BERNSTEIN-SATO POLYNOMIAL VERSUS COHOMOLOGY OF THE MILNOR FIBER FOR GENERIC HYPERPLANE ARRANGEMENTS

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ABSTRACT. Let $Q \in \mathbb{C}[x_1, \dots, x_n]$ be a homogeneous polynomial of degree $k > 0$. We establish a connection between the Bernstein-Sato polynomial $b_Q(s)$ and the degrees of the generators for the top cohomology of the associated Milnor fiber. In particular, the integer $u_Q = \max\{i \in \mathbb{Z} : b_Q(-(i+n)/k) = 0\}$ bounds the top degree (as differential form) of the elements in $H^{n-1}_{\mathrm{DR}}(Q^{-1}(1), \mathbb{C})$. The link is provided by the relative de Rham complex and $\mathcal{D}$-module algorithms for computing integration functors.

As an application we determine the Bernstein-Sato polynomial $b_Q(s)$ of a generic central arrangement $Q = \prod_{i=1}^k H_i$ of hyperplanes. We obtain in turn information about the cohomology of the Milnor fiber of such arrangements related to results of Orlik and Randell who investigated the monodromy.

We also introduce certain subschemes of the arrangement determined by the roots of $b_Q(s)$. They appear to correspond to iterated singular loci.

1. Introduction

Let f be a non-constant polynomial in n variables. In the 1960s, M. Sato introduced a-, b- and c-functions associated to a prehomogeneous vector space [32, 33]. The existence of b-functions associated to all polynomials and germs of holomorphic functions was later established in [2, 3].

The simplest interesting case of a b-function is the case of the quadratic form $f(x_1, \dots, x_n) = \sum_{i=1}^n x_i^2$. Let s be a new variable and denote by f^s the germ of the complex power of $f(x)$. One then has an identity

$$\left(\sum_{i=1}^{n} \frac{\partial^2}{\partial {x_i}^2} \right) \bullet f^{s+1} = 4(s+1)(s+n/2) f^s.$$

The b-function to $f(x)$ is here $b_f(s) = (s+1)(s+n/2)$. One may for general f use an equality of the type

$$P(s) \bullet f^{s+1} = b(s) f^s \tag{1.1}$$

to analytically continue f^s, and it was this application that initially caused I.N. Bernstein to consider $b_f(s)$. Today, the b-function of a polynomial is usually referred to as "Bernstein-Sato polynomial" and denoted $b_f(s)$.

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The Bernstein-Sato polynomial is always a multiple of $(s+1)$, and equality holds if f is smooth. The roots of $b_f(s)$ are always negative and rational [17]. It has been pointed out first in [24, 25] that there is an intimate connection between the singularity structure of $f^{-1}(0)$ and its Bernstein-Sato polynomial. The roots of $b_f(s)$ relate to a variety of algebro-geometric data like the structure of the embedded resolution of the pair $(\mathbb{C}^n, \mathrm{Var}(f))$, Newton polyhedra, Zeta functions, asymptotic expansions of integrals, Picard-Lefschetz monodromy, polar invariants and multiplier ideals: see, for example, [7, 16, 20, 22, 23, 37]. T. Yano systematically worked out a number of examples [40] and some interesting computations are given in [5]. A satisfactory interpretation of all roots of $b_f(s)$ for general f remains, however, outstanding. Indeed, until [28] there was not even an algorithm for the computation of $b_f(s)$ for an arbitrary polynomial f.

In this note we investigate the Bernstein-Sato polynomial when f defines a generic central hyperplane arrangement. By that we mean a reduced collection of k hyperplanes such that each subset of $\min\{k, n\}$ of the hyperplanes cuts out the origin. The paper is organized as follows. In this section we introduce the relevant notation. In the next section we find an upper bound for the Bernstein-Sato polynomial of a central generic arrangement. We shall compute a polynomial $b(s)$ that satisfies an identity of the type (1.1) using strongly that the arrangement is central and generic. In Section 3 we use some counting and Gröbner type arguments to obtain information about generators for the top cohomology of the Milnor fiber of such arrangements. We prove parts of a conjecture of Orlik and Randell on the cohomology of the Milnor fiber of a generic central arrangement. In particular, we determine in exactly which degrees the top cohomology lives, and we present a conjectured set of generators.

Malgrange [26] demonstrated that the Bernstein-Sato polynomial is the minimal polynomial of a certain operator on the sheaf of vanishing cycles. This says in essence that monodromy eigenvalues are exponentials of roots of $b_f(s)$. In the fourth section we prove roughly that for homogeneous f the degrees of the top Milnor fiber cohomology are roots of $b_f(s)$. This can in some sense be seen as a logarithmic lift of Malgrange's results. For generic central arrangements this links our results from Sections 2 and 3 and allows the determination of all roots of $b_f(s)$ and (almost) all multiplicities. We close Section 4 with an example of a non-generic arrangement, and finish in Section 5 with some statements and conjectures regarding the structure of the D_n-modules $R_n[f^{-1}]$ and $D_n[s] \bullet f^s$.

Notation 1.1. Throughout, we will work over the field of complex numbers $\mathbb{C}$. We should point out that this is mostly for keeping things simple as the Bernstein-Sato polynomial is invariant under field extensions.

In this note, for elements $\{f_1, \ldots, f_k\}$ of any ring A, $\langle f_1, \ldots, f_k\rangle$ denotes the left ideal generated by $\{f_1, \ldots, f_k\}$. If we mean a right ideal, we specify it by writing $\langle f_1, \ldots, f_k\rangle A$.

By R_n we denote the ring of polynomials $\mathbb{C}[x_1, \ldots, x_n]$ in n variables over $\mathbb{C}$, and by D_n we mean the ring of $\mathbb{C}$-linear differential operators on R_n, the n-th Weyl algebra. The ring D_n is generated by the partial derivative operators $\partial_i = \frac{\partial}{\partial x_i}$ and the multiplication operators x_i. One may consider R_n as a subring of D_n as well as a quotient of D_n (by the left ideal $\langle \partial_1, \ldots, \partial_n\rangle$). We denote by $\bullet$ the natural action of D_n on R_n via this quotient map, as well as induced actions of D_n on localizations of R_n.

We will have occasion to consider D_t, D_x and $D_{x,t}$ in some instances, where D_t is the Weyl algebra in the variable t, D_x the one in $x_1, \dots, x_n$ and $D_{x,t}$ is the Weyl algebra in $x_1, \dots, x_n$ and t.

The module of global algebraic differential n-forms on $\mathbb{C}^n$ is denoted Ω; it may be pictured as the quotient $D_n/\langle\partial_1, \dots, \partial_n\rangle D_n$. The left D_n-Koszul complex on D_n induced by the commuting vector fields $\partial_1, \dots, \partial_n$ is denoted $\Omega^\bullet$; it is a resolution for Ω as right D_n-module.

We shall use multi-index notation in R_n: writing x^α implies that $\alpha = (\alpha_1, \dots, \alpha_n)$ and stands for $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$. The same applies to elements of D_n, both for the polynomial and the differential components. If α is a multi-index, $|\alpha|$ denotes the sum of its components; if I is a set, then $|I|$ is its cardinality. Finally, if $k, r \in \mathbb{N}$ then $k \,|\, r$ signifies that k divides r while $k \not| \, r$ indicates that this is not the case.

1.1. **Bernstein-Sato polynomials.**

Definition 1.2. For $f \in R_n$ we define $J(f^s) \subseteq D_n[s]$ to be the annihilator of f^s via formal differentiation, this is a left ideal. We set

$$\mathcal{M} = D_n[s]/(J(f^s) + \langle f\rangle) = D_n \bullet f^s / D_n \bullet f^{s+1}.$$

By definition, the *Bernstein-Sato polynomial* $b_f(s)$ of f is the minimal polynomial of s on $\mathcal{M}$. So $b_f(s)$ is the monic polynomial of smallest degree satisfying a functional equation of the type (1.1) with $P(s) \in D_n[s]$.

Let $\tilde{\mathcal{M}} = D_n[s]/(J(f^s) + \langle f\rangle + D_n[s] \cdot \mathfrak{A})$ where $\mathfrak{A} \subseteq R_n$ is the Jacobian ideal of f, $\mathfrak{A} = \sum_{i=1}^n R_n \partial_i \bullet (f)$. Then $\tilde{\mathcal{M}}$ is isomorphic to $(s+1)\mathcal{M}$ and since $(s+1)$ divides $b_f(s)$ then the minimal polynomial of s on $\tilde{\mathcal{M}}$ is $\tilde{b}_f(s) = b_f(s)/(s+1)$.

Consider the module $D_n \bullet f^a$ for $a \in \mathbb{C}$ and write $J(f^a)$ for the kernel of the map $D_n \to D_n \bullet f^a$ induced by $P \mapsto P \bullet f^a$. There is a natural map $D_n \bullet f^{a+1} \hookrightarrow D_n \bullet f^a$ induced by $P \bullet f^{a+1} \mapsto Pf \bullet f^a$. Some roots of the Bernstein-Sato polynomial detect the failure of this map to be an isomorphism:

Lemma 1.3. *Let $a \in \mathbb{Q}$ be such that $b_f(a) = 0$ but $b_f(a-n) \neq 0$ for all positive natural numbers n. Then $D_n \bullet f^a \neq D_n \bullet f^{a+1}$.*

Proof. Suppose that a is as the hypotheses stipulate, and in addition assume that $D_n \bullet f^a = D_n \bullet f^{a+1}$. We will exhibit a contradiction.

Since $D_n \bullet f^{a+1} \to D_n \bullet f^a$ is an epimorphism, $D_n = \langle f\rangle + J(f^a)$. By the choice of a and Proposition 6.2 in [17], $J(f^a) = D_n \cap (J(f^s) + D_n[s] \cdot (s-a))$. Hence $D_n[s] = J(f^s) + \langle f\rangle + \langle s-a\rangle$. Multiplying by $b_f(s)/(s-a)$ we get $\langle b_f(s)/(s-a)\rangle \subseteq J(f^s) + \langle f\rangle + \langle b_f(s)\rangle$. Since $b_f(s) \in J(f^s) + \langle f\rangle$,

$$\frac{b_f(s)}{s-a} \in J(f^s) + \langle f\rangle.$$

That, however, contradicts the definition of $b_f(s)$ as the minimal polynomial in s contained in the sum on the right. □

1.2. **Isolated Singularities.**

Suppose that f has an isolated singularity and assume for simplicity that the singularity is at the origin. We give a short overview of what is known about the Bernstein-Sato polynomial in this case, following [19, 25, 40].

The module $\tilde{\mathcal{M}}$ is supported only at the origin, so by [17] the minimal polynomial of s on $\Omega \otimes_{D_n} \tilde{\mathcal{M}}$ is $\tilde{b}_f(s)$. If now f is homogeneous of degree k, $kf = \sum_{i=1}^n x_i\partial_i \bullet (f)$. Then $J(f^s)$ contains $\sum_{i=1}^n x_i\partial_i - ks$. The action of s on a homogeneous $g \in \Omega \otimes_{D_n} \tilde{\mathcal{M}} \cong R_n/\mathfrak{A}$ is easily seen to be multiplication by $(-n - \deg(g))/k$. Thus, the Bernstein-Sato polynomial of a homogeneous isolated singularity encodes exactly the degrees of non-vanishing elements in $R_n/\mathfrak{A}$.

Consider now the relative de Rham complex $\Omega_f^\bullet$ associated to the map $f : \mathbb{C}^n \to \mathbb{C}$. We shall denote the coordinate on $\mathbb{C}$ by t. The complex $\Omega_f^\bullet$ is the Koszul complex induced by left multiplication by $\partial_1, \dots, \partial_n$ on the $D_{x,t}$-module $\mathcal{N} = D_{x,t}/J_{n+1}(f)$ where $J_{n+1}(f)$ is the left ideal of $D_{x,t}$ generated by $t - f$ and the expressions $\partial_i + \partial_i \bullet (f)\partial_t$ for $1 \le i \le n$. The complex $\Omega_f^\bullet = \Omega^\bullet \otimes_{D_n} \mathcal{N}$ is a representative of the application of the de Rham functor $\int_f$ associated to the map f to the structure sheaf on $\mathbb{C}^n$, [13]. Its last nonzero cohomology module appears in degree n, $H^n(\Omega_f^\bullet) = \mathcal{N}/\{\partial_1, \dots, \partial_n\} \cdot \mathcal{N}$. This module is in a natural way a left D_t-module. For any $\alpha \in \mathbb{C}$, the cohomology of the derived tensor product of $\Omega_f^\bullet$ with $D_t/\langle t - \alpha\rangle D_t$ is the de Rham cohomology of the fiber at α. The identification of $\mathcal{N}/\{\partial_1, \dots, \partial_n, t - \alpha\}D_{x,t}$ with $H^{n-1}_{\mathrm{DR}}(\mathrm{Var}(f - \alpha))$ is explained in and before Lemma 4.11.

So one has an isomorphism

$$R_n/\mathfrak{A} \cong (D_t/\langle t - \alpha\rangle D_t) \otimes_{D_t} H^n(\Omega^\bullet \otimes_{D_n} \mathcal{N}) \cong H^{n-1}_{\mathrm{DR}}(f^{-1}(\alpha), \mathbb{C})$$

and the roots of $b_f(s)$ in fact represent the degrees of the cohomology classes of the Milnor fiber of f.

