s}(\alpha) + \psi^{s-i+1}_{1,s}(\gamma) + \phi^{s-i+1}_{1,s}(\beta), \delta^{s-i+1}_{H,s}(\beta) + \psi^{s-i+1}_{2,s}(\gamma), 0) = \delta^{s-i+1}_{C,s}(\alpha+\alpha'', \beta+\beta'', 0)$ Since $P^{\bullet}_{Z,\bullet}$ is a V_d-strict resolution, we can find $(\alpha',\beta',0)$ such that $\delta^{s-i+1}_{C,s}(\alpha',\beta',0) = \delta^{s-i+1}_{C,s}(\alpha+\alpha'',\beta+\beta'',\gamma)$ and $V_d deg(\alpha',\beta',0) \le V_d deg(a,b,0)$. ◻

Since by assumption $p_s^{s-i} = (a, b, 0)$, the lemma tells us that b equals $\delta_H(b_1)$ for some $b_1 \in P_{H,s}^{s-i+1}$, of V_d-degree at most e because $P_{H,s}^{\bullet}$ is V_d-strict. We replace p by $p := p - \delta_T(b_1)$. This erases a nonzero entry in p, does not increase its V_d-degree, keeps it in the image of δ_T and leaves components beyond the s-th invariant.

Lemma 5.18. *Let $(a, 0, 0) \in P_{C,s}^{s-i}$ be the image of $(\alpha, \beta, \gamma) \in P_{C,s}^{s-i+1}$ under $\delta_{C,s}^{s-i+1}$. Then there is $\alpha' \in P_{B,s}^{s-i+1}$ with $\delta_{C,s}^{s-i+1}(\alpha', 0, 0) = (a, 0, 0)$ and α' can be chosen to be of V_d-degree no bigger than $V_d deg(a)$.*

Proof. By the previous lemma, we can assume that $(a, 0, 0)$ is the image of $(\alpha, \beta, 0) \in P_{C,s}^{s-i+1}$. By construction, $\phi_{1,s}^{s-i+1}(\beta) \in \ker(P_{B,s}^{s-i} \to P_{B,s}^{s-i-1}) = \operatorname{im}(P_{B,s}^{s-i+1} \to P_{B,s}^{s-i})$. As $P_{B,s}^{\bullet}$ is a resolution , $\phi_{1,s}^{s-i+1}(\beta) = \delta_{B,s}^{s-i+1}(\alpha'')$ for some $\alpha'' \in P_{B,s}^{s-i+1}$. Hence $\delta_{C,s}^{s-i+1}(\alpha, \beta, 0) = (\delta_{B,s}^{s-i+1}(\alpha) + \phi_{1,s}^{s-i+1}(\beta), 0, 0) = \delta_{C,s}^{s-i+1}(\alpha, 0, 0) + \delta_{C,s}^{s-i+1}(\alpha'', 0, 0)$. This proves that $(a, 0, 0)$ is in fact the image of an element $(\alpha', 0, 0)$. Since $P_{B,s}^{\bullet}$ is a V_d-strict resolution, we can choose α' to be of V_d-degree at most $V_d deg(a)$. $\square$

Before lemma 5.18 we reduced the component of p that corresponds to $P_{C,s}^{s-i}$ to the form $(a, 0, 0)$. Lemma 5.18 tells us that since p is an image under δ_T, $a = \delta_{B,s}^{s-i+1}(\alpha') + (-1)^{s-1}\alpha$ where α' lives in $P_{B,s}^{s-i+1}$ (the first component of $P_{C,s}^{s-i+1}$) and $\alpha \in P_{B,s}^{s-i}$ (the third component of $P_{C,s-1}^{s-i}$).

Then the third component of p in $P_{C,s-1}^{s-i-1}$ must be $\delta_{B,s}^{s-i}(\alpha) = (-1)^{s-1}\delta_{B,s}^{s-i}(a)$. Replace p by $p - \delta_T((-1)^{s-1}a)$, this a positioned in the $P_{B,s}^{s-i-1}$-component of $P_{C,s-1}^{s-i-1}$.

The result is a p with zero component for $P_{C,s}^{s-i}$ and zero component in the third component of $P_{C,s-1}^{s-i-1}$. Furthermore, it differs from the original p by the δ_T-boundary of the element $a \in P_{C,s-1}^{s-i-1}$ of V_d-degree at most e. This completes the induction step and the proof of proposition 5.16. $\square$

We have proved

Theorem 5.19. *If $C^{\bullet}$ is a complex of left D_n-modules that is bounded below and above and presentations of all C^i in terms of generators and relations over*

D_n are given, then one can produce a V_d-strict complex of free D_n-modules that is quasi-isomorphic to $C^\bullet$. In fact, the complex in proposition 5.16 will do. $\square$

Remark 5.20. It is not true that the total complex associated to any double complex with V_d-strict rows and columns is V_d-strict. This would be equivalent to saying that all finite subsets of a free D_n-module form a Gröbner basis for any order refining V_d-degree. Consider for example the diagram

$$\begin{tikzcd} D_1[1] \arrow[r, "\cdot\,\partial_1"] & D_1[0] \\ 0 \arrow[u] \arrow[r] & D_1[1] \arrow[u, "\cdot(\partial_1 - 1)"'] \end{tikzcd}.$$

Here, $1 = 1 \cdot \partial_1 - 1 \cdot (\partial_1 - 1) \in \operatorname{im}(F_H^1(Tot^1[1,1]))$ but this element is not in $\operatorname{im}(F_H^0(Tot^1[1,1])) \subseteq F_H^0([0](Tot^0)))$, although it is of V_1-degree 0.

5.3 The Restriction of a Complex to a Subspace

For this section let $A^\bullet[\mathfrak{m}_\bullet]$ be a given bounded V_d-strict complex of finitely generated D_n-free modules. We shall assume further that the cohomology modules of $A^\bullet[\mathfrak{m}_\bullet]$ are specializable to $H = \operatorname{Var}(x_1, \ldots, x_d)$ (cf. definition 5.11). The main purpose of this section is to determine a suitable truncation of $A^\bullet[\mathfrak{m}_\bullet]$ such that the cohomology of $\Omega \otimes_{D_n} A^\bullet$ is captured by the "tensor product" of Ω with that truncation.

Recall that in [37] $A^\bullet$ is a V_n-strict resolution of a specializable module M and the truncation is determined by considering roots of the b-function $b(M)$ corresponding to restriction to the origin ([37], Algorithm 5.4).

5.3.1

Let H be the subspace defined by $x_1 = \ldots = x_d = 0$, and $A^\bullet[\mathfrak{m}_\bullet]$ a V_d-strict complex of free D_n-modules. As a first step, pick generators $\kappa_{i,l}$ for the modules $Z^i = Z^i(A^\bullet) = \ker(A^i \to A^{i+1}$ of $A^\bullet)$. To each of them is associated a degree in the V_d-filtration on $A^i[\mathfrak{m}_i]$ which we shall call $\lambda_{i,l}$.

Let a bar denote cosets of elements of Z^i in $H^i = H^i(A^\bullet) = Z^i/B^i$ where $B^i = \operatorname{im}(A^{i-1} \to A^i)$. Recall that we agreed to write $\theta_j = x_1\partial_1 + \ldots + x_j\partial_j$ for

$1 \leq j \leq n$. Since $D_n \cdot \overline{\kappa}_{i,l}$ is an H-specializable D_n-module, there is a b-function $b_{i,l}(\theta_d)$ associated to it which corresponds to the restriction of $D_n \cdot \overline{\kappa}_{i,l}[\mathfrak{m}_i]$ to $x_1 = \ldots = x_d = 0$, see [37]. Therefore, $b_{i,l}(\theta_d)\kappa_{i,l} \in F_H^{-1}(D_n) \cdot \kappa_{i,l} + \mathrm{im}(A^{i-1} \to A^i)$.

