Answers to questions from class

1. $\sum_{n=2}^{\infty} \frac{4 \cdot 6 \cdot 8 \cdots (2n)}{5^{n+1}(n+2)!} x^n$.

   Ratio test gives a ratio of $(2n+2)/5(n+3)|x|$. In the limit this is $2|x|/5$, so ROC is $5/2$.

   At $x = -5/2$ the series satisfies the alternating series test conditions (check that!!!), so it converges.

   At $x = 5/2$,

$$\begin{aligned}\sum_{2}^{\infty} \frac{4 \cdot 6 \cdot 8 \cdots (2n)}{5^{n+1}(n+2)!} x^n &= \frac{1}{10} \cdot \sum_{2}^{\infty} \frac{n! \cdot 2^n}{5^n (n+2)!} (5/2)^n \\ &= \frac{1}{10} \cdot \sum_{2}^{\infty} \frac{1}{(n+2)(n+1)}\end{aligned}$$

   which converges by p-test with $p = 2$.

2. Using the Taylor series for $\sin(x)$,

$$\begin{aligned}\int_0^x \sin(t^2) dt &= \int_0^x \left(t^2 - t^6/3! + t^{10}/5! - \cdots\right) dt \\ &= x^3/3 - x^7/(7 \cdot 3!) + x^{11}/(11 \cdot 5!) + \cdots .\end{aligned}$$

   Since $x \in [0, 1]$ is assumed, this satisfies the alternating series test (check!!!). In particular, the $(n+1)$st term in the series bounds the error one makes when replacing the actual integral by the nth partial sum.

   We want the error less than 10^{-3}, so we ask which of these terms in the series has absolute value less than that. At worst, $x = 1$ and so the third term is no more than $1/11 \cdot 120 < 10^{-3}$, so we can use the approximation $x^3/3 - x^7/(7 \cdot 3!)$.

3. $\frac{x^4}{x^4+y^2}$ along the y-axis is 0, but along the x-axis is 1. So at $(0, 0)$ there can be no limit.

4. The tangent plane at $P = (1, 1, 2)$ for the surface $x^2 + y^2 = z$ is perpendicular to the gradient of $x^2 + y^2 - z$ at $(1, 1, 2)$, and also passes through $(1, 1, 2)$. The gradient of the function at $(1, 1, 2)$ is $(2x, 2y, -1)$ at $(1, 1, 2)$ and this comes out to be $(2, 2, -1)$. So the equation of the tangent plane looks like $2x + 2y + (-1)z = c$, and plugging in c one finds $c = 2$.

5. With $R_1 = 5000\Omega$ and $R_2 = 1000\Omega$, $R = 1/(\frac{1}{R_1} + \frac{1}{R_2}) = (R_1 R_2)/(R_1 + R_2)$. Then $dR = (R_2 S - R_1 R_2) dR_1/S^2 + (R_1 S - R_1 R_2) dR_2/S^2$ where I abbreviate $R_1 + R_2$ by S.

   Then $dR_1 = 20$ and $dR_2 = 0$ leads to $dR = 20(R_2 S - R_1 R_2)/S^2 = 20(1000 * 6000 - 5000 * 1000)/6000^2 = 20/36$.

   On the other hand, $dR_1 = 0, dR_2 = 1$ leads to $dR = 1(5000 * 6000 - 5000 * 1000)/1000^2 = 25/36$. So the small change in R_2 has a greater effect overall than the 20 times bigger change in R_1.

6. $f = x^2 + kxy + y^2$ has Hesse matrix $\begin{pmatrix} 2 & k \\ k & 2 \end{pmatrix}$ and discriminant $4 - k^2$.

   If $|k| < 2$, this is positive and so f has a minimum (as $f_{xx} > 0$). If $|k| > 2$ it has a saddle.

   If $k = \pm 2$, then $f = (x \pm y)^2$ and so it has a whole line of minima along $x \pm y = 0$.

7. $a, b, c \geq 0$ means that the maximum of $f = ab^2c^3$ will not have any of $a, b, c = 0$ (since then $f = 0$ but $f(1, 1, 1) = 1$ is bigger). So, we are actually looking in the open set $a, b, c > 0$ with $a + b + c = 3$. In other words, we can ignore the boundary.

   Lagrange says, we need to satisfy the equations $g = 0$ and $\nabla f = \lambda \nabla g$.

   Write this out: $(b^2c^3, 2abc^3, 3ab^2c^2) = \lambda \cdot (1, 1, 1)$. As all components on the right are equal, so all those on the left. That is, $b^2c^3 = 2abc^3 = 3ab^2c^2$. Since $abc \neq 0$, this gives $b = 2a$, $c = 3a$. As $a + b + c = 3$, $6a = 3$ and hence $(a, b, c) = (1/2, 1, 3/2)$ realizes the maximum of f.

   (It is a maximum, since it cannot be a minimum as the the minimum is zero, realized in any point where $abc = 0$).

   Note: We also ought to check the places where the constraint is singular. However, the gradient of g is never zero, so no critical points can come that way.

8. If $g(x, y, z) = 0$ and we choose base variables x, y then this determines $z = z(x, y)$ as a function of x, y. Then $g(x, y, z(x, y))$ is tautologically 0.

   The chain rule for $(\partial/\partial y)_x$, viewing z as function of x, y on the constraint says: $8y + 18z(\partial z/\partial y)_x = 0$. Solve this for the derivative.

   Next use the chain rule for $(\partial/\partial y)_x$ on w. You get $(\partial w/\partial y)_x = x(-\sin(xy)\sin(yz) + \cos(xy)z\cos(yz)(\partial z/\partial y)_x + xz + xy(\partial z/\partial y)_x$. Plug in x, y, z as given, and $(\partial z/\partial y)_x$ as computed.