Aspects in complex hyperbolicity

Sai-Kee Yeung

Dedicated to Professor Yum-Tong Siu on the occasion of his 70th birthday

Abstract The purpose of this article is explain some aspects in complex hyperbolicity, through discussions of examples. We would focus our discussions on some recent results of Wing-Keung To and myself on Kobayashi hyperbolicity of some moduli space of polarized varieties, but would also mention some related results in complex hyperbolicity, as well as some examples for arithmetic problems related to hyperbolicity.

1 Introduction

Complex hyperbolicity is a notion in complex geometry which could be understood either from the point of view of value distribution of entire holomorphic curves in a complex manifold, or the point of view of existence of non-positive curved metric. The two commonly used notions are Brody hyperbolicity and Kobayashi hyperbolicity. A complex manifold M is said to be Brody hyperbolic if it does not contain the image of any non-trivial holomorphic map from $\mathbf{C}$. M is said to be Kobayashi hyperbolic if the Kobayashi metric on M is non-degenerate, cf. [44]. For simplicity, we regard a pseudo-metric as a metric in this exposition. The Kobayashi metric can be characterized as the largest among all the pseudo-distance functions δ_M on M satisfying $\delta_M(f(a), f(b)) \leq d_P(a,b)$ for all holomorphic maps $f : \Delta \to M$ and $a,b \in \Delta$, where Δ is the unit disc in $\mathbf{C}$ and d_P is the hyperbolic distance function on Δ, cf. [22]. It follows immediately that any Kobayashi hyperbolic manifold

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is Brody hyperbolic as well, since the image of any entire holomorphic curve on M would have degenerate Kobayashi semi-distance. For a compact manifolds, the two notions are equivalent, following a normal family argument as given by Brody reparametrization argument in [9]. For non-compact manifolds, there are examples of Brody hyperbolic manifolds which are not Kobayashi hyperbolic.

In recent years, interests in complex hyperbolicity have been kindled by conjectured parallelism between complex hyperbolicity and Mordellic properties in diophantine geometry, due to conjectures of Bombieri, Lang, Osgood and Vojta, cf. [54]. For a smooth projective variety V defined over a number field k, we say that V is Mordellic if the number of rational points in k is at most finite. In case that V is quasi-projective, we say that V is Mordellic if the number of integral points with respect to the infinity divisor is finite. A general conjecture of Lang [27] states that a smooth projective variety V defined over a number field k is complex hyperbolic if and only if it is Mordellic.

It is in general a difficult problem to prove that a complex manifold is complex hyperbolic, and even more so to prove Mordellic properties. The main purpose of this article is to consider some aspects of these topics through some explicit examples.

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1.1 $P^1_{\mathbf{C}}-\{0,1,\infty\}$ *revisited*

The example of $P^1_{\mathbf{C}}-\{0,1,\infty\}$ is historically among the first interesting examples in complex hyperbolicity. Little Picard Theorem concludes that $P^1_{\mathbf{C}}-\{0,1,\infty\}$ is hyperbolic.

To give a conceptually simple reason, we recall a handy criterion in the proof of hyperbolicity, the Schwarz Lemma of Ahlfors. Suppose M is a complex manifold equipped with a Hermitian metric h with holomorphic sectional curvature bounded from above by a negative constant. Ahlfors Schwarz Lemma states that a holomorphic map $f:\Delta\to M$ satisfies $f^*h\leq cg_P$, where g_P is the Poincaré metric on Δ and c is a positive constant. An immediate consequence is that M as above is complex hyperbolic.

Little Picard Theorem can be explained conceptually from Riemann Uniformization Theorem, which states that the universal covering of $P^1_{\mathbf{C}}-\{0,1,\infty\}$ is biholomorphic to the unit disk Δ in $\mathbf{C}$. Now the Poincaré metric on Δ has constant negative holomorphic sectional curvature -4, from which hyperbolicity follows after applying Ahlfors Schwarz Lemma.

Another observation is that $P^1_{\mathbf{C}}-\{0,1,\infty\}=\Delta/\Gamma_2$, where Γ_2 is the second congruence subgroup of $PSL(2,\mathbf{Z})$ of level 2. As such Δ/Γ_2 is naturally a covering of $\Delta/PSL(2,\mathbf{Z})$. On the other hand it is well-known that $\Delta/PSL(2,\mathbf{Z})$ can be considered as the parameter space of the space of all elliptic curves. Hence

$P^1_{\mathbf{C}} - \{0,1,\infty\} = \Delta/\Gamma_2$ naturally parametrizes a family of elliptic curves. This is the simplest of a moduli space which satisfies hyperbolic properties.