Definition 5.21. Let $b_H^{A^\bullet[\mathfrak{m}_\bullet]}(\theta_d) = b(\theta_d)$ be the least common multiple of all $b_{i,l}(\theta_d - \lambda_{i,l})$, the *$b$-function of H on $A^\bullet[\mathfrak{m}_\bullet]$*.

5.3.2

Consider the associated complex of graded $\mathrm{gr}_H^\bullet(D_n)$-modules $\oplus \frac{F_H^k(A^\bullet[\mathfrak{m}_\bullet])}{F_H^{k-1}(A^\bullet[\mathfrak{m}_\bullet])}$, the sum being taken over k.

Assume that the $\kappa_{i,l}$ form a V_d-strict Gröbner basis for $Z^i[\mathfrak{m}_i]$. Then for all $\zeta \in Z^i$ we can write $\zeta = \sum \alpha_{i,l}(\zeta)\kappa_{i,l}$ with $V_d deg(\alpha_{i,l}(\zeta)\kappa_{i,l}[\mathfrak{m}_i]) \leq V_d deg(\zeta[\mathfrak{m}_i])$. Hence the $\kappa_{i,l}$ are generators for $\mathrm{gr}_H^\bullet(H^i[\mathfrak{m}_i])$ and moreover $\mathrm{gr}_H^k(Z^i[\mathfrak{m}_i]) = \sum \mathrm{gr}_H^{k-\lambda_{i,l}}(D_n)\kappa_{i,l}$. Since

$$\mathrm{im}(F_H^k(A^{i-1}[\mathfrak{m}_{i-1}]) \to F_H^k(A^i[\mathfrak{m}_i])) = F_H^k(A^i[\mathfrak{m}_i]) \cap \mathrm{im}(A^{i-1} \to A^i)$$

we have $\mathrm{gr}_H^k(H^i(A^\bullet[\mathfrak{m}_i])) = \sum_l \mathrm{gr}^{k-\lambda_{i,l}}(D_n)\overline{\kappa}_{i,l}$.

Then observe the following:

$$\begin{aligned} b_H^{A^\bullet[\mathfrak{m}_\bullet]}(\theta_d + k)\,\mathrm{gr}_H^k(H^i(A^\bullet[\mathfrak{m}_\bullet])) &= b_H^{A^\bullet[\mathfrak{m}_\bullet]}(\theta_d + k)\sum \mathrm{gr}_H^{k-\lambda_{i,l}}(D_n)\overline{\kappa}_{i,l} \\ &= \sum \mathrm{gr}_H^{k-\lambda_{i,l}}(D_n) b_H^{A^\bullet[\mathfrak{m}_\bullet]}(\theta_d + \lambda_{i,l})\overline{\kappa}_{i,l} \\ &= 0 \qquad (5.5) \end{aligned}$$

because $b_H^{A^\bullet[\mathfrak{m}_\bullet]}(\theta_d + \lambda_{i,l})$ sends $\kappa_{i,l}$ into $F_H^{-1}(D_n) \cdot \kappa_{i,l} + \mathrm{im}(A^{i-1} \to A^i)$, which is zero in $\mathrm{gr}_H^{\lambda_{i,l}}[\mathfrak{m}_i](H^i(A^\bullet))$.

Remark 5.22. $b_H^{A^\bullet[\mathfrak{m}_\bullet]}(\theta_d)$ is a multiple of the b-function of $H^i(A^\bullet[\mathfrak{m}_i])$ with respect to the given shift vectors.

5.3.3

This paves the way for a result generalizing [37], Proposition 5.2. The proof is very similar to the one given there.

We need to introduce a number of Koszul complexes. Let $\mathcal{L}$ be a $\mathrm{gr}^\bullet_H(D_n)$-module and let $\mathcal{L}_k$, $k \in \mathbb{Z}$, be subgroups of $\mathcal{L}$ such that $x_i\mathcal{L}_k \subseteq \mathcal{L}_{k-1}$ for $1 \leq i \leq d$, $\mathcal{L}_k \cap \mathcal{L}_{k'} = 0$ for all k, k', and $\mathcal{L} = \bigcup_k \mathcal{L}_k$. In that case we will say that the $\mathcal{L}_k$ give an *H-filtration* for $\mathcal{L}$. For any integer k let $\mathcal{K}^\bullet(\mathcal{L}_\bullet, x_1, \ldots, x_d)[k]$ be the Koszul complex

$$0 \to \mathcal{L}_{k+d} \otimes_{\mathbb{Z}} \bigwedge^0 \mathbb{Z}^d \to \mathcal{L}_{k+d-1} \otimes_{\mathbb{Z}} \bigwedge^1 \mathbb{Z}^d \to \cdots \to \mathcal{L}_k \otimes_{\mathbb{Z}} \bigwedge^d \mathbb{Z}^d \to 0$$

equipped with the usual Koszul maps $\delta(u \otimes e_{i_1} \wedge \cdots \wedge e_{i_j}) = \sum_l x_l u \otimes e_l \wedge e_{i_1} \wedge \cdots \wedge e_{i_j}$.

Unifying all the graded pieces, let $\mathcal{K}^\bullet(\mathcal{L}_\bullet, x_1, \ldots, x_d)$ be the usual Koszul complex of $\mathcal{L}$ relative to $x_1, \ldots, x_d$.

More generally, for a complex of H-filtered $\mathrm{gr}^\bullet_H(D_n)$-modules $(\mathcal{L}^\bullet, \delta^\bullet)$ with a differential $\delta^\bullet$ that respects the H-filtration we define inductively the complex $\mathcal{K}^\bullet(\mathcal{L}^\bullet_\bullet, x_1, \ldots, x_d)[k]$ as the total complex of the double complex

$$\begin{array}{c} \mathcal{K}^\bullet(\mathcal{L}^\bullet_\bullet, x_1, \ldots, x_{d-1})[k] \\ (-1)^i x_d \cdot \Big\uparrow \\ \mathcal{K}^\bullet(\mathcal{L}^\bullet_\bullet, x_1, \ldots, x_{d-1})[k+1] \end{array}$$

where $\mathcal{K}^\bullet(\mathcal{L}^\bullet_\bullet, \emptyset)[k] = \mathcal{L}^\bullet_\bullet[k] = (\cdots \to \mathcal{L}^i_k \to \mathcal{L}^{i+1}_k \to \cdots)$ is the k-th piece of the H-filtration of the original complex. Thus $\mathcal{K}^\bullet(\mathcal{L}^\bullet_\bullet, x_1, \ldots, x_d)[k]$ is the component of the usual Koszul complex $\mathcal{K}^\bullet(\mathcal{L}^\bullet, x_1, \ldots, x_d)$ associated to $\mathcal{L}^\bullet$ and $x_1, \ldots, x_d$ that "ends" in the k-th filtration piece.

In the following theorem, which is also interesting in its own light, we will explain which H-filtered pieces of the complex $\mathcal{K}^\bullet(\mathcal{L}^\bullet, x_1, \ldots, x_d)$ are responsible for nontrivial cohomology.