For general f, the Bernstein-Sato polynomial is more complex, see Example 4.16 and the following remarks.

2. An upper bound for the Bernstein polynomial

Our goal is Theorem 2.13. We shall mimic some of the mechanism that makes the isolated singularity case so easy. It is clear that a literal translation is not possible, because $R_n/\mathfrak{A}$ has in general elements in infinitely many different degrees. However, we now introduce certain ideals in R_n that are intimately related to the Bernstein-Sato polynomial.

Definition 2.1. Let $q(s) \in \mathbb{C}[s]$. For a fixed $f \in R_n$ we define the ideal $\mathfrak{a}_{q(s)} \subseteq R_n$ as the set of elements $g \in R_n$

$$\left[g \in \mathfrak{a}_{q(s)}\right] \iff \left[\exists P(s) \in D_n[s] : P(s) \bullet f^{s+1} = q(s) g f^s\right].$$

We remark that $\mathfrak{a}_{q(s)} \subseteq \mathfrak{a}_{q(s)q'(s)}$, and if $q'(s)g \in \mathfrak{a}_{q(s)} \cdot R_n[s]$ then $g \in \mathfrak{a}_{q(s)q'(s)}$. The Jacobian ideal $\mathfrak{A}$ is contained in $\mathfrak{a}_{(s+1)}$, and $f \in \mathfrak{a}_{(1)}$.

The Bernstein-Sato polynomial of f is evidently the polynomial $b_f(s)$ of smallest degree such that $1 \in \mathfrak{a}_{b_f(s)}$.

Before we come to the computation of an estimate for $b_f(s)$ for generic arrangements we first consider general homogeneous polynomials and then arrangements in the plane.

2.1. **The homogeneous case.**

Assume now that $Q \in R_n$ is homogeneous.[1] We shall denote by $\mathfrak{m}$ the homogeneous maximal ideal of R_n. If $g \in \mathfrak{a}_{q(s)}$ then by definition $gQ^s \in \mathcal{M}$ is annihilated by $q(s)$. Since $b_Q(s)$ annihilates all of $\mathcal{M}$, finding $g \in \mathfrak{a}_{q(s)}$ is equivalent to finding eigenvectors of s on $\mathcal{M}$ to eigenvalues that are zeros of $q(s)$. In the isolated singularity case one only has to study the residues of $\mathfrak{a}_{q(s)}$ in $R_n/\mathfrak{A}$, and this goes as follows. Let $\delta_Q = \min_{k\in\mathbb{N}}\{\mathfrak{m}^{k+1} \subseteq \mathfrak{A}\}$. Then the homogeneous polynomial g with $0 \neq \overline{g} \in R_n/\mathfrak{A}$ is in $\mathfrak{a}_{q(s)}$ if and only if $(s+1)\prod_{i=\deg(g)}^{\delta_Q}\left(s+\frac{i+n}{\deg(Q)}\right)$ divides $q(s)$; this is proved in [40] based on results of Kashiwara.

For non-isolated homogeneous singularities Q we have a weak version of this:

Lemma 2.2. *If $R_n[s]\cdot\mathfrak{a}_{q(s)}$ contains $\mathfrak{m}^r g$ where $g = g(s) \in R_n[s]$ is homogeneous in $x_1, \dots, x_n$ then $g \in R_n[s]\cdot\mathfrak{a}_{q'(s)}$ where $q'(s) = q(s)\cdot\prod_{i=0}^{r-1}(s+(i+n+\deg(g))/k)$. In particular,*

$$\left[\mathfrak{m}^r \subseteq \mathfrak{a}_{q(s)}\right] \Longrightarrow \left[b_Q(s)\,|\,q(s)\cdot\prod_{i=0}^{r-1}\left(s+\frac{i+n}{k}\right)\right].$$

Proof. Let m be a monomial of degree $r-1$, so $x_i mg \in R_n[s]\cdot\mathfrak{a}_{q(s)}$. Then

$$\begin{aligned}\sum_{i=1}^{n}\partial_i \bullet (x_i mgQ^s) &= mg(\partial_1 x_1 + \ldots + \partial_n x_n)\bullet Q^s + \deg(mg)mgQ^s \\ &= mg(ks+n+\deg(mg))Q^s.\end{aligned}$$

As $\deg(m) = r-1$, $\left(s+\frac{r-1+n+\deg(g)}{k}\right)mg \in \mathfrak{a}_{q(s)}$. By decreasing induction on $\deg(m)$,

$$\prod_{i=1}^{r}\left(s+\frac{n+\deg(g)+r-i}{k}\right)g \in R_n[s]\cdot\mathfrak{a}_{q(s)}.$$

The final claim follows from the definition of $b_Q(s)$. □

Remark 2.3. Suppose that f is *w-quasi-homogeneous*, i.e., there are nonnegative numbers $w_1, \dots, w_n$ such that with $\xi = \sum_{i=1}^n w_i x_i \partial_i$ one has $f = \xi\bullet(f)$ and hence $\xi - s \in J(f^s)$. If $\mathfrak{n} \subseteq \mathfrak{a}_{q(s)}$ is a w-homogeneous $\mathfrak{m}$-primary ideal then one can show in the same manner that $b_f(s)$ divides the product of $q(s)$ and the minimal polynomial of ξ on $R_n/\mathfrak{n}$ evaluated at $-s-\sum_{i=1}^n w_i$. For example, $f = x^3+y^3+z^2w$ is (1/3,1/3,1/3,1/3)-homogeneous. One has $\mathfrak{a}_{(s+1)} = \langle x^2, y^2, z^2, zw\rangle$, which is of dimension 1, corresponding to the line of singularities $(0,0,0,w)$. One can see that the trick of Lemma 2.2 can be used to show that $\mathfrak{a}_{(s+1)(s+7/3)} = \langle x^2, xyz, y^2, z^2, zw\rangle$ since xyz is in the socle of $R_n/\mathfrak{a}_{(s+1)}$. Going one step further, $\mathfrak{a}_{(s+1)(s+7/3)(s+2)} = \langle x^2, xz, y^2, yz, z^2, zw\rangle$ and then z can be obtained in $\mathfrak{a}_{(s+1)(s+7/3)(s+2)(s+5/3)} = \langle x^2, y^2, z\rangle$. The new factors are always equal to $s+\sum_{i=1}^4(1/3)$ plus the degree of the new element in $\mathfrak{a}$.

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[1]Throughout we use Q for an instance of a homogeneous polynomial while f is used if no homogeneity assumptions are in force.

Now, however, nothing is in the socle and our procedure stops. On the other hand, f is also $(1/3,1/3,1/2,0)$-homogeneous and this can be used to show that

$$\begin{aligned}\mathfrak{a}_{(s+1)(s+7/3)(s+2)(s+5/3)(s+11/6)} &= \langle x,y,z\rangle,\\ \mathfrak{a}_{(s+1)(s+7/3)(s+2)(s+5/3)(s+11/6)(s+7/6)} &= R_n.\end{aligned}$$

In fact, $b_f(s) = (s+1)(s+7/3)(s+2)(s+5/3)(s+11/6)(s+7/6)$ and one can see again how the factors of $b_f(s)$ enlarge (if taken in the right order) the ideal $\mathfrak{a}$, by either saturating, or dropping dimension.

The trick for bounding $b_f(s)$ is therefore to find $q(s)$ such that $\mathfrak{a}_{q(s)}$ is zero-dimensional, and then to get a good estimate on the exponent r of Lemma 2.2 if $g=1$. The importance of the relation $k\cdot Q - \sum_{i=1}^n x_i\partial_i$ in the annihilator of Q^s for homogeneous Q of degree k justifies

Definition 2.4. The *Euler operator* is $\mathcal{E} = x_1\partial_1 + \ldots + x_n\partial_n$.

2.2. **Arrangements in the plane.**

One has the following folklore result:

Proposition 2.5. *Let $\{a_i\}_{3\le i\le k}$ be $k-2$ pairwise distinct nonzero numbers. Then the Bernstein-Sato polynomial of $Q = xy(x+a_3y)\cdots(x+a_ky)$ divides*

$$(s+1)\prod_{i=0}^{2k-4}\left(s+\frac{i+2}{k}\right).$$

Proof. Consider the partial derivatives Q_x and Q_y of Q and the homogeneous forms $x^iy^jQ_x$ and $x^iy^jQ_y$ where $i+j=k$. We claim that these $2(k+1)$ forms of degree $(k+1)+k$ are linearly independent (and hence that $\langle Q_x,Q_y\rangle$ contains all monomials of degree at least $2k+1$).

To see this, let $M = \{m_{a,b}\}_{0\le a,b\le 2k+3}$ be the matrix whose (a,b)-coefficient is the coefficient of $x^{2k-3-b}y^b$ in $x^{k-2-a}y^aQ_y$ if $a\le k-2$, and the coefficient of $x^{2k-3-b}y^b$ in $x^{2k-3-a}y^{a-k+1}Q_x$ if $a>k-2$. The determinant of M is the resultant of $Q_x(1,y/x)$ and $Q_y(1,y/x)$. These cannot have a common root since $\langle Q_x(x,y),Q_y(x,y)\rangle$ is $\langle x,y\rangle$-primary. Hence M is of full rank and $\mathfrak{m}^k\langle Q_x,Q_y\rangle = \mathfrak{m}^{2k+1}$. Since $Q_x,Q_y\in\mathfrak{a}_{(s+1)}$, $\mathfrak{m}^{2k+1}\subseteq\mathfrak{a}_{(s+1)}$. Lemma 2.2 implies the claim. $\square$

Of course, a central arrangement Q of lines in the plane is an isolated singularity. The interesting question was therefore the precise determination of δ_Q.

2.3. **Estimates in dimension $n>2$.**

For the remainder of this section, Q is a generic central arrangement $Q=\prod_{i=1}^k H_i$. In order to estimate $b_Q(s)$ for $n>2, k>n+1$ we will consider a mix of the two main ideas for $n=2$. Namely, we had $\mathfrak{m}^{2k+1}\subseteq\mathfrak{a}_{(s+1)}$. The point is that admitting $(s+1)$ as a factor of $b_Q(s)$ allowed to capture (set-theoretically) the singular locus of the arrangement. This in conjunction with Lemma 2.2 gave a bound for the Bernstein-Sato polynomial.

The plan is to devise a mechanism that starts with $\langle Q\rangle\subseteq\mathfrak{a}_1$ and uses iterated multiplication with $(s+1)$ to enlarge $\mathfrak{a}_{q(s)}$. Progress is measured by the dimension of (the variety of) $\mathfrak{a}_{q(s)}$. This approach works well for generic arrangements, while for non-generic arrangements or other singularities better tricks seem to be needed.

It is crucial to understand the difference between the Jacobian ideal of Q and the ideal generated by all $(n-1)$-fold products of distinct elements in $\mathcal{A}$, and more generally the difference between the ideal of the Jacobian ideal of the variety defined by all $(r+1)$-fold products of distinct elements of $\mathcal{A}$ and the ideal of all r-fold products of distinct elements of $\mathcal{A}$.

Definition 2.6. If $\mathcal{A} = \{H_1, \dots, H_k\}$ is a list of linear homogeneous polynomials and $\alpha \in \mathbb{N}^k$ we say that $\prod_{i=1}^k H_i{}^{\alpha_i}$ is an *$\mathcal{A}$-monomial.* If each α_i is either 0 or 1, we call the $\mathcal{A}$-monomial *squarefree.*

Definition 2.7. We define polynomials $\Delta_{J,I,N}(Q)$ for a given central arrangement $Q = H_1 \cdots H_k$. To this end let $N = \{\lambda_1, \dots, \lambda_n\} \subseteq \{1, \dots, k\}$ be a set of indices serving as a coordinate system. Let $v_{\lambda_1}, \dots, v_{\lambda_n}$ be n appropriate $\mathbb{C}$-linear combinations of $\partial_1, \dots, \partial_n$ such that $v_{\lambda_i} \bullet (H_{\lambda_j}) = \delta_{i,j}$.

Let $I \subseteq \{1, \dots, k\}$ with $|I| \geq k-n+1$. Set $\check{I} = \{1, \dots, k\} \setminus (I \cup N)$, $\hat{I} = I \cap N$, $H_I = \prod_{i \in I} H_i$. Observe that $|\hat{I}| = |I| - k + n + |\check{I}|$.

Let $\rho_N(I) := |\check{I}| + 1 \leq |\hat{I}|$ and pick $J \subseteq \hat{I}$ with $|J| = \rho_N(I)$. We define $\Delta_{J,I,N}(Q)$ to be the $\rho_N(I) \times \rho_N(I)$-determinant and linear combination of squarefree $\mathcal{A}$-monomials of degree $|I|-1$

$$(2.1)\ \Delta_{J,I,N}(Q) = \det \begin{pmatrix} v_{j_1} \bullet (H_{\check{\imath}_1}) & \cdots & v_{j_1} \bullet (H_{\check{\imath}_{|\check{I}|}}) & v_{j_1} \bullet (H_I) \\ \vdots & & \vdots & \vdots \\ v_{j_{|J|}} \bullet (H_{\check{\imath}_1}) & \cdots & v_{j_{|J|}} \bullet (H_{\check{\imath}_{|\check{I}|}}) & v_{j_{|J|}} \bullet (H_I) \end{pmatrix}$$

where $\check{I} = \{\check{\imath}_1, \dots, \check{\imath}_{|\check{I}|}\}$ and $J = \{j_1, \dots, j_{|J|}\}$. If $\check{I}$ is empty, $\Delta_{J,I,N}(Q)$ is just $\nu_{j_1}(H_I)$. We emphasize that $\Delta_{J,I,N}(Q)$ is defined only if $|I| > k-n$. For a given I, let $\Delta_I(Q)$ be the set of all $\Delta_{J,I,N}(Q)$, varying over all possible N, and for each N over all J satisfying $J \subseteq \hat{I}$ and $|J| = \rho_N(I)$.