The example and the above two observations lead naturally to two directions which lead to a lot of developments for complex hyperbolicity.

The first one is whether the complement of a divisor of high degree in $P^n_{\mathbf{C}}$ is hyperbolic. Clearly we may also ask for similar or more refined questions for other pairs of manifolds as well. The second one is whether a natural moduli space of some appropriate complex manifolds are hyperbolic or not.

The first direction is well-motivated and has generated a lot of research activities with a vast amount of literature. Since there are already good overviews of this direction in literature such as [45], we would only remark briefly on known results in the Section 2, but focus on the second direction as well as some arithmetic considerations in later sections.

In the following we explain some further motivations for the second direction.

1.2 The moduli space of curves $\mathscr{M}_g$ for $g \geqslant 2$

The moduli space of curves $\mathscr{M}_g$ as a topological space is the set of all equivalence classes of Riemann surfaces of genus g with the equivalent relation given by biholomorphism. It is known that $\mathscr{M}_g$ can be given the structure of a complex space with at worst orbifold singularities, or stacks. We may represent $\mathscr{M}_g$ as the quotient of a Teichmüller space $\mathscr{T}_g$ by the mapping class group Γ_g, and the Teichmüller space can be regarded as a bounded domain in $\mathbf{C}^{\mathbf{3g-3}}$. The complex structure on $\mathscr{M}_g$ can also be understood in terms of Kodaira-Spencer theory on deformation of complex structures.

On $\mathscr{M}_g$ a natural biholomorphic invariant metric is given by the Weil-Petersson metric. Let $t \in \mathscr{M}_g$ representing a Riemann surface M_t of genus g. A holomorphic tangent vector to $\mathscr{M}_g$ at t can be identified with the Kodaira-Spencer class in $H^1(M_t,\Theta)$, where Θ is the sheaf of holomorphic tangent vector fields on M_t. Denote by $\mathscr{H}^1(M_t,\Theta)$ the set of harmonic representative in $H^1(M_t,\Theta)$. Classically these are known as harmonic Beltrami differentials. Hence a tangent vector at x is represented by $\Phi_x \in \mathscr{H}^1(M_t,\Theta)$. The Weil-Petersson metric g_{WP} is represented by

$$(\Phi_1,\Phi_2)_{WP} := \int_{M_t} \langle \Phi_1,\Phi_2\rangle \omega_P \tag{1}$$

where $\langle\cdot,\cdot\rangle$ is the pointwise inner product with respect to the Poincaré metric g_P on M_t, and the integral is taken with respect to the volume form ω_P of the Poincaré metric. It follows from the work of Ahlfors that g_{WP} is Kähler. Furthermore, it is known from the work of Ahlfors ([2], [3]) and Royden [38] that the holomorphic sectional curvature of g_{WP} is negative with negative upper bound. In [58] Wolpert showed that the pointwise curvature of g_{WP} from the point of view of differential geometry can be expressed in closed form as

$$R^{(WP)}_{i\bar{j}k\bar{\ell}}(t) = 2\int_{M_t} ((\Box - 2)^{-1}\langle \Phi_i, \Phi_j\rangle) \cdot \langle \Phi_k, \Phi_\ell\rangle \omega_P \qquad (2)$$
$$+2\int_{M_t} ((\Box - 2)^{-1}\langle \Phi_k, \Phi_j\rangle) \cdot \langle \Phi_i, \Phi_\ell\rangle \omega_P.$$

Here $\Box$ is the Laplace-Beltrami operator. As a result, the holomorphic sectional curvature is bounded above by $-\frac{1}{2\pi(g-1)}$. Note that $2(g-1)$ is the degree of the canonical line bundle on M_t. We conclude that $\mathscr{M}_g$ is Kobayashi hyperbolic from Ahlfors Schwarz Lemma.