Theorem 5.23. *Suppose given is a complex of H-filtered $\mathrm{gr}^\bullet_H(D_n)$-modules $(\mathcal{L}^\bullet, \delta^\bullet)$ where the maps $\delta^i : \mathcal{L}^i \to \mathcal{L}^{i+1}$ between the $\mathcal{L}^i$ preserve the filtration. Assume that there is a polynomial $b(\theta)$ in $\mathbb{C}[\theta]$ that satisfies*

$$b(\theta_d + k) \cdot \ker(\mathcal{L}^i_k \to \mathcal{L}^{i+1}_k) \subseteq \mathrm{im}(\mathcal{L}^{i-1}_k \to \mathcal{L}^i_k)$$

for all k and all i. Let $k_0 \in \mathbb{Z}$ with $b(k_0) \neq 0$. Then $\mathcal{K}^\bullet(\mathcal{L}^\bullet_\bullet, x_1, \ldots, x_d)[k_0]$ is exact.

Proof. Let $H' = V(x_1, \dots, x_{d-1})$. The complex $\mathcal{K}^\bullet(\mathcal{L}^\bullet_\bullet, x_d)$ is H'-filtered in the obvious way. Here is the essential idea of the argument:

Lemma 5.24. *If $b(\theta_d + k)$ kills cohomology of $\mathcal{K}^\bullet(\mathcal{L}^\bullet_\bullet, \emptyset)[k]$ then $b^2(\theta_{d-1} + k)$ kills cohomology of $\mathcal{K}^\bullet(\mathcal{L}^\bullet_\bullet, x_d)[k]$.*

Proof. We may assume that $b(\theta_d)$ is not a constant since otherwise $\mathcal{L}^\bullet$ is exact and a spectral sequence argument shows that then $\mathcal{K}^\bullet(\mathcal{L}^\bullet, x_d)$ is exact as well.

So assume that $(u^{i+1}_{k+1}, u^i_k) \in \mathcal{L}^{i+1}_{k+1} \oplus \mathcal{L}^i_k$ is in $\mathcal{K}^i(\mathcal{L}^\bullet_\bullet, x_d)[k]$ and suppose this element is in the kernel of the differential in $\mathcal{K}^\bullet(\mathcal{L}^\bullet, x_d)$. For the convenience of the reader we illustrate the situation with the following diagram.

$$\begin{tikzcd}
\mathcal{L}^{i-1}_k \arrow[r, "\delta^{i-1}"] & \mathcal{L}^i_k \arrow[r, "\delta^i"] & \mathcal{L}^{i+1}_k \arrow[r, "\delta^{i+1}"] & \mathcal{L}^{i+2}_k \\
\mathcal{L}^{i-1}_{k+1} \arrow[r, "\delta^{i-1}"'] & \mathcal{L}^i_{k+1} \arrow[u, "(-1)^i x_d \cdot"] \arrow[r, "\delta^i"'] & \mathcal{L}^{i+1}_{k+1} \arrow[u, "(-1)^{i+1} x_d \cdot"'] \arrow[r, "\delta^{i+1}"'] & \mathcal{L}^{i+2}_{k+1}
\end{tikzcd}$$

Then we must have

$$\delta^{i+1}(u^{i+1}_{k+1}) = 0, \tag{5.6}$$

$$(-1)^{i+1} x_d u^{i+1}_{k+1} + \delta^i(u^i_k) = 0. \tag{5.7}$$

By hypothesis on b, $b(\theta_d + k + 1)u^{i+1}_{k+1} = \delta^i(u^i_{k+1})$ for some $u^i_{k+1} \in \mathcal{L}^i_{k+1}$. So $b(\theta_{d-1} + k + \partial_d x_d)u^{i+1}_{k+1} = \delta^i(u^i_{k+1})$ and therefore $b(\theta_{d-1}+k)u^{i+1}_{k+1} + \partial_d P x_d u^{i+1}_{k+1} = \delta^i(u^i_{k+1})$ for some V_d-homogeneous $P \in F^0_H(D_n) \backslash F^{-1}_H(D_n)$. Hence $b(\theta_{d-1} + k)u^{i+1}_{k+1} = \delta^i(u^i_{k+1} + (-1)^i \partial_d P u^i_k)$ using relation (5.7). Let us write this as

$$b(\theta_{d-1} + k)u^{i+1}_{k+1} = \delta^i(a^i_{k+1}), \tag{5.8}$$

$a^i_{k+1} \in (\mathcal{L}^i_{k+1} \oplus 0) \subset \mathcal{K}^{i-1}(\mathcal{L}^\bullet_\bullet, x_d)[k+1]$.

This implies that if (u^{i+1}_{k+1}, u^i_k) is in the kernel of δ^i_T, the differential on the total complex $\mathcal{K}^\bullet(\mathcal{L}^\bullet, x_d)$, then $b(\theta_{d-1} + k)(u^{i+1}_{k+1}, u^i_k)$ is, modulo the image of δ^{i-1}_T, congruent to an element $(0, v^i_k)$. Namely, $(0, v^i_k) = b(\theta_{d-1}+k) \cdot (u^{i+1}_{k+1}, u^i_k) - \delta^i_T(a^i_{k+1}, 0)$ which of course is also in the kernel of δ^i_T. So it suffices to show that any such kernel element $(0, v^i_k)$ satisfies $b(\theta_{d-1} + k)(0, v^i_k) \in \operatorname{im} \delta^{i-1}_T$.

Since $\delta_T^i(0, v_k^i) = 0$, we must have $\delta^i(v_k^i) = 0$. Hence $b(\theta_d + k)v_k^i = \delta^{i-1}(a_k^{i-1})$ for some $a_k^{i-1} \in \mathcal{L}_k^{i-1}$ by assumption on $b(\theta)$. Now $b(\theta_d + k)v_k^i = b(\theta_{d-1} + k)v_k^i + x_d Q \partial_d v_k^i$ for some V_d-homogeneous operator $Q \in F_H^0(D_n) \backslash F_H^{-1}(D_n)$. Therefore $b(\theta_{d-1} + k)v_k^i = \delta^{i-1}(a_k^{i-1}) - x_d Q \partial_d v_k^i$.

The element $Q\partial_d v_k^i \in \mathcal{L}_{k+1}^i$ satisfies $\delta^i(Q\partial_d v_k^i) = Q\partial_d \delta^i(v_k^i) = 0$. It follows that $b(\theta_{d-1} + k)(0, v_k^i) = \delta_T^{i-1}((-1)^i Q\partial_d v_k^i, a_k^{i-1})$. This concludes the proof of the lemma. $\square$

Now recall the inductive definition of $\mathcal{K}^\bullet(\mathcal{L}^\bullet, x_1, \ldots, x_d)$, which together with the lemma shows that if the cohomology of $\mathcal{K}^\bullet(\mathcal{L}_\bullet^\bullet, \emptyset)[k_0]$ is killed by $b(\theta_d + k_0)$ then the cohomology of $\mathcal{K}^\bullet(\mathcal{L}_\bullet^\bullet, x_1, \ldots, x_d)[k_0]$ is killed by $b^{(2^d)}(k_0)$.

The theorem now follows easily from the fact that $\mathbb{C}$ is a domain. $\square$

We come now to the final result of this section. The purpose is to exhibit an algorithm that computes the cohomology groups of the restriction of a complex.

We need to make a convention about the V_n-filtration on tensor products over D_n with $\tilde{\Omega}_d = D_n/(x_1, \ldots, x_d) \cdot D_n$.