Finally, put for $r > k-n$

$$\Delta_r(Q) = \langle \Delta_{J,I,N}(Q) : |I| = r \rangle + \langle H_I : |I| = r \rangle$$

and for all r

$$\Sigma_r(Q) = \langle H_I : |I| = r \rangle.$$

Remark 2.8. The ideal $\Sigma_{r-1}(Q)$ describes set-theoretically the locus where simultaneously $k-r+2$ of the H_i vanish (for the case $r < k-n+2$ see see Lemma 2.9), while $\Delta_r(Q)$ is the Jacobian ideal of the variety to $\Sigma_r(Q)$. It is clear that

$$\Sigma_{r-1}(Q) \supseteq \Delta_r(Q) = \langle \{\Delta_{J,I,N}(Q) : |I| = r\} \rangle + \Sigma_r(Q) \supseteq \Sigma_r(Q).$$

The following is easily checked (since Q is generic):

Lemma 2.9. *If $r \leq k-n+1$ then $\Sigma_r(Q) = \mathfrak{m}^r = \Delta_r(Q)$.* □

We can describe the "difference" of $\Delta_r(Q)$ and $\Sigma_{r-1}(Q)$ as follows:

Proposition 2.10. *Let $k \geq r \geq k-n+1$. Then*

$$\operatorname{ann}_{R_n}\left(\frac{\Sigma_{r-1}(Q)}{\Delta_r(Q)}\right) \supseteq \mathfrak{m}^{k-n}.$$

Proof. First let $k = n$ so that $n \geq r \geq 1$. In this case $\mathcal{A}$-monomials and monomials are the same concepts. Then $\Sigma_{r-1}(Q)$ is the ideal of all squarefree ($\mathcal{A}$-)monomials of degree $r-1$, and $\Delta_r(Q)$ is the ideal of all squarefree ($\mathcal{A}$-)monomials of degree r as well as all partial derivatives of these monomials. Clearly then in this case $\Delta_r(Q) = \Sigma_{r-1}(Q)$.

We shall prove the claim by induction on $k-n$ and we assume now that $k>n$. Let H_I be a squarefree $\mathcal{A}$-monomial of degree r. We must show that $mH_I/H_i \in \Delta_r(Q)$ for all $i\in I$ and all $m\in\mathfrak{m}^{k-n}$.

Pick $N\subseteq\{1,\dots,k\}$ with $|N|=n$ and write $m=\sum_{j\in N} m_jH_j$, $m_j\in\mathfrak{m}^{k-n-1}$. Consider the summands $m_jH_jH_I/H_i$ in mH_I/H_i. If $j=i$ or if $j\notin I$, then certainly $H_jH_I/H_i\in\Delta_r(Q)$. Thus we are reduced to showing that if $i\neq j\in I$ then $m_jH_jH_I/H_i\in\Delta_r(Q)$.

Note that if $k\geq r\geq k-n+1$ then $k-1\geq r-1\geq k-1-n+1$. Since H_I/H_j is a squarefree $\mathcal{A}\setminus\{H_j\}$-monomial of degree $(r-1)$ we may use the induction hypothesis on the arrangement to Q/H_j with $k-1\geq n$ factors. Hence for $i\neq j\in I$ there are $q_j\in\Delta_{r-1}(Q/H_j)$ such that $m_jH_I/H_iH_j=q_j$. Then $m_jH_jH_I/H_i={H_j}^2q_j$ so that it suffices to show that

$$[q_j\in\Delta_{r-1}(Q/H_j)]\Longrightarrow\left[{H_j}^2q_j\in\Delta_r(Q)\right].$$

It is sufficient to check this for q_j being equal to one of the two types of generators for $\Delta_{r-1}(Q/H_j)$, namely $H_{I'}$ and the determinants $\Delta_{J',I',N'}(Q/H_j)$, where as usual $J',I',N'\subseteq\{1,\dots,k\}\setminus\{j\}$, $|N'|=n$ and $|I'|=r-1$. If $q_j=H_{I'}$ then ${H_j}^2q_j=H_jH_{I'\cup\{j\}}\in\Delta_r(Q)$. So assume that $q_j=\Delta_{J',I',N'}(Q/H_j)$.

Multiplication of $\Delta_{J',I',N'}(Q/H_j)$ by ${H_j}^2$ can be achieved by multiplying the last column of the defining matrix (2.1) of $\Delta_{J',I',N'}(Q/H_j)$ by ${H_j}^2$. Let in that context $j_t\in N'$ and v_{j_t} be the corresponding derivation relative to N'. Then

$${H_j}^2v'_{j_t}\bullet(H_{I'})=H_jv'_{j_t}\bullet(H_{I'\cup\{j\}})-H_{I'\cup\{j\}}v'_{j_t}\bullet(H_j).$$

Thus, ${H_j}^2\Delta_{J',I',N'}(Q/H_j)=H_j\Delta_{J',I',N'}(Q)$ modulo $\langle H_{I'\cup\{j\}}\rangle$. As $H_{I'\cup\{j\}}\in\Delta_r(Q)$, ${H_j}^2\Delta_{r-1}(Q/H_j)\subseteq\Delta_r(Q)$. The proposition follows hence by induction. $\square$

Recall that $\Delta_I(Q)$ is the collection of all $\Delta_{J,I,N}(Q)$ for fixed I. We now relate the ideals $\Sigma_{r-1}(Q)$ and $\Delta_r(Q)$ to ideals $\mathfrak{a}_{q(s)}$ and give hence the latter ideals geometric meaning.

Lemma 2.11. *Fix integers $r\geq k-n+2$, and t. Suppose $\mathfrak{m}^tH_I\subseteq\mathfrak{a}_{q(s)}$ for some I with $|I|=r$. Then $\mathfrak{m}^{t+1}\Delta_I(Q)\subseteq\mathfrak{a}_{(s+1)q(s)}$. In particular,*

$$\left[\mathfrak{m}^t\Sigma_r(Q)\subseteq\mathfrak{a}_{q(s)}\right]\Longrightarrow\left[\mathfrak{m}^{t+1}\Delta_r(Q)\subseteq\mathfrak{a}_{(s+1)q(s)}\right].$$

Proof. Pick a specific $\Delta_{J,I,N}(Q)$ and a monomial m of degree t. In particular, this means that a coordinate system $H_{\lambda_1},\dots,H_{\lambda_n}$ and derivations $v_{\lambda_1},\dots,v_{\lambda_n}$ have been chosen. Consider the effect of v_j on mH_IQ^s for $j\in J\,(\subseteq I\cap N)$:

$$\begin{aligned}\frac{v_j\bullet(mH_IQ^s)}{Q^s} &= \frac{1}{Q}\left(v_j\bullet(m)H_IQ+mQv_j\bullet(H_I)+smH_Iv_j\bullet(Q)\right)\\ &= v_j\bullet(m)H_I+(s+1)mv_j\bullet(H_I)+\sum_{i\in\{1,\dots,k\}\setminus I} smH_I\frac{v_j\bullet(H_i)}{H_i}\end{aligned}$$

The sum has only poles of order one. These poles occur exactly along all hyperplanes in $\check{I}$ since $v_j\bullet(H_i)=0$ if $i\neq j, i\in N$. (Note that $j\in J\subseteq I$ is not index of a summand.) The $|\check{I}|+1$ distinct elements of J give rise to that many expressions of the type shown. Hence there is a nontrivial $\mathbb{C}$-linear combination of the $v_j\bullet(mH_IQ^s)$ without poles; by construction this linear combination is in $\mathfrak{a}_{q(s)}$. It is easy to see that the desired expression results in $(s+1)m\Delta_{J,I,N}(Q)+V(m)H_I$

where $V(m)$ is a linear combination in the $v_j \bullet (m)$. As $x_i v_j \bullet (m) H_I \in \mathfrak{m}^t H_I \subseteq \mathfrak{a}_{q(s)}$, $x_i m \Delta_{J,I,N}(Q) \in \mathfrak{a}_{(s+1)q(s)}$ for all i, J, N and so $\mathfrak{m}^{t+1}\Delta_I(Q) \subseteq \mathfrak{a}_{(s+1)q(s)}$.

To prove the final assertion, note that $\Delta_r(Q)$ is generated by all $\Delta_I(Q)$, $|I| = r$ and all H_I, $|I| = r$. One then only needs to observe that all H_I with $|I| = r$ are already in $\Sigma_r(Q)$. $\square$

One can now conclude alternately from 2.10 and 2.11 that

$$\begin{array}{rcl}
\Sigma_k(Q) & \subseteq & \mathfrak{a}_1, \\
\mathfrak{m}\Delta_k(Q) & \subseteq & \mathfrak{a}_{(s+1)}, \\
\mathfrak{m}^{k-n+1}\Sigma_{k-1}(Q) & \subseteq & \mathfrak{a}_{(s+1)}, \\
& \vdots & \\
\mathfrak{m}^{(k-n+1)(n-2)+1}\Delta_{k-n+2}(Q) & \subseteq & \mathfrak{a}_{(s+1)^{n-1}}, \\
\mathfrak{m}^{(k-n+1)(n-1)}\Sigma_{k-n+1}(Q) & \subseteq & \mathfrak{a}_{(s+1)^{n-1}},
\end{array}$$

and since $\Sigma_{k-n+1}(Q) = \mathfrak{m}^{k-n+1}$, $\mathfrak{m}^{(k-n+1)n} \subseteq \mathfrak{a}_{(s+1)^{n-1}}$. It is very intriguing how in the above sequence of containments an extra factor of $(s+1)$ in $q(s)$ allows each time to reduce the dimension of $\mathfrak{a}_{q(s)}$ and in fact to enlarge $\mathfrak{a}_{q(s)}$ to an ideal with radical equal to the singular locus of $\mathfrak{a}_{q(s)}$. One might compare this to the example in Remark 2.3.

The remainder of this section is devoted to decreasing substantially the exponent of $\mathfrak{m}$ in the final row of the display above.

Proposition 2.12. *For all $r \in \mathbb{N}$ with $k - n + 1 \leq r \leq k + 1$,*

$$\mathfrak{m}^{2k-n-1} \cap \Sigma_{r-1}(Q) \subseteq \mathfrak{a}_{(s+1)^{k-r+1}}.$$

Proof. We shall proceed by decreasing induction on r. We know that

$$\mathfrak{m}^{2k-n-1} \cap \Sigma_k(Q) \subseteq \Sigma_k(Q) = \langle Q \rangle \subseteq \mathfrak{a}_{(s+1)^0}.$$

Assume then that $k - n + 1 \leq r \leq k$ and that $\mathfrak{m}^{2k-n-1} \cap \Sigma_r(Q) \subseteq \mathfrak{a}_{(s+1)^{k-r}}$. Since $\Sigma_r(Q) \subseteq \mathfrak{m}$ is homogeneous of degree r, this implies that

$$\mathfrak{m}^{2k-n-1-r} \cdot \Sigma_r(Q) \subseteq \mathfrak{a}_{(s+1)^{k-r}}.$$

We need to show that $\mathfrak{m}^{2k-n-1} \cap \Sigma_{r-1}(Q) \subseteq \mathfrak{a}_{(s+1)^{k-r+1}}$ in order to get the induction going. For this, we consider $\Delta_r(Q)$. Let Δ be a generator of $\Delta_r(Q)$. Either $\Delta = H_I$ and $|I| = r$, in which case $\Delta \in \Sigma_r(Q)$. Or, $\Delta = \Delta_{J,I,N}(Q)$ with $|I| = r$. In that case, Lemma 2.11 together with $\mathfrak{m}^{2k-n-1-r} \cdot \Sigma_r(Q) \subseteq \mathfrak{a}_{(s+1)^{k-r}}$ implies that $\mathfrak{m}^{2k-n-r} \cdot \Delta \subseteq \mathfrak{a}_{(s+1)^{k-r+1}}$. Therefore our hypotheses imply that

$$\mathfrak{m}^{2k-n-1} \cap \Delta_r(Q) \subseteq \mathfrak{a}_{(s+1)^{k-r+1}}.$$

But then,

$$\begin{aligned}
\mathfrak{m}^{2k-n-1}\cap\Sigma_{r-1}(Q) &= \mathfrak{m}^{2k-n-1-(r-1)}\Sigma_{r-1}(Q)\\
&\quad (\Sigma_{r-1}(Q) \text{ is homogeneously generated in degree } r-1)\\
&= \mathfrak{m}^{k-r}\mathfrak{m}^{k-n}\Sigma_{r-1}(Q)\\
&\subseteq \mathfrak{m}^{k-r}(\Delta_r(Q)\cap\mathfrak{m}^{k-n+r-1})\\
&\quad (\text{by Proposition 2.10})\\
&\subseteq \mathfrak{m}^{2k-n-1}\cap\Delta_r(Q)\\
&\subseteq \mathfrak{a}_{(s+1)^{k-r+1}}.
\end{aligned}$$

□

This proposition says that sufficiently high degree parts of the ideal defining the higher iterated singular loci of $\mathcal{A}$ are contained in certain $\mathfrak{a}_{q(s)}$. It gives quite directly a bound for the Bernstein-Sato polynomial:

Theorem 2.13. *The Bernstein-Sato polynomial of the central generic arrangement $Q = H_1\cdots H_k$ divides*

$$(s+1)^{n-1}\prod_{i=0}^{2k-n-2}\left(s+\frac{i+n}{k}\right). \tag{2.2}$$

Proof. The previous proposition shows (with $r = k-n+2$) that $\mathfrak{m}^{2k-n-1}\cap\Sigma_{k-n+1}\subseteq \mathfrak{a}_{(s+1)^{n-1}}$. By Lemma 2.9, $\Sigma_{k-n+1}(Q) = \mathfrak{m}^{k-n+1}$. Thus, $\mathfrak{m}^{2k-n-1}\subseteq\mathfrak{a}_{(s+1)^{n-1}}$. We conclude now as in Lemma 2.2. □

In the next two sections we show that this estimate is in essence the correct answer.