1.3 Algebraic geometric results in moduli spaces of higher dimensional varieties

The results of Sect. 1.1 show that moduli spaces of smooth projective curves with ample or flat canonical line bundle are Kobayashi hyperbolic. The result has the following algebraic geometric interpretation. Suppose that $\pi : M \to P^1_{\mathbf{C}}$ is an algebraic family of generically smooth curves. Then there are at least 3 singular fibers. The reason is that $P^1_{\mathbf{C}}$ minus three or more points is hyperbolic, but $P^1_{\mathbf{C}}$ minus two or fewer points is not hyperbolic, since it contains $\mathbf{C}^*$, or image of $\mathbf{C}$ after the exponential mapping. Similarly if the base curve is an elliptic curve, there is at least one singular fiber.

From an algebraic geometric point of view, it is interesting to know if the above observation is true also for family of higher dimensional varieties of general type. In fact, results in this direction have been proved by Migliorini [32] and Kovacs [25], [26]. A typical result is that for a $P^1_{\mathbf{C}}$ family of canonically polarized projective algebraic varieties, there are at least 3 singular fibers in the family. As $P^1_{\mathbf{C}}$ minus 3 points is hyperbolic, the above result is the consequence of a result of Viehweg and Zuo [57], that the base manifold of any family of non-trivial canonically polarized projective algebraic manifolds is Brody hyperbolic. For a precise statement, we refer the readers to statement in .

Two questions arise naturally. The first one is whether the base manifold above is Kobayashi hyperbolic as well. As mentioned before, the notion of Kobayashi hyperbolicity is strictly stronger than Brody hyperbolicity for non-compact manifolds. The second is whether similar results hold for families of polarized Kähler Ricci flat manifolds. We would address the above the problems from Section 3 to Section 4.

2 Some tools in the study of complex hyperbolicity

The purpose of this section is to explain some examples in the first direction mentioned in the introduction of this paper. We would also explain some techniques known in the study of complex hyperbolicity which directly or indirectly motivate the discussions in the later sections.

A far reaching generalization of Little Picard Theorem is Nevanlinna theory. Nevanlinna theory provides the formalism and techniques for the study of entire holomorphic curve, which is the image of an entire holomorphic map from $\mathbf{C}$ in a complex manifold. In particular, the classical Little Picard Theorem is a consequence of the formulation that the defect of any entire holomorphic curve on $P^1_{\mathbf{C}}$ is at most $2+\varepsilon$ for arbitrarily small $\varepsilon > 0$, which itself is consequence of the Second Main Theorem of Nevanlinna.

A direct generalization of Little Picard Theorem to higher dimensions is the question of complex hyperbolicity or its analogs for $P^n_{\mathbf{C}} - D$, where D is the union of a finite number of hyperplanes in general positions. Among many interesting results, we just mention a few below. For statements in value distribution of entire holomorphic curves, there is the result of H. Cartan on Truncated Second Main Theorem [10], a direct generalization of the result of Nevanlinna to higher dimensional projective spaces. There is also the result of Ahlfors [1], who introduced the notion of associated curves and studied their defects. In an inhomogeneous representation of an entire holomorphic curve $f : \mathbf{C} \to \mathbf{P^n_C}$, the associated curve can be considered as a wedge product of the mapping f and its successive derivatives, $f \wedge f' \wedge \cdots f^{(k)}$, regarded as a map from $\mathbf{C}$ taking values in the Grassmanian resulted. Ahlfors obtained Second Main Theorem and defect relations for the associated maps iteratively. Hyperbolicity of complement of $2n+1$ hyperplanes in generic positions in $P^n_{\mathbf{C}}$ is proved by Greens. Replacing $2n+1$ hyperplanes by $2n+1$ hypersurfaces, the result is also known due to Ru and other people, cf. [40] and the references there. for results in this direction.

The more difficult situation is the question of hyperbolicity of $P^n_{\mathbf{C}} - D$ for a generic divisor D of high degree. For a generic D, defined by a polynomial of degree d, $G(X_1, \cdots X_{n+1}) = 0$ in $P^n_{\mathbf{C}}$, we may consider the branch cover M of $P^n_{\mathbf{C}} - D$ defined by

$$T^d = G(X_1, \cdots X_{n+1}) \tag{3}$$

in $P^{n+1}_{\mathbf{C}}$. In this way, a hyperbolicity problem on $P^n_{\mathbf{C}} - D$ is reduced to the corresponding problem on a hypersurface M on $P^{n+1}_{\mathbf{C}}$. The statement can be made precise. In particular, Kobayashi conjectured that $P^n_{\mathbf{C}} - D$ is hyperbolic if $\deg D \geqslant 2n+1$, similarly for a generic hypersurface $D \subset P^{n+1}$ of degree at least $2(n+1)$. Analogous results for a general manifold have been conjectured by Lang and Vojta, cf. [27], [54]. The precise degree is expected to be dictated by the geometry of the manifold involved, such as the degree of the canonical class.