Definition 5.25. If $A[\mathfrak{m}]$ is a free H-filtered D_n-module with shift vector $\mathfrak{m}$ then $\tilde{\Omega}_d \otimes_{D_n} A[\mathfrak{m}]$ is filtered by $F_H^k(\tilde{\Omega}_d \otimes_{D_n} A[\mathfrak{m}]) :=$ the span of $\{\overline{P} \otimes_{D_n} Q | V_d deg(P) + V_d deg(Q[\mathfrak{m}]) \leq k\}$. $\tilde{\Omega}_d = \mathbb{C}[\partial_1, \ldots, \partial_d]\langle x_{d+1}, \partial_{d+1}, \ldots, x_n, \partial_n\rangle$ as right D_n-module, thus $F_H^k(\tilde{\Omega}_d \otimes_{D_n} A[\mathfrak{m}])$ is the free $\mathbb{C}\langle x_{d+1}, \partial_{d+1}, \ldots, x_n, \partial_n\rangle$-module on the symbols $\{(P_1, \ldots, P_{\dim_{D_n}(A)}) | P_j \in \mathbb{C}[\partial_1, \ldots, \partial_d], \deg_\partial(P_j) \leq k - \mathfrak{m}(j)\ \forall j\}$.

Before we state our theorem we would like to point out that a complex of graded $\mathrm{gr}_H^\bullet(D_n)$-modules is a special case of an H-filtered complex. In particular, 5.23 applies in that situation.

Theorem 5.26. *Let $(A^\bullet[\mathfrak{m}_\bullet], \delta^\bullet)$ be a V_d-strict complex of free D_n-modules. The restriction of $A^\bullet[\mathfrak{m}_\bullet]$ to $H = V(x_1, \ldots, x_d)$, interpreted as a complex of modules over $\mathbb{C}\langle x_{d+1}, \partial_{d+1}, \ldots, x_n, \partial_n\rangle$, can be computed as follows:*

1. *Relative to the induced filtration on $H^i(A^\bullet[\mathfrak{m}_\bullet])$ compute the b-functions $b_i(s)$ for the restriction of $H^i(A^\bullet[\mathfrak{m}_\bullet])$ to $x_1 = \ldots = x_d = 0$.*

2. *Let $b(s)$ be the least common multiple of all the b_i. Find integers k_0, k_1 with $(b(k) = 0, k \in \mathbb{Z}) \Rightarrow (k_0 \le k \le k_1)$.*

3. *$\tilde{\Omega}_d \otimes^L_{D_n} A^\bullet$ is quasi-isomorphic to the complex*

$$\cdots \to \frac{F_H^{k_1}(\tilde{\Omega}_d \otimes_{D_n} A^i[\mathfrak{m}_i])}{F_H^{k_0-1}(\tilde{\Omega}_d \otimes_{D_n} A^i[\mathfrak{m}_i])} \to \frac{F_H^{k_1}(\tilde{\Omega}_d \otimes_{D_n} A^{i+1}[\mathfrak{m}_{i+1}])}{F_H^{k_0-1}(\tilde{\Omega}_d \otimes_{D_n} A^{i+1}[\mathfrak{m}_{i+1}])} \to \cdots \quad (5.9)$$

This is a complex of free finitely generated $\mathbb{C}\langle x_{d+1}, \partial_{d+1}, \ldots, x_n, \partial_n\rangle$-modules.

Proof. Let us first consider the complex $\mathrm{gr}^\bullet_H(A^\bullet[\mathfrak{m}_\bullet])$ with differential induced from $\delta^\bullet$. The hypotheses on $b(s)$ imply by theorem 5.23 that the complex $\mathcal{K}^\bullet(\mathrm{gr}^\bullet_H(A^\bullet[\mathfrak{m}_\bullet]), x_1, \ldots, x_d)[k]$ is exact for $k \notin [k_0, k_1]$.

Let us define $\mathcal{K}^\bullet(A^\bullet[\mathfrak{m}_\bullet], x_1, \ldots, x_d)\{k\}$ to be the complex

$$\cdots \to F^k(A^i[\mathfrak{m}_i]) \to F^k(A^{i+1}[\mathfrak{m}_{i+1}]) \to \cdots$$

if $d = 0$ and to be the total complex of the double complex

$$\begin{array}{c} \mathcal{K}^\bullet(A^\bullet[\mathfrak{m}_\bullet], x_1, \ldots, x_{d-1})\{k\} \\ \left\uparrow{\scriptstyle (-1)^i x_d \cdot}\right. \\ \mathcal{K}^\bullet(A^\bullet[\mathfrak{m}_\bullet], x_1, \ldots, x_{d-1})\{k+1\} \end{array}$$

for $d > 0$. We also write $\mathcal{K}^\bullet(A^\bullet[\mathfrak{m}_\bullet], x_1, \ldots, x_d)\{k/k'\}$ for the complex whose j-th entry is the j-th entry of $\mathcal{K}^\bullet(A^\bullet[\mathfrak{m}_\bullet], x_1, \ldots, x_d)\{k\}$ modulo the j-th entry of the complex $\mathcal{K}^\bullet(A^\bullet[\mathfrak{m}_\bullet], x_1, \ldots, x_d)\{k'\}$ and whose maps are induced from $\mathcal{K}^\bullet(A^\bullet[\mathfrak{m}_\bullet], x_1, \ldots, x_d)\{k\}$.

Lemma 5.27. *$\mathcal{K}^\bullet(A^\bullet[\mathfrak{m}_\bullet], x_1, \ldots, x_d)\{k\}$ is exact for $k < k_0$.*

Proof. We observe first that for sufficiently small k this is a consequence of the fact that operators of negative V_d-degree in $D_n[0]$ are necessarily right multiples of x_j for $1 \le j \le d$ which are zero in $\tilde{\Omega}_d \otimes D_n[0]$. Let $\overline{\alpha}$ be a cohomology class in $\mathcal{K}^\bullet(A^\bullet[\mathfrak{m}_\bullet], x_1, \ldots, x_d)\{k\}$, α being of V_d-degree at most k. Thus, $\overline{\alpha}$ represents a cohomology class in $\mathcal{K}^\bullet(A^\bullet[\mathfrak{m}_\bullet], x_1, \ldots, x_d)\{k/k-1\} =$

$\mathcal{K}^\bullet(\mathrm{gr}_H^\bullet(A^\bullet[\mathfrak{m}_\bullet]), x_1, \dots, x_d)[k]$. Since that latter complex is acyclic, α is the sum of an element α_0 of V_d-degree at most $k-1$ and a boundary $\delta^i(\alpha_1)$. As $A^\bullet[\mathfrak{m}_\bullet]$ is V_d-strict, α_1 can be chosen to be of V_d-degree at most k which proves that $\overline{\alpha} = \overline{\alpha}_0$ in $\mathcal{K}^\bullet(A^\bullet, x_1, \dots, x_d)\{k\}$. Since $A^\bullet[\mathfrak{m}_\bullet]$ is V_d-adapted, α_0 represents also a cohomology class in $\mathcal{K}^\bullet(A^\bullet[\mathfrak{m}_\bullet], x_1, \dots, x_d)\{k-1\}$. By induction on k we can assume that this cohomology class is zero, proving that α_0 is in fact an image of an element of V_d-degree at most $k-1$. This proves the lemma. □

In our next lemma, let $\mathcal{K}^\bullet(A^\bullet[\mathfrak{m}_\bullet], x_1, \dots, x_d)\{\infty\}$ stand for the Koszul complex $\mathcal{K}^\bullet(A^\bullet[\mathfrak{m}_\bullet], x_1, \dots, x_d)$.