3. Remarks on a conjecture by Orlik and Randell

Let $Q:\mathbb{C}^n\to\mathbb{C}$ be a homogeneous polynomial map, denote by X_α the preimage $Q^{-1}(\alpha)$ for $\alpha\in\mathbb{C}\setminus\{0\}$ and let X be the fiber over zero. As Q is homogeneous the X_α are all isomorphic and smooth. Let $\tilde{\mathbb{C}}^\times$ be the universal cover of $\mathbb{C}^\times = \mathbb{C}\setminus\{0\}$, and $\tilde{X}$ the fiber product of $\tilde{\mathbb{C}}^\times$ and $\mathbb{C}^n\setminus X$ over $\mathbb{C}^\times$. Then $(\alpha,x)\to(\alpha+2\pi,x)$ is a diffeomorphism of $\tilde{X}$ and therefore induces an isomorphism μ on the cohomology $H^*(X_\alpha,\mathbb{C})$, the Picard-Lefschetz monodromy [6, 13, 15]. If in addition X has an isolated singularity then X_α is homotopy equivalent to a bouquet of $(n-1)$-spheres [27] and so the only (reduced) cohomology of the fiber is in degree $n-1$. The roots of the minimal polynomial $a_\mu(s)$ of μ are in that case obtained from the roots of the Bernstein-Sato polynomial of Q by $\lambda\to e^{2\pi i\lambda}$ [25]. The multiplicities remain mysterious, however. If X is not an isolated singularity, the X_α have cohomology in degrees other than $n-1$ and the monodromy acts on all these cohomology groups. The monodromy is then not so nicely related to the Bernstein-Sato polynomial and not well understood.

3.1. The conjecture.

The natural projection $R_n\twoheadrightarrow R_n/\langle Q-\alpha\rangle$ induces a map of differentials $\Omega\to\Omega_\alpha$ which in turn induces a surjective map of de Rham complexes $\pi:\Omega^\bullet\twoheadrightarrow\Omega^\bullet_\alpha$ where Ω_α are the $\mathbb{C}$-linear differentials on $R_n/\langle Q-\alpha\rangle$ and $\Omega^\bullet_\alpha$ is the de Rham complex on X_α. It is an interesting and open question to determine explicit formulæ for generators of the cohomology of $\Omega^\bullet_\alpha$, i.e. forms on $\mathbb{C}^n$ that restrict to generators of $H^i(\Omega^\bullet_\alpha)$, $i \le n-1$. If X has an isolated singularity then the Jacobian ideal $\mathfrak{A}$ is Artinian, the dimension of the vector space $R_n/\mathfrak{A}$ equals $\dim_{\mathbb{C}}(H^{n-1}_{\mathrm{DR}}(X_\alpha,\mathbb{C}))$, and the elements of $R_n/\mathfrak{A}$ can be identified with the classes in $H^{n-1}_{\mathrm{DR}}(X_\alpha,\mathbb{C})$. Namely, $\overline{g} \in R_n/\mathfrak{A}$ corresponds to $g\omega$ where

$$\omega = \sum_{i=1}^{n}(-1)^{i-1}x_i\, dx_1 \wedge \ldots \wedge \widehat{dx_i} \wedge \ldots \wedge dx_n \tag{3.1}$$

and the hat indicates omission.

For the remainder of this section let Q be a reduced polynomial describing a central generic arrangement, $Q = H_1 \cdots H_k$. We let as before $\mathcal{A} = \{H_1, \ldots , H_k\}$. In [31] (Proposition 3.9) it is proved that every cohomology class in $H^{n-1}(\Omega^\bullet_\alpha)$ is of the form $\overline{\pi(g\omega)}$ for some $g \in R_n$, and that

$$\dim_{\mathbb{C}}(H^{n-1}(\Omega^\bullet_\alpha)) = \binom{k-2}{n-2} + k\binom{k-2}{n-1}.$$

The authors make a conjecture which states roughly that g may be chosen to be homogeneous and that Milnor fibers of central generic arrangements have a cohomology description similar to the isolated singularity case.

By $(R_n)_r$ we denote the homogeneous elements in R_n of degree r. The following vector space is central to the ideas of Orlik and Randell.

Definition 3.1. We denote by μ a subset of $\mathcal{A}$ of cardinality $n-1$. We write then $J_\mu(a)$ with $a \in R_n$ for the Jacobian determinant associated to $H_{\mu_1}, \ldots , H_{\mu_{n-1}}, a$. We also denote by Q_μ the product of all H_i with $i \notin \mu$, its degree is hence $k-n+1$. In our previous notation, Q_μ was H_I with $I = \mathcal{A} \setminus \mu$.

With these notations, let E be the vector space in R_n generated by all elements of the form

$$\deg(a)a\, J_\mu(Q_\mu) - kQ_\mu\, J_\mu(a), \tag{3.2}$$

varying over all homogeneous $a \in R_n$. It is not an R_n-ideal.

Conjecture 3.2 (Orlik-Randell, [31])**.** Consider the fiber $X_1 = \mathrm{Var}(Q-1)$. There is a finite dimensional homogeneous vector space $U \subset R_n$ such that

(1) $R_n = E \oplus (\mathbb{C}[Q] \otimes U)$;
(2) the map $U \to H^{n-1}(X_1,\mathbb{C})$ given by $g \to \overline{\pi(g\omega)}$ is an isomorphism, and $\Omega^{n-1}_\alpha = \pi(U\omega) \oplus d\Omega^{n-2}_\alpha$.
(3) The dimensions u_r of U_r, the graded pieces of U of degree r, are as follows:

$$u_r = \begin{cases} \binom{r+n-1}{n-1} & \text{for } 0 \le r \le k-n, \\ \binom{k-2}{n-1} & \text{for } k-n+1 \le r \le k-1, \\ \binom{k-2}{n-1} - \binom{r-k+n-1}{n-1} & \text{for } k \le r \le 2k-n-2. \end{cases}$$

In this section we will prove that if k does not divide $r-k+n$ then the dimension of $(R_n/E)_r$ is bounded by $\binom{k-1}{n-1}$ and that strict inequality holds if additionally $r > k$. In the next section we will see that $(R_n)_r = E_r + \langle Q\rangle_r$ for $r \ge 2k-n-1$. This will imply that $(R_n/(E + \langle Q-1\rangle))_r$ is nonzero exactly if $0 \le r \le 2k-n-2$, and that for $k-n-1 \le r \le k$ its dimension is exactly as the conjecture by Orlik and Randell predicts.

It is worth pointing out that the vector space E is too small if Q is an arrangement that is not generic. For example, with $Q = xyz(x+y)(x+z)$ as in Example 4.16 one obtains that the dimension of $(R_n/E + \langle Q\rangle)_r$ is 2 whenever r is at least 5.

3.2. **Generators for U.**
We now consider the question of finding generators for U. By Lemma 2.9, $(R_n)_{k-n+1}$ is generated by the set of all Q_μ as a vector space. Then $(R_n)_r$ is for $r > k-n+1$ generated by $\mathfrak{m}^{r-k+n-1} \cdot \Sigma_{k-n+1}(Q)$. We claim that we may pick vector space generators $G = \{g_i\}$ for $(R_n)_r$, $r > k-n+1$, such that

a) each g_i is an $\mathcal{A}$-monomial,
b) each g_i is a multiple of some Q_μ.

To see this, observe that $(R_n)_r = \left(\mathfrak{m}^{r-k+n-1}\right)_{r-k+n-1} \cdot (\Sigma_{k-n+1}(Q))_{k-n+1}$. Since $\mathcal{A}$ is essential, Lemma 2.9 completes the argument. We call an element of R_n satisfying these two conditions a *standard product.*

We shall now prove that there are no more than $\binom{k-2}{n-1}$ standard products necessary to generate $(R_n/(E+\langle Q-1\rangle))_r$. For $k-n+1 \le r < k$ this is exactly the number stipulated by Conjecture 3.2. We will do this by showing that the relations in E may be used to eliminate the majority of all summands in a typical element of $(R_n)_r/E_r$. In order to do this, we need to study the nature of the relations in E. To get started, note that

$$[H_j \in \mathcal{A} \setminus \mu] \Longrightarrow [J_\mu(H_j) \neq 0]\,.$$

We will now show that every generator (3.2) of E induces a syzygy between $k-n+1$ squarefree $\mathcal{A}$-monomials of degree $k-n$.

Lemma 3.3. *Let $a \in (R_n)_r$ be an $\mathcal{A}$-monomial of positive degree r such that $k \not| \, r$, and pick $n-1$ distinct factors μ of Q. Consider the corresponding element*

$$\deg(a) a J_\mu(Q_\mu) - kQ_\mu J_\mu(a) \tag{3.3}$$

of E. In this expression (using the product rule for computing the Jacobian) the first term contributes $k-n+1$ summands of the form $\deg(a)a\frac{Q_\mu}{H_i}J_\mu(H_i)$ where H_i runs through the factors of Q_μ. Similarly the second term contributes $\deg(a)$ summands of the form $kQ_\mu\frac{a}{a_i}J_\mu(a_i)$ with a_i running through the factors of a. We claim that all nonzero summands in the latter set (apart from constant factors) appear as nonzero summands in the former set. Moreover, for each summand that is nonzero on both sides the coefficients are different.

Proof. There are two main cases: $a_i \in \mu$ and $a_i \notin \mu$. If $a_i \in \mu$ then $J_\mu(a_i)$ is a determinant with a repeated column, and hence the summand $Q_\mu\frac{a}{a_i}J_\mu(a_i)$ is zero. On the other hand, $H_i \notin \mu$ gives a summand $\deg(a)a\frac{Q_\mu}{H_i}J_\mu(H_i) \neq 0$. So the left term in (3.3) gives $k-n+1$ nonzero $\mathcal{A}$-monomials with nonzero coefficients. If $a_i \notin \mu$, then $a_i = H_j$ (say), and $\frac{Q_\mu}{H_j}a = Q_\mu\frac{a}{a_i}$. Let t be the multiplicity of H_j in a, $a = a' \cdot {H_j}^t$. In (3.3) the first term contributes $\deg(a)Q_\mu\frac{a}{H_j}J_\mu(H_j)$ while the second yields t times $-kQ_\mu\frac{a}{H_j}J_\mu(H_j)$ by the product rule. So the total number of copies of $\frac{aQ_\mu}{H_j}J_\mu(H_j)$ in (3.3) is $\deg(a) - kt$.

As k is not a divisor of $\deg(a) = r$, each generator of E_r gives rise to a relation between exactly $k-n+1$ of our generators of $(R_n)_r$, corresponding to the divisors of Q_μ. ☐

Remark 3.4. Suppose that in a linear combination of $\mathcal{A}$-monomials the previous lemma is used to eliminate $Q_\mu \frac{a}{H_i} J_\mu(H_i)$. Then the replacing $\mathcal{A}$-monomials are of the form $Q_\mu \frac{a}{H_j} J_\mu(H_j)$ where $H_j \notin \mu$.

We now show how to use Lemma 3.3 to limit the dimension of $(R_n)_r/E_r$.

Proposition 3.5. *Let $r \in \mathbb{N}$, $k-n+1 \leq r$, and $k \not| (r-k+n)$. The (cosets of) $\mathcal{A}$-monomials of the form*

$$H_{i_1} \cdots H_{i_{k-n-1}} H_{k-1} {H_k}^{r-k+n}, \quad i_1 < \ldots < i_{k-n-1} < k-1 \tag{3.4}$$

span $(R_n/E)_r$ and therefore generate $\left(H^{n-1}_{\mathrm{DR}}(Q^{-1}(1), \mathbb{C})\right)_r$.