The Kobayashi Conjecture as mentioned has led to a lot of research activities, though still not solved. In [48], Siu and myself showed that the conjecture is true for $n = 2$ if the degree of D is very large. The degree was lowered greatly by the work of McQuillen [28] and Demailly-Elgoul [14]. In higher dimensions, a breakthrough comes from [44], see also [45], where Siu introduced the method of slanted vector fields on moduli of hypersurfaces and showed that the statements of the conjectures of Kobayashi were true if the degree of D is sufficiently large. Reasonable effective bounds of the method in dimensions 2 and 3 have been given as in Paun [37] and Rousseau [39]. For the analogous problem of algebraic degeneracy of the image of entire holomorphic curves, results with bounds on the degree of D has been obtained by Diverio-Merker-Rousseau [16], see also [30].

As used in [48], there are in general the following steps in proving complex hyperbolicity. The first step is the construction of some non-trivial jet differentials ω vanishing on some ample divisor on M. Once this is available, a suitable Schwarz Lemma for holomorphic jet differentials implies that the image of an entire holomorphic curve $f : \mathbf{C} \to \mathbf{M}$ satisfies $f^*\omega = 0$. This means that the image of f is confined in the sense its jets satisfy a differential equation coming from ω. The second step involves further restriction of the image, by either repeating the construction of sections on the restriction of the jet bundles on the Zariski closure of the image of f in the jet space, or by showing that there are a lot of freedom in the choice of the jet differentialsω so that their common vanishing set could be shown to be small when projected down to the manifold M. For the first step, the usual method is by Riemann-Roch together with estimates of higher cohomology groups as given in [21], [12], [45], or by explicit Siegel Lemma type argument as in [48], Section 2, and [45]. See the results of [13], [31] and the references there for more recent works. For the second step, a direct computation using Riemann-Roch type theorem on the image of jets of entire holomorphic curve works only in special situation or low dimensions. The method of slanted vector fields introduced in [44] is general but the degree involved is still quite large at this stage. The latter approach is parallel to the restrictions of rational curves on a very general hypersurface of large degree in projective spaces as studied by Clemens [11], Ein [17] and Voisin [53].

The Schwarz Lemma for holomorphic jet differentials was proved for special two jet in dimension 2 in [48], here the word special means that the jet differential has invariant form under reparametrization, and is generalized to all situations in [50], [12] and [45].

As mentioned earlier, the formulation of complex hyperbolicity of a complex manifold M in terms of geometry either in the form of M or a pair (M,D) has been generalized by Lang and Vojta, cf. [27], [54]. Apart from $P^n_{\mathbf{C}}$, one may for example ask the same question for ample divisors M in Abelian varieties A and the complement of an ample divisor D in an Abelian variety A. For the case of $M \subset A$, the results of Bloch essentially implies that the Zariski closure of an entire holomorphic curve in M is a translation of some sub-Abelian variety, cf. [6], [45]. The modern use of jet differentials as well as some analogues of Schwarz Lemma can be traced to the work of [6]. The case of $A \backslash D$ has been a conjecture of Lang and was settled in [49], [50]. The corresponding result for semi-Abelian varieties were discussed in [36]. The situation of Abelian varieties has an analogue in arithmetics, which would be discussed in Sect. 5.

We mention that the use of generalized Weil-Petersson metric on iterated Kodaira-Spencer class in [51] to be explained in Sect. 3 is motivated by this version of Schwarz lemma. The formulation of telescoping estimates of the curvature expressions mentioned in Subsect. 3.3 is motivated by the formulation of Ahlfors for associated curves as mentioned above.