Lemma 5.28. *$\mathcal{K}^\bullet(A^\bullet[\mathfrak{m}_\bullet], x_1, \dots, x_d)\{\infty/k\}$ is exact for $k \geq k_1$.*

Proof. We show first that $\mathcal{K}^\bullet(A^\bullet[\mathfrak{m}_\bullet], x_1, \dots, x_d)\{k/k_1\}$ is exact for $k \geq k_1$.

The complex $\mathcal{K}^\bullet(A^\bullet[\mathfrak{m}_\bullet], x_1, \dots, x_d)\{k/k_1\}$ is filtered by the subcomplexes $\mathcal{K}^\bullet(A^\bullet[\mathfrak{m}_\bullet], x_1, \dots, x_d)\{k'/k_1\}$ with $k_1 \leq k' \leq k$. Since each of the quotient complexes $\mathcal{K}^\bullet(A^\bullet[\mathfrak{m}_\bullet], x_1, \dots, x_d)\{k'/k'-1\}$ is exact, so must be the complex $\mathcal{K}^\bullet(A^\bullet[\mathfrak{m}_\bullet], x_1, \dots, x_d)\{k/k_1\}$.

If now $\overline{\alpha}$ is a cohomology class of $\mathcal{K}^\bullet(A^\bullet[\mathfrak{m}_\bullet], x_1, \dots, x_d)\{\infty/k_1\}$ and represented by the element α of V_d-degree k, then α also represents a class in $\mathcal{K}^\bullet(A^\bullet[\mathfrak{m}_\bullet], x_1, \dots, x_d)\{k/k_1\}$. Since the latter complex is exact, $\overline{\alpha}$ is the class of an image and hence is the zero class. □

Now we turn to the proof of the theorem. Consider the exact sequence of complexes

$$\begin{array}{c}
\mathcal{K}^\bullet(A^\bullet[\mathfrak{m}_\bullet], x_1, \dots, x_d)\{\infty/k_1\} \\
\uparrow \\
\mathcal{K}^\bullet(A^\bullet[\mathfrak{m}_\bullet], x_1, \dots, x_d)\{\infty\} \\
\uparrow \\
\mathcal{K}^\bullet(A^\bullet[\mathfrak{m}_\bullet], x_1, \dots, x_d)\{k_1\}.
\end{array}$$

Lemma 5.28 shows that the middle and bottom complex are quasi-isomorphic.

Similarly,

$$\begin{array}{c}
\mathcal{K}^\bullet(A^\bullet[\mathfrak{m}_\bullet], x_1, \ldots, x_d)\{k_1/k_0 - 1\} \\
\uparrow \\
\mathcal{K}^\bullet(A^\bullet[\mathfrak{m}_\bullet], x_1, \ldots, x_d)\{k_1\} \\
\uparrow \\
\mathcal{K}^\bullet(A^\bullet[\mathfrak{m}_\bullet], x_1, \ldots, x_d)\{k_0 - 1\}
\end{array}$$

is exact and lemma 5.27 shows that the top two complexes are quasi-isomorphic. Therefore, both $\mathcal{K}^\bullet(A^\bullet[\mathfrak{m}_\bullet], x_1, \ldots, x_d)\{k_1/k_0 - 1\}$ and $\mathcal{K}^\bullet(A^\bullet[\mathfrak{m}_\bullet], x_1, \ldots, x_d)$ represent the same object in the derived category.

We finish the proof with the remark that the cohomology of the complex $\mathcal{K}^\bullet(A^\bullet[\mathfrak{m}_\bullet], x_1, \ldots, x_d)\{k_1/k_0 - 1\}$ is precisely the cohomology of the complex (5.9) and point out that the i-th module of that complex is the free $\mathbb{C}\langle x_{d+1}, \partial_{d+1}, \ldots, x_n, \partial_n\rangle$-module on the symbols $\{(P_1, \ldots, P_{\dim_{D_n}(A^i)}) | P_j \in \mathbb{C}[\partial_1, \ldots, \partial_d], k_0 - \mathfrak{m}_i(j) \le \deg_\partial(P_j) \le k_1 - \mathfrak{m}_i(j)\}$. □

Remark 5.29. The complex (5.9) is a complex in the category of finitely generated left modules over the Weyl algebra in the $n - d$ variables $x_{d+1}, \ldots, x_n$. Thus, the cohomology objects of (5.9) are finitely generated modules over the Weyl algebra $\mathbb{C}\langle x_{d+1}, \partial_{d+1}, \ldots, x_n, \partial_n\rangle$ and computable by standard Gröbner basis procedures. In particular, if $d = n$ then the resulting complex is a complex of finite-dimensional $\mathbb{C}$-vectorspaces.

5.4 Computation of de Rham Cohomology

Recall that $X = \mathbb{C}^n, Y = V(f_1, \ldots, f_r), U = X \setminus Y$. In this section we apply the results of the the previous sections to obtain an algorithm that computes $H^\bullet_{dR}(U, \mathbb{C})$. We proved in section 5.1 that

$$H^i_{dR}(U, \mathbb{C}) = H^{i-n}(\Omega \otimes^L_{D_n} MV^\bullet) = H^{i-n}(\mathcal{K}^\bullet(MV^\bullet, \partial_1, \ldots, \partial_n)).$$

In order to cope with the problem that we would like to compute the cohomology of $\mathcal{K}^\bullet(MV^\bullet, \partial_1, \ldots, \partial_n)$ and not $\mathcal{K}^\bullet(MV^\bullet, x_1 \ldots, x_n)$, we make use of the Fourier transform.

Ω is sent to $\tilde{\Omega} = D_n/(x_1, \dots, x_n)\cdot D_n$ by the n-th Fourier transform $\mathcal{F} = \mathcal{F}_n$. Thus, $H^i(\mathcal{K}^\bullet(MV^\bullet, \partial_1, \dots, \partial_n))$ is isomorphic to $H^i(\mathcal{K}^\bullet(\tilde{MV}^\bullet, x_1, \dots, x_n))$ as a vector space, where $\tilde{MV}^\bullet$ is the image of the complex $MV^\bullet$ under the n-th Fourier transform.

Then $\tilde{MV}^\bullet$ may be replaced by a free V_n-strict complex $A^\bullet[\mathfrak{m}_\bullet]$ as constructed in section 5.2 and the cohomology of $\tilde{MV}^\bullet$ is holonomic ([24]) and therefore specializable to the origin.

Let $b(s) \in K[s]$ be the polynomial found in subsection 5.3.1 for $d = n$, the b-function for the complex $A^\bullet[\mathfrak{m}_\bullet]$ for restriction to the origin. Then $b(\theta_d + k)\,\mathrm{gr}^k_H(H^i(A^\bullet[\mathfrak{m}_\bullet])) = b(\theta_d + k)\cdot H^i(\mathrm{gr}^k_H(A^\bullet[\mathfrak{m}_\bullet]))$ is zero according to (5.5). Therefore by theorem 5.23, $b(k)$ kills the degree k pieces of the cohomology of $\mathcal{K}^\bullet(\mathrm{gr}^\bullet_H(A^\bullet[\mathfrak{m}_\bullet]), x_1, \dots, x_n)$. In other words, if k_0, k_1 are integers with $\{b(s) = 0, s \in \mathbb{Z}\}$ only if $s \in [k_0, k_1] \cap \mathbb{Z}$, then $\mathcal{K}^\bullet(\mathrm{gr}^\bullet_H(A^\bullet[\mathfrak{m}_\bullet]), x_1, \dots, x_n)[k]$ is exact if $k \notin [k_0, k_1] \cap \mathbb{Z}$.