Proof. Let $P \in (R_n)_r$ be a standard product. We prove that it may be replaced by a linear combination of $\mathcal{A}$-monomials of the stipulated form. Here are three ways of modifying a linear combination of $\mathcal{A}$-monomials modulo E:

(1) If P uses $l > k-n+1$ distinct factors of $\mathcal{A}$ we can write $P = P'Q_\mu$ for a suitable μ and we can assume that $H_k \in \mu$. That means that $H_k \not| Q_\mu$ and the multiplicity of H_k in P' is of course at most $r-k+n-1$. Let $i_0 = \min\{i : H_i \notin \mu\}$ and $\mu' = \mu \cup \{H_{i_0}\} \setminus \{H_k\}$, so $Q_{\mu'} = H_k Q_\mu / H_{i_0}$. Consider the element of E given by $(r-k+n)P'H_{i_0} J_{\mu'}(Q_{\mu'}) - kQ_{\mu'} J_{\mu'}(P'H_{i_0})$. It is a linear dependence modulo E between $P'H_{i_0}Q_{\mu'}/H_k = P$ on one side and terms of the form $P'H_{i_0}Q_{\mu'}/H_i = P'H_kQ_\mu/H_i$ for $H_k \neq H_i \in \mathcal{A} \setminus \mu'$ on the other, with no coefficient equal to zero. It follows that $P = P'Q_\mu$ may, modulo E, be replaced by a linear combination of standard products with a higher power of H_k in each of them than in P and l or $l-1$ distinct factors. Note that each replacing $\mathcal{A}$-monomial has multiplicity of H_k at most $r-k+n$.
(2) Suppose now that P has exactly $k-n+1$ distinct factors, but that H_k is not one of them. Let Q_μ be the product of all distinct factors of P, and set $P = P'Q_\mu$. Let $i_0 = \min\{i : H_i \notin \mu\}$ and set $\mu' = \mu \cup \{H_{i_0}\} \setminus \{H_k\}$. The relation $(r-k+n)H_{i_0}P'J_{\mu'}(Q_{\mu'}) - kQ_{\mu'}J_{\mu'}(P'H_{i_0})$ allows to replace P by a linear combination of standard products with $k-n+1$ or $k-n+2$ distinct factors (depending on the multiplicity of H_{i_0} in P') such that H_k divides each of the new standard products.
(3) Now assume that P is a standard product with exactly $k-n+1$ distinct factors and assume furthermore that H_k divides P with multiplicity $l < r-k+n$. Let μ be such that Q_μ divides P. Since the arrangement is generic, the $n-1$ elements of μ, together with H_k, span the maximal ideal and thus if $i_0 = \min\{i : {H_i}^2 \mid P\}$ then one factor H_{i_0} of P may be replaced by an appropriate linear combination in H_k and the elements of μ. This creates a linear combination of $(n-1)$ standard products with $k-n+2$ distinct factors in each summand where H_k has multiplicity l, and one $\mathcal{A}$-monomial with $k-n+1$ factors where the H_k-degree is $l+1$.

Starting with any standard product of degree r, using these steps in appropriate order will produce a linear combination of standard products with exactly $k-n+1$ factors and multiplicity $r-k+n$ in H_k. This is because after every execution of Step 1 and 2, the degree in H_k goes up, and after each execution of Step 3 we may do Step 1 at least once on the $n-1$ standard products with $k-n+1$ factors.

Now let $P = H_{i_1} \cdots H_{i_{k-n}} {H_k}^{r-k+n}$ with $i_1 < i_2 \cdots < i_{k-n} < k-1$. Let μ be such that $Q_\mu = H_{i_1} \cdots H_{i_{k-n}} H_{k-1}$, in particular $H_k \in \mu$. Then

$$E \ni (r-k+n){H_k}^{r-k+n} J_\mu(Q_\mu) - kQ_\mu J_\mu({H_k}^{r-k+n}) = (r-k+n){H_k}^{r-k+n} J_\mu(Q_\mu)$$

allows to replace P by a sum of $\mathcal{A}$-monomials each of which has $k-n+1$ distinct $\mathcal{A}$-factors, and each of which is divisible by $H_{k-1}{H_k}^{r-k+n}$ (note that the only term that might fail to have H_{k-1} in it disappears because $J_\mu({H_k}^{r-k+n}) = 0$ as $H_k \in \mu$). Thus, modulo E, P is equivalent to a linear combination of $\mathcal{A}$-monomials of type (3.4).

The condition $k \not| \, (r-k+n)$ is needed because otherwise Lemma 3.3 does not work. $\square$

Remark 3.6. Note that there are exactly $\binom{k-2}{k-n-1} = \binom{k-2}{n-1}$ $\mathcal{A}$-monomials of type (3.4). It follows that $\dim(R_n/E)_r \leq \binom{k-2}{n-1}$ unless k divides $r-k+n$. Also, if $r = k-n$ the conjecture says that $\binom{k-2}{n-1}$ generators for $(R_n/E)_r$ are not enough. So in a sense this is an optimal estimate. In the following section we will see that $(R_n)_r = E_r$ along $Q^{-1}(1)$ for $r > 2k-n-2$. We finish this section with a lemma that will be used in the next section to prove that $(R_n)_r \neq E_r$ for $k \leq r \leq 2k-n-2$.

Lemma 3.7. *If $r \geq k$ and $k \not| \, (r-k+n)$ then $\dim_{\mathbb{C}}(R_n/E + \langle Q\rangle)_r \leq \binom{k-2}{n-1} - 1$.*

Proof. The proof of Proposition 3.5 contains a procedure to turn $Q{H_k}^{r-k}$ into a sum of $\mathcal{A}$-monomials of the form (3.4). One may do so using only Step 1 of that proof. In fact, if $P = H_1 \cdots H_{k-n-1} H_{i_1} \cdots H_{i_j} H_{k-1} {H_k}^{r-k+n-j}$ for $k-n-1 < i_1 < \cdots < i_j < k-1$, then the relation (3.3) induced by $Q_\mu = H_1 \cdots H_{k-n-1} H_{i_1} H_k$ and $a = P/Q_\mu$ allows to replace P by a sum of $\mathcal{A}$-monomials each of which is divisible by $H_{k-1}{H_k}^{r-k+n+1-j}$, each of which has only H_k as repeated factor, and precisely one of which is a nonzero multiple of $H_1 \cdots H_{k-n-1}$. Therefore rewriting $Q{H_k}^{r-k}$ only using Step 1 (and only $Q_\mu = H_1 \cdots H_{k-n-1} \cdot H_{i_1} H_k$ with $k-n \leq i_1 < k$) gives a relation modulo E between the products of (3.4) where the coefficient for $H_1 \cdots H_{k-n-1} H_{k-1} {H_k}^{r-k+n}$ is nonzero. Hence in particular, $(R_n/E + \langle Q\rangle)_r$ has dimension at most $\binom{k-2}{n-1} - 1$. $\square$

We have shown that filtering $R_n/(E+\langle Q-1\rangle)$ by degree, the r-th graded piece has dimension at most $\binom{k-2}{n-1} - 1$ unless k divides $r-k+n$. Moreover, $(R_n/E)_r = (R_n)_r$ for $r \leq k-n$.

4. Integration, Restriction and Bernstein-Sato polynomials

If Q is radical and describes a generic arrangement then we have seen that

- $b_Q(s)$ is a divisor of $(s+1)^{n-1} \prod_{i=0}^{2k-n-2} (s + \frac{i+n}{k})$;
- $\dim_{\mathbb{C}}(R_n/E + \langle Q\rangle)_r \leq \binom{k-2}{n-1}$ if $k \not| \, (r-k+n)$;
- the inequality of the previous item is strict if in addition $r < k-n$ or $r \geq k$.

We now prove, among other things, that each homogeneous generator of the top cohomology group $H^{n-1}_{\mathrm{DR}}(Q^{-1}(1), \mathbb{C})$ of the Milnor fiber gives for all homogeneous polynomials Q rise to a root of $b_Q(s)$ just as it is the case for homogeneous isolated singularities.

4.1. **Restriction and Integration.**

A central part of this section is occupied by effective methods for D-modules. In fact, we shall use in an abstract way algorithms that were pioneered by Oaku [28] and since have become the centerpiece of algorithmic D-module theory.

We shall first explain some basic facts about restriction and integration functors. Much more detailed explanations may be found in [28, 29, 30] and [38]. In particular, we only consider the situation of $n+1$ variables $x_1, \ldots, x_n, t$ and explain restriction to $t=0$ and integration along $\partial_1, \ldots, \partial_n$.

Definition 4.1. Let $\tilde{\Omega}_t = D_{x,t}/t \cdot D_{x,t}$ and $\Omega_\partial = D_{x,t}/\{\partial_1, \ldots, \partial_n\} \cdot D_{x,t}$.

The *restriction of the $D_{x,t}$-complex $A^\bullet$ to the subspace $t=0$* is the complex $\rho_t(A^\bullet) = \tilde{\Omega}_t \otimes^L_{D_{x,t}} A^\bullet$ considered as a complex in the category of D_x-modules.

The *integration of $A^\bullet$ along $\partial_1, \ldots, \partial_n$* is the complex $\mathrm{DR}(A^\bullet) = \Omega_\partial \otimes^L_{D_{x,t}} A^\bullet$ considered as a complex in the category of D_t-modules.

In the sequel we describe tools that may be used to compute restriction and integration.

Definitions 4.2. On the ring $D_{x,t}$ the *V_t-filtration* $F^l_t(D_{x,t})$ is the $\mathbb{C}$-linear span of all operators $x^\alpha \partial^\beta t^a \partial_t^b$ for which $a + l \geq b$. More generally, on a free $D_{x,t}$-module $A = \bigoplus_{j=1}^r D_{x,t} \cdot e_j$ we set

$$F^l_t(A[\mathfrak{m}]) = \sum_{j=1}^r F_t^{l-\mathfrak{m}(j)}(D_{x,t}) \cdot e_j,$$

where $\mathfrak{m}$ is any element of $\mathbb{Z}^r$ called the *shift vector*. A shift vector is tied to a fixed set of generators. The *V_t-degree* of an operator $P \in A[\mathfrak{m}]$ is the smallest $l = V_t\text{-deg}(P[\mathfrak{m}])$ such that $P \in F^l_t(A[\mathfrak{m}])$.

If M is a quotient of the free $D_{x,t}$-module $A = \bigoplus_1^r D_{x,t} \cdot e_j$, $M = A/I$, we define the V_t-filtration on M by $F^l_t(M[\mathfrak{m}]) = F^l_t(A[\mathfrak{m}]) + I$ and for submodules N of A by intersection: $F^l_t(N[\mathfrak{m}]) = F^l_t(A[\mathfrak{m}]) \cap N$.

Definitions 4.3. A complex of free $D_{x,t}$-modules

$$\cdots \to A^{i-1} \xrightarrow{\phi^{i-1}} A^i \xrightarrow{\phi^i} A^{i+1} \to \cdots$$

is said to be *V_t-strict* with respect to the shift vectors $\{\mathfrak{m}_i\}$ if

$$\phi^i\left(F^l_t(A^i[\mathfrak{m}_i])\right) \subseteq F^l_t(A^{i+1}[\mathfrak{m}_{i+1}])$$

and also

$$\mathrm{im}(\phi^{i-1}) \cap F^l_t(A^i[\mathfrak{m}_i]) = \mathrm{im}(\phi^{i-1}|_{F^l_t(A^{i-1}[\mathfrak{m}_{i-1}])})$$

for all i, l.

Set $\theta = t\partial_t$, the Euler operator for t. A $D_{x,t}$-module $M[\mathfrak{m}] = A[\mathfrak{m}]/I$ is called *specializable to $t=0$* if there is a polynomial $b(s)$ in a single variable such that

$$b(\theta + l) \cdot F^l_t(M[\mathfrak{m}]) \quad \subseteq \quad F^{l-1}_t(M[\mathfrak{m}]) \tag{4.1}$$

for all l (cf. [18, 29]). Holonomic modules are specializable. Introducing

$$\mathrm{gr}^l_t(M[\mathfrak{m}]) = (F^l_t(M[\mathfrak{m}]))/(F^{l-1}_t(M[\mathfrak{m}])),$$

this can be written as

$$b(\theta + l) \cdot \mathrm{gr}^l_t(M[\mathfrak{m}]) = 0.$$

The monic polynomial $b(\theta)$ of least degree satisfying an equation of the type (4.1) is called the *b-function for restriction of $M[\mathfrak{m}]$ to $t=0$.*

By [30] (Proposition 3.8) and [38] every complex admits a V-strict resolution. In the theorems to follow, the meaning of filtration on restriction and integration complexes is as in [38], Definition 5.6.

Theorem 4.4 ([28, 30, 38]). *Let $(A^\bullet[\mathfrak{m}_\bullet],\delta^\bullet)$ be a V_t-strict complex of free $D_{x,t}$-modules with holonomic cohomology. The restriction $\rho_t(A^\bullet[\mathfrak{m}_\bullet])$ of $A^\bullet[\mathfrak{m}_\bullet]$ to $t=0$ can be computed as follows:*

1. *Compute the b-function $b_{A^\bullet[\mathfrak{m}_\bullet]}(s)$ for restriction of $A^\bullet[\mathfrak{m}_\bullet]$ to $t=0$.*
2. *Find an integer l_1 with $\left[b_{A^\bullet[\mathfrak{m}_\bullet]}(l)=0, l\in\mathbb{Z}\right]\Rightarrow[l\leq l_1]$.*
3. *$\rho_t(A^\bullet[\mathfrak{m}_\bullet])$ is quasi-isomorphic to the complex*

$$\cdots\to F_t^{l_1}(\tilde\Omega_t\otimes_{D_{x,t}}A^i[\mathfrak{m}_i])\to F_t^{l_1}(\tilde\Omega_t\otimes_{D_{x,t}}A^{i+1}[\mathfrak{m}_{i+1}])\to\cdots \tag{4.2}$$

This is a complex of free finitely generated D_x-modules and a representative of $\rho_t(A^\bullet[\mathfrak{m}_\bullet])$. Moreover, if a cohomology class in $\rho_t(A^\bullet[\mathfrak{m}_\bullet])$ has V_t-degree d then d is a zero of $b_{A^\bullet[\mathfrak{m}_\bullet]}(s)$. □

In order to compute the integration along $\partial_1,\ldots,\partial_n$ one defines a filtration by

$$F_\partial^l(D_{x,t})=\{x^\alpha\partial^\beta t^a\partial_t^b : |\alpha|\leq|\beta|+l\}.$$

With $\tilde{\mathcal{E}}=-\partial_1x_1-\ldots-\partial_nx_n$ the *b-function for integration* of the module M is the least degree monic polynomial $\tilde b(s)$ such that

$$\tilde b(\tilde{\mathcal{E}}+l)\cdot F_\partial^l(M)\subseteq F_\partial^{l-1}(M).$$

Then the integration complex $\mathrm{DR}(M)$ of M is quasi-isomorphic to

$$\cdots\to\tilde F_\partial^{l_1}(\Omega_\partial\otimes_{D_{x,t}}A^i[\mathfrak{m}_i])\to\tilde F_\partial^{l_1}(\Omega_\partial\otimes_{D_{x,t}}A^{i+1}[\mathfrak{m}_{i+1}])\to\cdots \tag{4.3}$$

where $A^\bullet[\mathfrak{m}_\bullet]$ is a V_∂-strict resolution of M, and l_1 is the largest integral root of $\tilde b(s)$. Again, cohomology generators have V_∂-degree equal to a root of $\tilde b(s)$.

4.2. Bernstein-Sato polynomial and the relative de Rham complex.

Following Malgrange [24], we consider for $f\in R_n$ the symbol f^s as generating a $D_{x,t}$-module contained in the free $R_n[f^{-1},s]$-module $R_n[f^{-1},s]f^s$ via

$$t\bullet\frac{g(s)}{f^j}f^s=\frac{g(s+1)}{f^{j-1}}f^s,\qquad \partial_t\bullet\frac{g(s)}{f^j}f^s=\frac{-sg(s-1)}{f^{j+1}}f^s.$$

Then the left ideal $J_{n+1}(f)=\langle t-f,\{\partial_i+\partial_i\bullet(f)\partial_t\}_{i=1}^n\rangle\subseteq D_{x,t}$ is easily seen to consist of operators that annihilate f^s. Moreover, $-\partial_t t$ acts as multiplication by s. Since $J_{n+1}(f)$ is maximal, it actually contains all annihilators of f^s. It turns out, that $J_{n+1}(f)$ describes the $\mathcal{D}$-module direct image of R_n under the embedding $x\to(x,f(x))$:

Lemma 4.5. *For all $f\in R_n$,*

$$D_{x,t}/J_{n+1}(f)\cong H^1_{t-f}(R_{x,t}),$$

generated by $\frac{1}{t-f}$.

Proof. Consider $\tau = \frac{1}{t-f} \in R_{x,t}[(t-f)^{-1}]$. It is obviously annihilated by $\{\partial_i + \partial_i \bullet (f)\partial_t\}_{i=1}^n$. Moreover, $\frac{t-f}{t-f} \in R_{x,t}$ so that $(t-f)(\tau \mod R_{x,t}) = 0 \in H^1_{t-f}(R_{x,t})$. Hence $J_{n+1}(f)$ annihilates the coset of τ in $H^1_{t-f}(R_{x,t})$.

The polynomial $t-f$ is free of singularities and so its Bernstein-Sato polynomial is $s+1$. Hence τ generates $R_{x,t}[(t-f)^{-1}]$. Therefore the coset of τ generates the local cohomology module. Since this module is nonzero, the coset of τ cannot be zero. Hence its annihilator cannot be $D_{x,t}$. As $J_{n+1}(f)$ is maximal we are done. □

We now connect the ideas of Malgrange with algorithmic methods pioneered by Oaku, and Takayama, to show that the ideal $J_{n+1}(f)$ is intimately connected with the Bernstein-Sato polynomial of f:

Theorem 4.6. *Let Q be a homogeneous polynomial of degree $k > 0$ with Bernstein-Sato polynomial $b_Q(s)$. Then $b_Q((-s-n)/k)$ is a multiple of the b-function for integration of $J_{n+1}(Q) = \langle t-Q, \{\partial_i + \partial_i \bullet (Q)\partial_t\}_{i=1}^n\rangle$ along $\partial_1, \dots, \partial_n$.*

Proof. It is well-known, that $J_{n+1}(Q) \cap D_x[s] = \operatorname{ann}_{D_x[s]}(Q^s)$. Hence in particular, $J_{n+1}(Q)$ contains $\mathcal{E} - ks = \mathcal{E} + k\partial_t t$.

To be a Bernstein polynomial means that $b_Q(s) \in J_{n+1}(Q) \cap D_x[s] + D_x[s] \cdot Q$. Write $b_Q(s) = j + P(s)Q$ with $j \in J_{n+1}(Q) \cap D_x[s]$, $P(s) \in D_x[s]$.

The ideal $J_{n+1}(Q)$ is $(1,k)$-homogeneous if we set $\deg(x_i) = 1$, $\deg(\partial_i) = -1$, $\deg(t) = k$, $\deg(\partial_t) = -k$. Since $b_Q(s)$ is $(1,k)$-homogeneous of degree 0, we may assume that j (and hence $P(s)Q$) is also $(1,k)$-homogeneous of degree 0. Writing $P(s) = \sum_{i=0}^l P_i s^i$ with $P_i \in D_x$, we see that each P_i is of $(1,k)$-degree $-k$. This implies, as $P_i \in D_x$, that $P_i \in F_\partial^{-k}(D_x)$. Note that as $t - Q \in J_{n+1}(Q)$, $b_Q(s) = P(s)t$ modulo $J_{n+1}(Q)$. So $P(s)t = P(-\partial_t t)t \in F_\partial^{-k}(D_{x,t})$ and $b_Q(-\partial_t t) \in J_{n+1}(Q) + F_\partial^{-k}(D_{x,t})$.

Also, $b_Q(-\partial_t t)$ is modulo $J_{n+1}(Q)$ equivalent to $b_Q((-\tilde{\mathcal{E}} - n)/k)$ because $\mathcal{E} + k\partial_t t \in J_{n+1}(Q)$. Thus

$$b_Q\left(\frac{-\tilde{\mathcal{E}} - n}{k}\right) \in J_{n+1}(Q) + F_\partial^{-k}(D_{x,t}),$$

proving that $b_Q(-(s+n)/k)$ is a multiple of the b-function for integration of $D_{x,t}/J_{n+1}(Q)$ along $\partial_1, \dots, \partial_n$. □

Combining Theorem 4.6 with Theorem 4.4 and its integration counterpart, one obtains

Corollary 4.7. *The only possible V_∂-degrees for the generators of the cohomology of $\operatorname{DR}(D_{x,t}/J_{n+1}(Q))$ are those specified by the roots of $b_Q(-(s+n)/k)$.* □

4.3. **Restriction to the fiber.**

Let Q be a homogeneous polynomial of positive degree. We now consider the effect of restriction to $t-1$ of the relative de Rham complex $\operatorname{DR}(D_{x,t}/J_{n+1}(Q))$. This is computed as the cohomology of the tensor product over D_t of $\operatorname{DR}(D_{x,t}/J_{n+1}(Q))$ with $(D_{x,t} \stackrel{(t-1)\cdot}{\longrightarrow} D_{x,t})$. We shall concentrate on the highest cohomology group. It equals $D_{x,t}/(J_{n+1}(Q) + \{\partial_1, \dots, \partial_n, t-1\}D_{x,t})$.

Theorem 4.8. *The quotient*

$$U \quad := \quad D_{x,t}/(J_{n+1}(Q) + \{\partial_1, \dots, \partial_n, t-1\}D_{x,t}) \tag{4.4}$$

is spanned by polynomials $g \in R_n$. *One may choose a* $\mathbb{C}$*-basis for* U *in such a way that*

- *all basis elements are in* R_n *and homogeneous;*
- *no basis element may be replaced by an element of smaller degree (homogeneous or not).*

We call such a basis a homogeneous degree minimal basis.

The degree of any element g *of a degree minimal basis satisfies*

$$b_Q(-(\deg(g)+n)/k) = 0$$

and then the usual degree of g *is the* V_∂*-degree of the class of* g.

Proof. Clearly U is spanned by the cosets of $R_n[\partial_t]$. Let $g \in R_n$ be homogeneous of degree d. In U we have $gt^a\partial_t^b = gt^b\partial_t^b = g\prod_{j=0}^{b-1}(t\partial_t - j)$ for all $a, b \in \mathbb{N}$. Now observe that $\mathcal{E} + k\partial_t t \in J_{n+1}(Q)$ implies that in U

$$\begin{aligned} 0 = \partial_t^b g(\mathcal{E} + k\partial_t t) &= \partial_t^b(\mathcal{E} - d + k\partial_t t)g \\ &= \partial_t^b(-n - d + k\partial_t t)g \\ &= \left((-n-d+k(b+1))\partial_t^b + k\partial_t^{b+1}\right) g. \end{aligned}$$

By induction this shows that in U

$$\partial_t^b g = t^b\partial_t^b g = \prod_{i=1}^{b}\left(\frac{n+d}{k} - i\right) g. \tag{4.5}$$

Hence U is spanned by the cosets of R_n. As $\partial_t^b g$ and $\prod_{i=1}^b\left(\frac{n+d}{k} - i\right) g$ have the same V_∂-degree, minimal V_∂ -deg representatives for all $u \in U$ can be chosen within R_n.

Now let $u' \in R_n$ be homogeneous and let $0 \neq u \in D_{x,t}$ be a V_∂-degree minimal representative of the class of u' in U. By the previous paragraph, without affecting V_∂-degree, u can be assumed to be in R_n. Then obviously the V_∂-degree agrees with the usual degree.

Therefore by definition $\deg(u) = V_\partial\text{-}\deg(u) \le V_\partial\text{-}\deg(u') = \deg(u')$. Hence for any $u' \in U$ we have

$$\min\{\deg(u) : u \in R_n, u = u' \in U\} = \min\{V_\partial\text{-}\deg(u) : u' = u \in U\}$$

and the equality can be realized by one and the same element $u \in R_n$ on both sides. If this u is nonzero in U, then clearly $u \in H^n(\mathrm{DR}(D_{x,t}/J_{n+1}(Q)))$ is also nonzero since this module surjects onto U. The V_∂-degree of u within $H^n(\mathrm{DR}(D_{x,t}/J_{n+1}(Q)))$ cannot be smaller than the V_∂-degree of u in the bigger coset when considered in U, and hence is just the usual degree of u. By Corollary 4.7, $b_Q(-(\deg(u)+n)/k) = 0$. This implies that U is finite dimensional. Hence any $\mathbb{C}$-basis for U may be turned into a degree minimal one.

It remains to show that the basis can be picked in a homogeneous minimal way. Note that $J_{n+1}(Q)+\{\partial_1, \ldots, \partial_n, t-1\}D_{x,t}$ is $\mathbb{Z}/\langle k\rangle$-graded (by $x_i \mapsto 1, \partial_i \mapsto -1$ and $t, \partial_t \mapsto 0$); so U is $\mathbb{Z}/\langle k\rangle$-graded and U has a degree minimal $\mathbb{Z}/\langle k\rangle$-graded basis. If u is in a $\mathbb{Z}/\langle k\rangle$-graded minimal degree basis but not homogeneous, the degrees of its graded components only differ by multiples of k. Write $u = u_a + u_{a+1} + \ldots + u_b$ with $a, b \in \mathbb{N}$ and u_j the component of u in degree jk. Then since $t - Q$ is in $J_{n+1}(f)$, we have in U the equality $u = \sum_{j=a}^b u_j Q^{b-j}$. The right hand side is homogeneous and both of usual and of V_∂-degree $\deg(u)$. Hence $\mathbb{Z}/\langle k\rangle$-graded minimal degree bases for U can be changed into homogeneous minimal degree bases without changing the occurring degrees (all of which we proved to be roots of $b_Q(-(s+n)/k)$). $\square$

Remark 4.9. An important hidden ingredient of the above theorem is the fact that the b-function for restriction to $t-1$ of both $D_{x,t}/J_{n+1}(f)$ and $H^n\,\mathrm{DR}(D_{x,t}/J_{n+1}(f))$ is $(t-1)\partial_t$ whenever f is w-homogeneous. Namely, if $f=\sum_{i=1}^n w_ix_i\partial_i \bullet f$ then with $\xi=\sum_{i=1}^n w_ix_i\partial_i$ we have $(\partial_t t+\xi)\bullet f^s=0$. Consider then the equation

$$(t-1)\partial_t = \underbrace{(t-1)(\partial_t t+\xi)}_{A} - \underbrace{(t-1)(\partial_t(t-1)+\xi)}_{B}.$$

Obviously, $A\in J_{n+1}(f)$ and $B\in F^{-1}_{t-1}(D_{x,t})\cap F^{-1}_{\partial,t-1}(D_{x,t})$. These are the required conditions to be a b-function for restriction to $t-1$ of $D_{x,t}/J_{n+1}(f)$ respectively $H^n(\mathrm{DR}(D_{x,t}/J_{n+1}(f)))$.