Theorem 5.30. *Let $X = \mathbb{C}^n, Y = V(f_1, \dots, f_r), U = X \setminus Y$. Then $H^i_{dR}(U, \mathbb{C})$ equals $H^{i-n}(\Omega \otimes^L_{D_n} MV^\bullet)$. Moreover, the cohomology of $\Omega \otimes^L_{D_n} MV^\bullet$ can be computed as follows:*

1. *Compute $MV^\bullet = (0 \to \check{C}^1 \to \dots \to \check{C}^r \to 0)$ as in algorithm 5.10 as complex of finitely generated D_n-modules.*

2. *Compute a V_n-strict D_n-free complex*

$$\cdots \to A^{r-1}[\mathfrak{m}_{r-1}] \to A^r[\mathfrak{m}_r] \to 0$$

   quasi-isomorphic to $\tilde{MV}^\bullet$, where $\tilde{MV}^\bullet$ denotes the image of $MV^\bullet$ under the n-th Fourier automorphism $\mathcal{F}_n$.

3. *Relative to the induced filtration on $H^i(A^\bullet[\mathfrak{m}_\bullet])$ compute the b-functions $b_i(s)$ for the restriction of $H^i(A^\bullet[\mathfrak{m}_\bullet])$ to the origin $x_1 = \dots = x_n = 0$.*

4. *Let $b(s)$ be the least common multiple of all the b_i. Find integers k_0, k_1 with $(b(k) = 0, k \in \mathbb{Z}) \Rightarrow (k_0 \le k \le k_1)$.*

5. $\Omega \otimes^L_{D_n} MV^\bullet$ *is quasi-isomorphic to the complex*

$$\cdots \to \frac{F_H^{k_1}(\tilde{\Omega} \otimes_{D_n} A^i[\mathfrak{m}_i])}{F_H^{k_0-1}(\tilde{\Omega} \otimes_{D_n} A^i[\mathfrak{m}_i])} \to \frac{F_H^{k_1}(\tilde{\Omega} \otimes_{D_n} A^{i+1}[\mathfrak{m}_{i+1}])}{F_H^{k_0-1}(\tilde{\Omega} \otimes_{D_n} A^{i+1}[\mathfrak{m}_{i+1}])} \to \cdots \tag{5.10}$$

which is a complex of finite dimensional $\mathbb{C}$-vector spaces.

Proof. We already remarked that $H^i(\mathcal{K}^\bullet(MV^\bullet, \partial_1, \ldots, \partial_n))$ is isomorphic to $H^i(\mathcal{K}^\bullet(\tilde{MV}^\bullet, x_1, \ldots, x_n))$ over $\mathbb{C}$, and so we only need to show that the latter can be computed from (5.10).

Almost tautologically,

$$\cdots \to \mathrm{gr}^\bullet_H(\tilde{MV}^i[\mathfrak{m}_i]) \to \cdots \to \mathrm{gr}^\bullet_H(\tilde{MV}^{r-1}[\mathfrak{m}_{r-1}]) \to \mathrm{gr}^\bullet_H(\tilde{MV}^r[\mathfrak{m}_r]) \to 0$$

is D_n-quasi-isomorphic to

$$\cdots \to \mathrm{gr}^\bullet_H(A^i[\mathfrak{m}_i]) \to \cdots \to \mathrm{gr}^\bullet_H(A^{r-1}[\mathfrak{m}_{r-1}]) \to \mathrm{gr}^\bullet_H(A^r[\mathfrak{m}_r]) \to 0$$

and hence the same holds after application of $\tilde{\Omega} \otimes^L_{D_n} (-)$. By theorem 5.26 we arrive at the desired conclusion. □

Remark 5.31.

5.31.1. The quotient $\frac{F_H^{k_1}(\tilde{\Omega} \otimes_{D_n} A^i[\mathfrak{m}_i])}{F_H^{k_0-1}(\tilde{\Omega} \otimes_{D_n} A^i[\mathfrak{m}_i])}$ should be thought of as arrays of polynomials in $\partial_1, \ldots, \partial_n$ of degrees bounded between $k_0 - \mathfrak{m}_i(j)$ and $k_1 - \mathfrak{m}_i(j)$.

5.31.2. Since Gröbner bases do not change under field extensions, we may interpret (5.10) as a complex of K-vectorspaces and evaluate the dimension of its $(i-n)$-th cohomology group which will equal the complex dimension of $H^i_{dR}(U, \mathbb{C})$.

5.31.3. $\mathcal{K}^{r-i}(\mathrm{gr}^\bullet_H(A^\bullet[\mathfrak{m}_\bullet]), x_1, \ldots, x_n)$ involves terms from $A^r, \ldots, A^{r-i}$, but not from A^j with $j < r - i$. We deduce: if $MV^\bullet$ is exact at $r - i$ and beyond, then $\Omega \otimes^L_{D_n} MV^\bullet$ is exact at $j \geq n + r - i$. That follows by considering $b(s) \cong 1$, which kills the last i cohomology terms in $MV^\bullet$; inspecting the proof of theorem 5.23 one sees that then 1 also kills the last i cohomology terms in $\mathcal{K}^\bullet(A^\bullet[\mathfrak{m}_\bullet], x_1, \ldots, x_n)$.

It follows the well-known

Corollary 5.32. $H^i_{dR}(U, \mathbb{C}) = 0$ *if* $i \geq n + \mathrm{cd}(f_1, \ldots, f_r)$. □

5.5 De Rham Cohomology with Supports

Let Y, Z be two Zariski-closed subsets of X. In this section we are concerned with finding an algorithm that computes the de Rham cohomology groups $H^\bullet_{dR,Z}(U,\mathbb{C})$ of $U = X\backslash Y$ with coefficients in $\mathbb{C}$ and supports in Z.

$H^\bullet_{dR,Z}(U,\mathbb{C})$ is defined as follows. Recall the de Rham complex $\Omega^\bullet(U)$ on U. Usual de Rham cohomology is defined as the hypercohomology of $\Omega^\bullet(U)$. De Rham cohomology with supports is defined as the hypercohomology with supports in Z of $\Omega^\bullet(U)$. In other words, $H^\bullet_{dR,Z}(U,\mathbb{C}) = H^\bullet(R\Gamma_Z(U,\Omega^\bullet(U)))$.

As was pointed out by Hartshorne, there is a natural exact sequence

$$\cdots \to H^i_{dR,Z}(U,\mathbb{C}) \to H^i_{dR}(U,\mathbb{C}) \to H^i_{dR}(U\backslash Z,\mathbb{C}) \to H^{i+1}_{dR,Z}(U,\mathbb{C}) \to \cdots \tag{5.11}$$

which indicates that $H^\bullet_{dR,Z}(U,\mathbb{C})$ measures the change in cohomology due to the removal of $Z\cap U$ from U.

For the entire section let us assume that $Y = \mathrm{Var}(F), F = (f_1,\dots,f_r)$ and $Z = \mathrm{Var}(G), G = (g_1,\dots,g_s)$. Write $F\cdot G = \{f_i\cdot g_j\}$. As before we will write F_I for $\prod_{i\in I} f_i$ and G_J for $\prod_{j\in J} g_j$. There is a natural map of Mayer-Vietoris complexes

$$MV^\bullet(F\cdot G, F) \to MV^\bullet(F\cdot G)$$

given by the natural projection

$$\bigoplus_{|I|+|J|+|K|=l} R_n[F_I\cdot G_J]^{-1}\otimes_R R_n[F_K{}^{-1}] \to \bigoplus_{|I|+|J|=l} R_n[F_I\cdot G_J{}^{-1}]$$

sending each summand with $|K|>0$ to zero. This map corresponds to the embedding $X\backslash(Y\cup Z)\hookrightarrow X\backslash Y$. Clearly the map is surjective and the kernel is the subcomplex of $MV^\bullet(F\cdot G,F)$ consisting of those pieces which contain at least one factor from F. It is not hard to check that this kernel is exactly $MV^\bullet(F)\otimes_{R_n}\check{C}^\bullet(F\cdot G)$, $\check{C}^\bullet(F\cdot G)$ being the Čech complex to $F\cdot G$ given by $\bigotimes_{i,j}(0\to R_n \xrightarrow{nat} R_n[f_i\cdot g_j{}^{-1}]\to 0)$.

Notice that the sequences

$$0 \to \left(MV^\bullet(F)\otimes_{R_n}\check{C}^\bullet(F\cdot G)\right)^i \to MV^i(F\cdot G,F)\to MV^i(F\cdot G)\to 0$$

are all split exact. Then there is a sequence of complexes

$$0 \to \Omega^\bullet \otimes_{D_n} \left(MV^\bullet(F) \otimes_{R_n} \check{C}^\bullet(F \cdot G)\right) \to \\ \to \Omega^\bullet \otimes_{D_n} MV^\bullet(F \cdot G, F) \to \Omega^\bullet \otimes_{D_n} MV^\bullet(F \cdot G) \to 0 \qquad (5.12)$$

with split exact rows. In other words, we have a short exact sequence of complexes.

As was explained earlier, the cohomology of $\Omega^\bullet \otimes_{D_n} MV^\bullet(F \cdot G, F)$ is $H^i_{dR}(X\backslash Y, \mathbb{C})$ while the cohomology of $\Omega^\bullet \otimes_{D_n} MV^\bullet(F \cdot G)$ is $H^i_{dR}(X\backslash(Y \cup Z), \mathbb{C})$ and the map on cohomology is induced by the natural inclusion.

Comparison of the long exact sequence (5.11) with the long exact sequence that results from the short exact sequence of complexes (5.12) shows that the cohomology of $\Omega^\bullet \otimes_{D_n} (MV^\bullet(F) \otimes_{R_n} \check{C}^\bullet(F \cdot G))$ is exactly $H^i_{dR,Z}(X\backslash Y, \mathbb{C})$.

Computationally this situation is very bad: de Rham cohomology of $X\backslash Y$ and $X\backslash Z$ come from the Mayer-Vietoris complex of F and G respectively while here we have (approximately) the Mayer-Vietoris complex of $F \cup F \cdot G$. We shall try to improve this situation now. As a first step in that direction we point out that the long exact sequence (5.11) shows that for complements of affine closed varieties $H^i_{dR,Z}(X\backslash Y, \mathbb{C})$ is in fact nothing but the relative cohomology group $H^i(X\backslash Y, X\backslash(Y \cup Z); \mathbb{C})$.

Consider the space $X\backslash(Y \cap Z)$ and its open covering by the two sets $X\backslash Y$ and $X\backslash Z$. It follows from [14], Example 17.1, that this is an exact triad for homology with integer coefficients, and from [9], Theorem 11.4, that the same holds for cohomology with coefficients in $\mathbb{C}$. This means that the natural inclusion of pairs

$$(X\backslash Y, X\backslash(Y \cup Z)) \hookrightarrow (X\backslash(Y \cap Z), X\backslash Z)$$

induces an isomorphism $H^i(X\backslash Y, X\backslash(Y \cup Z); \mathbb{C}) \cong H^i(X\backslash(Y \cap Z), X\backslash Z; \mathbb{C})$. This implies that instead of $H^i_{dR,Z}(X\backslash Y, \mathbb{C})$ we may calculate $H^i_{dR,Z}(X\backslash(Y \cap Z), \mathbb{C})$ since the groups are isomorphic.

Now consider the natural projection of complexes

$$MV^\bullet(F, G) \to MV^\bullet(G)$$

given by $\bigoplus_{|I|+|J|=l} R_n[F_I{}^{-1}] \otimes_{R_n} R_m[G_J \to \bigoplus_{|I|=l} R_n[F_I{}^{-1}]$ induced by the inclusion $X \backslash Z \hookrightarrow X \backslash (Y \cap Z)$. As before, this induces a short exact sequence of complexes

$$0 \to MV^\bullet(F) \otimes_{R_n} \check{C}^\bullet(G) \to MV^\bullet(F, G) \to MV^\bullet(G) \to 0.$$

Tensoring over D_n with the complex $\Omega^\bullet$ we discover that the cohomology of $\Omega^\bullet \otimes_{D_n} (MV^\bullet(F) \otimes_{R_n} \check{C}^\bullet(G))$ is $H^\bullet_{dR,Z}(X \backslash (Y \cap Z)) \cong H^\bullet_{dR,Z}(X \backslash Y)$. Now the complexity is down to the level of computing $H^\bullet_{dR}(X \backslash (Y \cap Z))$. It follows

Algorithm 5.33. Input: polynomials $F = \{f_1, \dots, f_r\}$ defining Y, and $G = \{g_1, \dots, g_s\}$ defining Z; $i \in \mathbb{N}$.

Output: The de Rham cohomology groups of $U = X \backslash Y$ with supports in Z, $H^i_{dR,Z}(X \backslash Y, \mathbb{C})$, which equal the relative cohomology groups $H^i(X \backslash Y, X \backslash (Y \cup Z); \mathbb{C})$.

Begin

1. Compute the complex $MV^\bullet(F) \otimes_{R_n} \check{C}^\bullet(G)$ as a complex of left D_n-modules as in [47], algorithm 5.1.

2. Compute a free V_n-strict complex $A^\bullet[\mathfrak{m}_\bullet]$ that is quasi-isomorphic to the image of $MV^\bullet(F) \otimes_{R_n} \check{C}^\bullet(G)$ under the n-th Fourier automorphism as in section 5.2, theorem 5.19.

3. Find the cohomology groups of $A^\bullet[\mathfrak{m}_\bullet]$ and compute the b-functions of the cohomology groups under the given shifts. Let k_0 and k_1 be lower and upper bounds of the roots of the b-functions.

4. Replace each D_n in $A^\bullet$ by $k[\partial_1, \dots, \partial_n] = \tilde{\Omega}$ and restrict the complex to the components between V_n-degree $k_0 - 1$ and k_1.

5. Take $(i-n)$-th cohomology of the resulting complex of vector spaces and return its dimension d_i.

6. $H^i_{dR,Z}(X \setminus Y, \mathbb{C}) \cong \mathbb{C}^{d_i}$.

End.

References

[1] D. Bayer and M. Stillman. Macaulay: a system for computation in algebraic geometry and commutative algebra. *zariski.harvard.edu via ftp*, (login: anonymous; password: any; cd Macaulay), 1982-1990.

[2] J. Bernstein. Modules over the ring of differential operators. *Funct. Anal. Appl.*, 2:1–16, 1971.