Corollary 4.10. *Let $Q=H_1\cdots H_k$ define a central generic arrangement. Then U has a homogeneous basis of polynomials of degree at most $2k-n-2$.*

Proof. By Theorem 2.13, $b_Q(s)$ has its zero locus inside $\{-n/k,\dots,(-2k+2)/k\}$. Then by Theorem 4.8 the degrees of a minimal degree basis for U are bounded above by $2k-n-2$. $\square$

4.4. De Rham cohomology from $\mathcal{D}$-module operations.

For any f, the complex $\mathrm{DR}(D_{x,t}/J_{n+1}(f))$ carries the de Rham cohomology of the fibers of the map $\mathbb{C}^n\ni x\to f(x)\in\mathbb{C}$, since it is the result of applying the de Rham functor to the composition of maps $x\to(x,f(x))$ and $(x,y)\to(y)$ (see [13]). The de Rham functor for the embedding corresponds to the functor that takes the D_x-module M to the $D_{x,t}$-complex $M\otimes_{D_x}(D_{x,t}\stackrel{\cdot(f-t)}{\longrightarrow}D_{x,t})$, while the projection corresponds to the formation of the Koszul complex induced by left multiplication by $\partial_1,\dots,\partial_n$. The cohomology of the fiber $Q^{-1}(1)$ is obtained as the restriction to $t-1$.

With the shifts in cohomological degree, $U=D_{x,t}/(J_{n+1}(Q)+\langle\partial_1,\dots,\partial_n,t-1\rangle\cdot D_{x,t})$ thus encodes the top de Rham cohomology of $Q^{-1}(1)$. For homogeneous Q the correspondence between these two spaces is at follows. Write $dX=dx_1\wedge\ldots\wedge dx_n$ and $\widehat{dX_i}=dx_1\wedge\ldots\wedge\widehat{dx_i}\wedge\ldots\wedge dx_n$ where the hat indicates omission. An element g in U determines the form $g\,dX$ on $\mathbb{C}^n$. Under the embedding $Q^{-1}(1)\hookrightarrow\mathbb{C}^n$, the form $g\,dX$ restricts ($\mathcal{D}$-module theoretically) to the form G which satisfies $dQ\wedge G=g\,dX$. Let us compute G. Since G is an $(n-1)$-form, $G=\sum_{i=1}^n g_i\,\widehat{dX_i}$. Thus, $dQ\wedge G=\sum_{i=1}^n(-1)^i\partial_i\bullet(Q)g_i\,dX$. On the other hand, along $Q^{-1}(1)$, $g\,dX=gQ\,dX=\frac{g}{k}\sum_{i=1}^n x_i\partial_i\bullet(Q)dX$. Thus by comparison, $kg_i=(-1)^ix_ig$. With ω as in (3.1), the $(n-1)$-form on $Q^{-1}(1)$ encoded by $g\in U$ is $G=g\omega/k$. We show now that all forms in $H^{n-1}_{\mathrm{DR}}(Q^{-1}(1),\mathbb{C})$ are captured by U.

Lemma 4.11. *If Q is homogeneous, then $H^{n-1}_{\mathrm{DR}}(Q^{-1}(1),\mathbb{C})$ is generated by $R_n\cdot\omega$.*

Proof. This is trivial for $n=1$, so we assume that $n>1$. Consider the map $R_n\to H^{n-1}_{\mathrm{DR}}(Q^{-1}(1),\mathbb{C})$ given by $g\to g\omega$. Suppose $g\omega=0$. Then

$$g\omega=(Q-1)h+d(G)+A\wedge dQ$$

for $h = \sum(-1)^{i+1} h_i \, \widehat{dX_i} \in \Omega^{n-1}$, $G = \sum g_{i,j} dx_1 \wedge \ldots \wedge \widehat{dx_i} \wedge \ldots \wedge \widehat{dx_j} \wedge \ldots \wedge dx_n \in \Omega^{n-2}$, $A \in \Omega^{n-2}$. Multiply by dQ to get

$$kQg\,dX = \left(\sum_{i=1}^{n} (Q-1)\partial_i \bullet (Q) h_i + \sum_{i,j=1}^{n} \partial_i \bullet (Q) \partial_j \bullet (g_{i,j}) - \partial_j \bullet (Q) \partial_i \bullet (g_{i,j}) \right) dX$$

in $\Omega^n = R_n\, dX$. Now look at this in U. Note that $kQg = ktg = kg$ and $(Q-1)\partial_i \bullet (Q)h_i = (t-1)\partial_i \bullet (Q) h_i = 0$ in U. So (in U)

$$kg = \sum_{i,j=1}^{n} \partial_i \bullet (Q) \partial_j \bullet (g_{i,j}) - \partial_j \bullet (Q) \partial_i \bullet (g_{i,j}) \tag{4.6}$$

We would like this to be zero in U; in fact it will turn out to vanish term by term.

We may assume that $g_{i,j}$ is homogeneous by looking at the graded pieces g of (4.6). So to simplify notation let h be a homogeneous polynomial in R_n. In the remainder of this proof we shall use a subscript to denote derivatives: $h_i = \partial_i \bullet (h)$. Then in U we have

$$\begin{aligned} 0 = -\partial_j t h_i + \partial_i t h_j &= -th_{i,j} + th_{j,i} + (-th_i \partial_j + th_j \partial_i) \\ &= th_i Q_j \partial_t - th_j Q_i \partial_t \\ &= (h_i Q_j - h_j Q_i) t \partial_t \\ &= (h_i Q_j - h_j Q_i) \frac{n + \deg(h) - 2}{k} \qquad \text{by (4.5).} \end{aligned}$$

If $\deg(h) > 0$, this implies the vanishing of $h_i Q_j - h_j Q_i \in U$. But if $\deg(h) = 0$ there is nothing to prove in the first place. Therefore the sum (4.6) is zero. Hence if $g\omega = 0$ in $H^{n-1}_{\mathrm{DR}}(Q^{-1}(1), \mathbb{C})$ then $kg = 0$ in U. So $R_n \twoheadrightarrow U$ factors as $R_n \twoheadrightarrow R_n\omega \twoheadrightarrow U = H^{n-1}_{\mathrm{DR}}(Q^{-1}(1), \mathbb{C})$. $\square$

Our considerations prove in view of Corollary 4.10:

Theorem 4.12. *Let $Q \in R_n$ be a homogeneous polynomial of degree k. The de Rham cohomology group $H^{n-1}_{\mathrm{DR}}(Q^{-1}(1), \mathbb{C})$ is isomorphic to $U \cdot \omega$. There is a homogeneous basis for U with degrees bounded by*

$$u_Q = \max\{ i \in \mathbb{Z} : b_Q(-(i+n)/k) = 0 \}.$$

If Q defines a generic arrangement of hyperplanes, $u_Q \leq 2k - n - 2$. $\square$

4.5. Non-vanishing of $H^{n-1}_{\mathrm{DR}}(Q^{-1}(1), \mathbb{C})$ and roots of $b_Q(s)$.

We now establish the existence of a non-vanishing $g \in H^{n-1}_{\mathrm{DR}}(Q^{-1}(1), \mathbb{C})$ in all degrees $0 \leq \deg(g) \leq 2k - n - 2$ for generic central arrangements Q. This will certify each root of (2.2) as root of $b_Q(s)$.

The primitive k-th root ζ_k of unity acts on $\mathbb{C}^n$ by $x_i \to \zeta_k x_i$. This fixes Q and hence the ideal $J_{n+1}(Q)$. Therefore it gives an automorphism of the de Rham complex and hence the induced map on cohomology separates $H^{n-1}_{\mathrm{DR}}(Q^{-1}(1), \mathbb{C})$ into eigenspaces, $U = \bigoplus_{\bar{i} \in \mathbb{Z}/k\mathbb{Z}} M_{\bar{i}}$ which are classified by their degree modulo k.

From [31] we know the monodromy of Q. In particular, $M_{\bar{i}}$ is a $\binom{k-2}{n-1}$-dimensional vector space unless $\bar{i} = \overline{k-n}$. Write U_i for the elements in U with (homogeneous) minimal degree representative of degree precisely i. Since elements of U have degree at most $2k-n-2$, we find that

$$\dim(U_i)+\dim(U_{i+k}) = \dim(U_i+U_{i+k}) = \binom{k-2}{n-1} \quad \text{for} \quad 0\le i\le k-n-1;$$

$$U_i = M_{\bar{i}} \text{ and } \dim(U_i) = \binom{k-2}{n-1} \quad \text{for} \quad k-n<i\le k-1.$$

Moreover, $U_{k-n} = M_{\overline{k-n}} = (R_n)_{k-n}$ of dimension $\binom{k-1}{n-1}$.

Since $R_n/(E+\langle Q-1\rangle)$ surjects onto U, Lemma 3.7 shows that neither U_i nor U_{i+k} is zero-dimensional for $0\le i\le k-n-1$. So one has

Theorem 4.13. *For a generic hyperplane arrangement Q the vector space*

$$\left(H^{n-1}_{\mathrm{DR}}(Q^{-1}(1),\mathbb{C})\right)_r \neq 0 \quad \textit{for} \quad 0\le r\le 2k-n-2.$$

It is zero for all other i. □

One can now use the non-vanishing to certify roots of $b_Q(s)$ as such:

Corollary 4.14. *The Bernstein-Sato polynomial of a generic central arrangement $Q=\prod_{H_i\in\mathcal{A}} H_i$ of degree k is*

$$(s+1)^r \prod_{i=0}^{2k-n-2}\left(s+\frac{i+n}{k}\right)$$

where $r=n-1$ or $r=n-2$.

Proof. By the previous theorem, $U_i\neq 0$ for $0\le i\le 2k-n-2$. A minimal degree basis for U must therefore contain elements of all these degrees. By the last part of Theorem 4.8, $b_Q(s)$ is a multiple of $\prod_{i=0}^{2k-n-2}\left(s+\frac{i+n}{k}\right)$. On the other hand, Theorem 2.13 proves that $b_Q(s)$ divides the displayed expression with $r=n-1$. This proves everything apart from the multiplicity of $(s+1)$.

Let $\vec{x}\neq\vec{0}$ be any point of the arrangement where precisely $n-1$ planes meet. The Bernstein-Sato polynomial of Q is a multiple of the local Bernstein-Sato polynomial at $\vec{x}$ (which is defined by the same type of equation as $b_Q(s)$ but where $P(s)$ is in the localization of $D_x[s]$ at the maximal ideal defining $\vec{x}$). Since the local Bernstein-Sato polynomial at a normal crossing of $n-1$ smooth divisors is $(s+1)^{n-1}$, the theorem follows. □

We are quite certain, that the exponent r in Corollary 4.14 is $n-1$, but we do not know how to show that. In fact, we believe that the elements $g\in R_n$ whose cosets in

$$(s+1)^{n-2}\prod_{i=0}^{2k-n-2}\left(s+\frac{i+n}{k}\right)\cdot\frac{D_n[s]\bullet f^s}{D_n[s]\bullet f^{s+1}}$$

are zero are precisely the elements of $\mathfrak{m}^{k-n+1}$.

Conjecture 4.15. If $k\le r\le 2k-n-2$ we believe that the space $(R_n/(E+\langle Q-1\rangle))_r$ is spanned by the expressions in (3.4) for which $i_1<(n-1)+(r-k)$. If $k-n<r<k$, the expressions in Proposition 3.5 are known to span U. If $r\le k-n$ we believe that $U_r=(R_n)_r$.

This is in accordance with [31] as there are exactly as many such expressions as Conjecture 3.2 predicts for the dimension of $(H^{n-1}_{\mathrm{DR}}(Q^{-1}(1),\mathbb{C}))_r$.

Example 4.16. Consider the non-generic arrangement given by $Q = xyz(x+y)(x+z)$. With the D-module package [21] of *Macaulay 2* [14] one computes its Bernstein-Sato polynomial as

$$(s+1)(s+\frac{2}{3})(s+\frac{3}{3})(s+\frac{4}{3})(s+\frac{3}{5})(s+\frac{4}{5})(s+\frac{5}{5})(s+\frac{6}{5})(s+\frac{7}{5}).$$

Therefore the b-function for integration of J_{n+1} along $\partial_1, \ldots, \partial_n$ is a divisor of

$$(s-2)(s-\frac{1}{3})(s-2)(s-\frac{11}{3})(s-0)(s-1)(s-2)(s-3)(s-4).$$

This indicates that the degrees of the top cohomology of the Milnor fiber $Q^{-1}(1)$ are at most 4. It also shows that in this case these degrees do not suffice to determine the roots of $b_Q(s)$. In fact, the degrees of no class in any $H^i_{\mathrm{DR}}(Q^{-1}(1), \mathbb{C})$ will explain the roots $-2/3$ and $-4/3$ in $b_Q(s)$.

However, consider a point $P \neq 0$ on the line $x = y = 0$. This line is the intersection of three participating hyperplanes, x, y and $x+y$. In P the variety of Q has a homogeneous structure as well, so the local Bernstein-Sato polynomial of Q at P is a multiple of the minimal polynomial of the local Euler operator on the cohomology of the Milnor fiber of Q at P. In fact, at P the variety of Q is a generic arrangement in the plane, times the affine line. Without difficulty one verifies then that the Milnor fiber has top cohomology in degrees 0, 1 and 2, and that $b_{Q,P}(s) = (s+2/3)(s+1)^2(s+4/3)$.