[3] J.-E. Björk. *Rings of Differential Operators.* North Holland, 1978.

[4] M. Brodmann. Einige Ergebnisse aus der lokalen Kohomologietheorie und ihre Anwendung. *Osnabrücker Schriften zur Mathematik*, 5:439–445, 1983.

[5] M. P. Brodmann and R. Y. Sharp. *Local Cohomology.* Cambridge University Press, 1998.

[6] W. Bruns and R. Schwänzl. The number of equations defining a determinantal variety. *Bull. London Math. Soc.*, 22:439–445, 1990.

[7] B. Buchberger. Ein Algorithmus zum Auffinden der Basiselemente des Restklassenringes nach einem nulldimensionalen Polynomideal. *Math. Inst Universität Innsbruck*, Doctoral Dissertation, 1965.

[8] F. Castro. Calcul effectifs pour les idéaux d'opérateurs différentiels. *Travaux en Cours*, 24:1–19, Herman, Paris, 1987.

[9] S. Eilenberg and N. Steenrod. *Foundations of Algebraic Topology.* Princeton University Press, 1952.

[10] D. Eisenbud. *Commutative Algebra with a View toward Algebraic Geometry.* Springer Verlag, 1995.

[11] G. Faltings. Über lokale Kohomologiegruppen hoher Ordnung. *Journal für die Reine und Angewandte Mathematik*, 313:43–51, 1980.

[12] A. Galligo. Some algorithmic questions on ideals of differential operators. *Lecture Notes in Comp. Science*, 204:413–421, 1985.

[13] T. Georgiou, A. Tannenbaum, and U. Walther. Computational Algebraic Geometry and Switching Surfaces in Optimal Control. *IEEE Transactions on Automatic Control*, (submitted) 1999.

[14] M. Greenberg and J. Harper. *Algebraic Topology, a first course*. Mathematical Lecture Notes Series 58. Addison Wesley, 1981.

[15] R. Hartshorne. Cohomological Dimension of Algebraic Varieties. *Ann. of Math.*, 88:403–450, 1968.

[16] R. Hartshorne. On the de Rham Cohomology of Algebraic Varieties. *Publ. Math. Inst. Hautes Sci.*, 45:5–99, 1975.

[17] R. Hartshorne and R. Speiser. Local cohomological dimension in characteristic p. *Ann. Math.(2)*, 105:45–79, 1977.

[18] H. Hironaka. Resolution of singularities of an algebraic variety over a field of characteristic zero. *Ann. Math.*, 79:109–326, 1964.

[19] R. Hotta and T. Tanisaki. *D-modules and algebraic groups (in Japanese).* Springer, Tokyo, 1994.

[20] C. Huneke and G. Lyubeznik. On the vanishing of local cohomology modules. *Invent. Math.*, 102:73–93, 1990.

[21] C. Huneke and R. Y. Sharp. Bass numbers of local cohomology modules. *Transactions of the AMS (2)*, 339:765–779, 1993.

[22] A. Kandri-Rody and V. Weispfenning. Non-commutative Gröbner bases in algebras of solvable type. *J. Symb. Comp.*, 9:1–26, 1990.

[23] M. Kashiwara. B-functions and holonomic systems, rationality of B-functions. *Invent. Math.*, 38:33–53, 1976.

[24] M. Kashiwara. On the holonomic systems of linear partial differential equations, II. *Invent. Math.*, 49:121–135, 1978.

[25] Y. Laurent and P. Schapira. Images inverses des modules différentiels. *Compositio Math.*, 61:229–251, 1987.

[26] G. Lyubeznik. Finiteness Properties of Local Cohomology Modules: an Application of D-modules to Commutative Algebra. *Invent. Math.*, 113:41–55, 1993.

[27] G. Lyubeznik. F-modules: applications to local cohomology and D-modules in characteristic $p > 0$. *Journal für die Reine und Angewandte Mathematik*, 491:65–130, 1997.

[28] F. Macaulay. *Algebraic Theory of Modular Systems*. Cambridge Tracts. Cambridge University Press, 1916.

[29] B. Malgrange. Le polynôme de Bernstein d'une singularité isolée. *Lecture Notes in Mathematics, Springer Verlag*, 459:98–119, 1990.

[30] J. Montaner. Characteristic cycles of local cohomology modules. *Preprint*, 1997.

[31] M. Mustaţă. Local cohomology at monomial ideals. *Preprint*, 1998.

[32] T. Oaku. Computation of the characteristic variety and the singular locus of a system of differential equations with polynomial coefficients. *Japan Journal of Industrial and Applied Mathematics*, 11:485–497, 1994.

[33] T. Oaku. An algorithm for computing B-functions. *Duke Math. Journal*, 87:115–132, 1997.

[34] T. Oaku. Algorithms for b-functions, restrictions, and algebraic local cohomology groups of D-modules. *Adv. in Appl. Math.*, 19:61–105, 1997.

[35] T. Oaku. Gröbner bases for D-modules on a non-singular affine algebraic variety. *Tôhoku Math. J.*, 48:575–600, 1997.

[36] T. Oaku and N. Takayama. An algorithm for de Rham cohomology groups of the complement of an affine variety via

D-module computation. *Journal of Pure and Applied Algebra*, `http://xxx.lanl.gov/list/math.AG/9801114`, (to appear) 1998.

[37] T. Oaku and N. Takayama. Algorithms for D-modules – restriction, tensor product, localization and local cohomology groups. `http://xxx.lanl.gov/list/math.AG/9805006`, 1998.

[38] T. Oaku, N. Takayama, and U. Walther. A localization algorithm for D-modules. *Journal of Symbolic Computation*, `http://xxx.lanl.gov/list/math.AG/9811030`, (submitted) 1998.

[39] A. Ogus. Local cohomological dimension of algebraic varieties. *Ann. of Mathematics*, 98:327–365, 1973.

[40] C. Peskine and L. Szpiro. Dimension projective finie et cohomologie locale. *Inst. Hautes Études Sci. Publ. Math.*, 42:47–119, 1973.

[41] M. Saito, B. Sturmfels, and N. Takayama. *Gröbner deformations of hypergeometric differential equations.* Springer, 1999.

[42] B. Sturmfels. *Gröbner bases and convex polytopes.* AMS University Lecture Series. Providence, 1996.

[43] T. Noro et. al. Risa/Asir, a computer algebra system. `http://endeavor.fujitsu.co.jp/pub/isis/asir`, 1993, 1995.

[44] N. Takayama. Gröbner basis and the problem of contiguous relations. *Japan J. Appl. Math.*, 6:147–160, 1989.

[45] N. Takayama. Kan, a computer package for symbolic computations in Weyl algebras. `www.math.s.kobe-u.ac.jp/KAN/`, 1991–.

[46] U. Walther. Locol, a computer program for symbolic computations in local cohomology. `walther@math.umn.edu`, 1996-1998.

[47] U. Walther. Algorithmic Computation of Local Cohomology Modules and the Local Cohomological Dimension of Algebraic Varieties. *Journal of Pure*

and Applied Algebra, `http://xxx.lanl.gov/list/math.AG/9710004`, (to appear) 1997.

[48] U. Walther. Algorithmic computation of de Rham cohomology of complements of complex affine varieties. *Journal of Symbolic Computation*, `http://xxx.lanl.gov/list/math.AG/9807176`, (submitted) 1998.

[49] K. Yanagawa. Bass numbers of local cohomology modules with supports in monomial ideals. *Preprint*, 1999.