The global Bernstein-Sato polynomial of Q is the least common multiple of all local Bernstein-Sato polynomials $b_{Q,P}(s)$. Hence $b_Q(s)$ must be a multiple of $(s+2/3)(s+1)^2(s+4/3)$ and so all roots of $b_Q(s)$ come in one way or another from cohomology degrees on Milnor fibers. This prompts the following question:

Problem 4.17. Let Q be a locally quasi-homogeneous polynomial in R_n (as for example a hyperplane arrangement). Is it true that every root of $b_Q(s)$ arises through the action of an Euler operator on the top de Rham cohomology of the Milnor fiber of Q at some point of the arrangement?

This is of course true for isolated quasi-homogeneous singularities. If Q is an arrangement then by the local-to-global principle one may restrict to central arrangements. We have proved here that the question has an affirmative answer for generic arrangements.

One more remark is in order. The cohomology we have used to describe the Bernstein-Sato polynomial is the one with coefficients in the constant sheaf $\mathbb{C}$, which may be viewed as the sheaf of solutions of the D_n-ideal $\langle \partial_1, \ldots, \partial_n \rangle$ describing R_n on $\mathbb{C}^n$. Relating holonomic D_n-modules to locally constant sheaves on $\mathbb{C}^n$ is the point of view of the Riemann-Hilbert correspondence, [4]. There are, however, other natural locally constant sheaves on $\mathbb{C}^n \setminus Q^{-1}(0)$ induced by D_n-modules than just the constant sheaf. For example, for every $a \in \mathbb{C}$ the D_n-ideal $\mathrm{ann}_{D_n}(f^a)$ induces such a sheaf as the sheaf of its local solutions. For most exponents a this is of course a sheaf without global sections on $Q \neq 0$, and more generally without any cohomology. For suitable exponents, however, this is different; it is sufficient to consider the case where $a + \mathbb{Z}$ contains a root of the Bernstein-Sato polynomial. Perhaps one can characterize the Bernstein-Sato polynomial as the polynomial of smallest degree such that $s = -\partial_t t$ annihilates the V_∂-degree of every cohomology class in $H^i(\Omega \otimes^L_{D_n} \int_\iota \mathcal{P})$ for every D_n-module $\mathcal{P}$ defining a locally constant system on $\mathbb{C}^n \setminus Q^{-1}(0) \overset{\iota}{\hookrightarrow} \mathbb{C}^{n+1}$. Another possibility is given by the cyclic covers introduced by Cohen and Orlik [12].

5. Miscellaneous Results

In this section we collect some results and conjectures concerning the structure of the module $D_n \bullet Q^s$ associated to central arrangements.

5.1. **Arbitrary arrangements.**

We begin with a fact pointed out to us by A. Leykin.

Theorem 5.1 (Leykin). *The only integral root of the Bernstein-Sato polynomial of any arrangement $\mathcal{A}$ is -1.*

Proof. By Lemma 1.3 it will be sufficient to show that if $Q = \prod_{H_i \in \mathcal{A}} H_i$ then $R_n[Q^{-1}]$ is generated by $1/Q$ since this implies that $D_n \bullet (Q^{-1}) = D_n \bullet (Q^{-r})$ for all $r \in \mathbb{N}$.

Since the Bernstein-Sato polynomial is the least common multiple of the local Bernstein-Sato polynomials, we may assume that $\mathcal{A}$ is central. We may also assume that $\mathcal{A} \subseteq \mathbb{C}^n$ is not contained in a linear subspace of $\mathbb{C}^n$.

The claim is true for a normal crossing arrangement. We proceed by induction on the difference $k - n > 0$ where $k = \deg(Q)$. Since the local cohomology module $H^k_{\mathfrak{m}}(R_n)$ vanishes, $R_n[Q^{-1}] = \sum_{i=1}^k R_n[(Q/H_i)^{-1}]$. Moreover, by induction $R_n[(Q/H_i)^{-1}]$ is generated by H_i/Q as D_n-module. Since obviously H_i/Q is in the D_n-module generated by $1/Q$, the theorem follows. $\square$

Remark 5.2. Note that the same argument proves the following. Let $g_1, \dots, g_k \in R_n$ and set $G = \prod_{i=1}^k g_i$. If $R_n[(G/g_i)^{-1}]$ is generated by $(g_i/G)^m$ for $i = 1, \dots, k$, and $H^k_{\langle g_1,\dots,g_k\rangle}(R_n) = 0$ then $R_n[G^{-1}]$ is generated by $1/G^m$. That is to say, if the smallest integral root of $b_{G/g_i}(s)$ is at least $-m$, and if $H^k_{\langle g_1,\dots,g_k\rangle}(R_n) = 0$ then the smallest integral root of $b_G(s)$ is at least $-m$. By Grothendieck's vanishing theorem this last condition is always satisfied if $k > n$.

We now give some combinatorial results on the localization module $R_n[Q^{-1}]$. The following really is a general fact about finite length modules.

Proposition 5.3. *Let $M = \sum_{i=1}^k M_i$ be a holonomic D_n-module. Then the holonomic length satisfies*

$$\ell(M) = \sum_{i=1}^{k} (-1)^{i+1} \sum_{|I|=i} \ell(M_I)$$

where $M_I = \bigcap_{j \in I} M_j$.

Proof. ℓ is additive in short exact sequences. Hence $\ell(M) = \ell(M_1) + \ell(M/M_1)$. In order to start the induction, one needs to look at the case $k = 2$ which is the second isomorphism theorem.

Also, by induction,

$$\begin{aligned}
\ell(M)-\ell(M_1)=\ell(M/M_1) &= \ell(\sum_{i>1}(M_j+M_1)/M_1)\\
&= \sum_{i=1}^{k-1}(-1)^{i+1}\sum_{|I|\subseteq\{2,\dots,k\}}\ell(\bigcap_{1<j\in I}(M_j+M_1)/M_1)\\
&= \sum_{i=1}^{k-1}(-1)^{i+1}\sum_{|I|\subseteq\{2,\dots,k\}}\ell(M_I/(M_1\cap M_I))\\
&= \sum_{i=1}^{k-1}(-1)^{i+1}\sum_{|I|\subseteq\{2,\dots,k\}}[\ell(M_I)-\ell(M_{I\cup\{1\}})]
\end{aligned}$$

The terms $\ell(M_I)$ in the last sum are all the summands in the sum of the theorem without the index 1. The terms $\ell(M_{I\cup\{1\}})$ together with $\ell(M_1)$ make up all those who do use the index 1. $\square$

Proposition 5.4. *In the context of Theorem 5.1, let $M_I = R_n\left[\prod_{j\notin I} H_j{}^{-1}\right]$. The length of $M = R_n[Q^{-1}]$ is determined recursively as follows where $H^i_{\mathcal{A}}(-)$ is local cohomology with supports in the ideal $\langle H_1,\dots,H_k\rangle$.*

- *If $H^k_{\mathcal{A}}(R_n)=0$ then $\ell(M)=\sum(-1)^i\sum_{|I|=i}\ell(M_I)$.*
- *If $H^k_{\mathcal{A}}(R_n)\neq 0$ then $\ell(M)=\sum(-1)^i\sum_{|I|=i}\ell(M_I)+1$.*

This information can be obtained from the intersection lattice.

Proof. In the first case the Čech complex shows that $M=\sum_{i=1}^k M_i$ and hence all that needs to be shown is that the two usages of the symbol M_I here and in Proposition 5.3 agree. In other words, we must show that

$$R_n\left[\prod_{j\in\{1,\dots,k\}\setminus I} H_j{}^{-1}\right]\cap R_n\left[\prod_{j'\in\{1,\dots,k\}\setminus I'} H_{j'}{}^{-1}\right] = R_n\left[\prod_{j\in\{1,\dots,k\}\setminus(I\cup I')} H_j{}^{-1}\right]$$

for all index sets I, I'. This, however, is clear.

If $H^k_{\mathcal{A}}(R_n)\neq 0$ then the H_i form a regular sequence and hence we know that this local cohomology module is of length one, a suitable generator being annihilated by all H_i. The formula follows by considering $\sum_{|I|=1} M_I$ and $0\to\sum_{|I|=1}M_I\to M\to H^k_{\mathcal{A}}(R_n)\to 0$. $\square$

Remark 5.5. There are substantially more general results by Àlvarez-Montaner, García-López, and Zarzuela-Armengou. In fact, Propositions 5.3 and 5.4 can be modified to apply to the characteristic cycle of $R_n[Q^{-1}]$. This idea is discussed in [1] and then used to express the lengths of the modules $H^r_{\mathcal{A}}(R_n)$ in terms of Betti numbers obtained from the intersection lattice (even for subspace arrangements).

5.2. Some conjectures.

We now close with conjectures on the generators of $J(Q^s)$ and $\mathrm{ann}_{D_n}(Q^{-1})$.

Definition 5.6. For a central arrangement $\mathcal{A}=\{H_1,\dots,H_k\}$ and $Q=H_1\cdots H_k$ we define the ideals $I(\mathcal{A})$ and $I_s(\mathcal{A})$ as follows. Let $H_1,\dots,H_n$ be linearly independent. Choose vector fields v_i with constant coefficients such that $v_i\bullet(H_j)=\delta_{i,j}$ for all $1\le i,j\le n$.

Form $P_{i,j}(Q) = \frac{H_i H_j}{H_1 \cdots H_n}(v_i \bullet (Q) v_j - v_j \bullet (Q) v_i) \in D_n$; $P_{i,j}(Q)$ kills Q^s. Let

$$I(Q) = \left\langle \left\{ P_{i,j}(Q') \frac{Q}{Q'} : Q' \mid Q \right\} \right\rangle \subseteq D_n.$$

We define $I_s(Q)$ recursively. If $\deg(Q) = 1$, set $I_s(Q) = I(Q)$. If $\deg(Q) > 1$,

$$I_s(Q) = \left\langle \left\{ Q''^{s+1} P_{i,j}(Q') Q''^{-s} : Q = Q'Q'' \right\} \right\rangle \subseteq D_n[s].$$

It is apparent that $I_s(Q)$ kills Q^s and $I(Q)$ kills $1/Q$.

Conjecture 5.7. For any central arrangement Q,

(1) the annihilator $\operatorname{ann}_{D_n}(Q^{-1})$ is $I(Q) + \langle \mathcal{E} + k \rangle$;
(2) the annihilator $\operatorname{ann}_{D_n[s]}(Q^s)$ is $I_s(Q) + \langle \mathcal{E} - ks \rangle$.

There is certainly a considerable amount of redundancy in these generators. Particularly for generic arrangements much smaller sets can be taken. The importance of the conjecture lies perhaps more in the fact that all operators shown are order one. We make some remarks about this now.

T. Torelli [35] has proved that $\operatorname{ann}(Q^{-1})$ is generated in order one for the union of a generic arrangement with a hyperbolic arrangement. A divisor $\operatorname{div}(f)$ on $\mathbb{C}^n$ is called *free* if the module of logarithmic derivations $\operatorname{der}(\log f) = \{\delta \in \operatorname{der}(R_n) : \delta(f) \in \langle f \rangle\}$ is a locally free R_n-module. It is called *Koszul-free* if one can choose a basis for the logarithmic derivations such that their top order parts form a regular sequence in $\operatorname{gr}_{(0,1)}(D_n)$. The complex of logarithmic differentials $\Omega^\bullet(\log f)$ consists (in the algebraic case) of those differential forms $\omega \in \Omega^\bullet(R_n[f^{-1}])$ for which both $f\omega$ and $fd\omega$ are regular forms on $\mathbb{C}^n$. It is a subcomplex of $\Omega^\bullet(R_n[f^{-1}])$ and (algebraic) Logarithmic Comparison is said to hold if the inclusion is a quasi-isomorphism.

Let $\tilde{I}^{\log f}$ be the subideal of $\operatorname{ann}(1/f)$ generated by the order one operators introduced by L. Narvaez-Macarro [36] and put $I^{\log f} = D_n \cdot \operatorname{der}(\log f)$. F.J Castro and J.M. Ucha [10, 11], using results and ideas of F.J. Calderon [8], proved that if f is Koszul-free then the map from $\Omega^\bullet(\log f)$ to the (holomorphic) de Rham complex of the (holonomic) module $\tilde{M}^{\log f} = D_n/\tilde{I}^{\log f}$ is a quasi-isomorphism, and $\tilde{I}^{\log f}$ and $I^{\log f}$ are holonomically dual. Hence if f is Koszul-free and $\tilde{M}^{\log f}$ regular holonomic, then $\tilde{M}^{\log f} = R_n[f^{-1}]$ if and only if (holomorphic) Logarithmic Comparison holds. In his paper [35] Torelli conjectures that if f is reduced (but not necessarily Koszul-free) then (holomorphic) Logarithmic Comparison holds for f if and only if $\operatorname{ann}(1/f) = \tilde{I}^{\log f}$.

Terao conjectured in [34] that (algebraic) Logarithmic Comparison holds for any central arrangement (and more) and there is a proof in the analytic case for free quasi-homogeneous divisors in [9]. This can via Torelli's conjecture be seen as counterpart to our conjecture. Wiens and Yuzvinsky have proved in [39] Terao's conjecture for arrangements in $\mathbb{C}^{\leq 4}$, and all tame arrangements.

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