3 Moduli of canonically polarized manifolds

In the following two sections, we would explain results in the second direction mentioned in the Subsection 1.1. From a differential geometric point of view, it would be desirable to generalize the computation of curvature for a family of Riemann surfaces of fixed genus as given in Subsection 1.2 to a family of higher dimensional manifolds, from which complex hyperbolicity would follow naturally. We consider a holomorphic family $\pi : \mathscr{X} \to S$ of compact canonically polarized complex manifolds over a complex manifold S. By this we assume that $\pi : \mathscr{X} \to S$ is a surjective holomorphic map of maximal rank between two complex manifolds $\mathscr{X}$ and S, and each fiber $M_t := \pi^{-1}(t)$, $t \in S$, is a compact complex manifold such that K_{M_t} is ample. From results of Aubin [5] and Yau [59], every compact complex manifold with ample canonical line bundle admits a Kähler-Einstein metric of negative Ricci curvature, which is unique up to a positive multiplicative constant. Hence on each M_t, the Ricci curvature tensor of the Kähler-Einstein metric g satisfies $R_{\alpha\bar{\beta}}(t) = kg_{\alpha\bar{\beta}}(t)$ for some constant $k < 0$.

The formulation of the problem and the first breakthrough are given by the paper of Siu in [44].

3.1 Curvature formula of Siu for the Weil-Petersson metric

For a holomorphic family $\pi : \mathscr{X} \to S$ of complex manifolds, holomorphic tangent vectors at $t \in S$ are represented by Kodaira-Spencer map $\rho_t : T_t S \to H^1(M_t, TM_t)$. We say that π is effectively parametrized if each ρ_t is injective. One can easily define Weil-Petersson metric with respect to the Kähler metric in the same way as in the case of one dimensional fibers. The curvature formula for higher dimensional fibers is however complicated and the sign is difficult to determine.

Here are some details. Consider an effectively parametrized family $\pi : \mathscr{X} \to S$ of canonically polarized manifolds. For $t \in S$ and a local tangent vector field u (of type $(1,0)$) on an open subset U of S, there is a unique lifting of u such that $\Phi(u(t))$ is the harmonic representative of Kodaira-Spencer class $\rho_t(u(t))$ for each $t \in S$, which is called the canonical lifting or horizontal lifting of u, cf. [43], [41] When $u = \partial/\partial t^i$ is a coordinate vector field, we will simply denote its canonical lifting by $v_i := v_{\partial/\partial t^i}$ and the associated harmonic Kodaira-Spencer representative by $\Phi_i := \Phi(\partial/\partial t^i)$. The Weil-Petersson metric $h^{(WP)} = \sum_{i,j=1}^n h^{(WP)}_{i\bar{j}} dt^i \otimes d\bar{t}^j$ on S is defined as in equation (1) by

$$h^{(WP)}_{i\bar{j}}(t) := \int_{M_t} \langle \Phi_i, \Phi_j \rangle \frac{\omega^n}{n!}, \tag{4}$$

where $\langle \Phi_i, \Phi_j \rangle := (\Phi_i)^{\gamma}_{\bar{\alpha}} \overline{(\Phi_j)^{\delta}_{\bar{\beta}}} g_{\gamma\bar{\delta}} g^{\bar{\alpha}\beta}$ denotes the pointwise Hermitian inner product on tensors.

It follows from Koiso's result [24] that $h^{(WP)}$ is Kähler. Let $R^{(WP)}$ denote the curvature tensor. By [43], p. 296, the components of the curvature tensor $R^{(WP)}$ of $h^{(WP)}$ with respect to normal coordinates (of $h^{(WP)}$) at a point $t \in S$ are given by

$$
\begin{aligned}
R^{(WP)}_{i\bar{j}k\bar{\ell}}(t) = k \int_{M_t} ((\Box - k)^{-1}\langle \Phi_i, \Phi_j\rangle) \cdot \langle \Phi_k, \Phi_\ell\rangle \frac{\omega^n}{n!} \\
+k \int_{M_t} ((\Box - k)^{-1}\langle \Phi_k, \Phi_j\rangle) \cdot \langle \Phi_i, \Phi_\ell\rangle \frac{\omega^n}{n!} \\
+k \int_{M_t} \langle (\Box - k)^{-1} \mathscr{L}_{v_i}\Phi_k, \mathscr{L}_{v_j}\Phi_\ell\rangle \frac{\omega^n}{n!} \\
+ \int_{M_t} \langle H(\Phi_i \otimes \Phi_k), H(\Phi_j \otimes \Phi_\ell)\rangle \frac{\omega^n}{n!}.
\end{aligned} \tag{5}
$$

Here by normal coordinates of $h^{(WP)}$ at the point $t \in S$, we mean $h^{(WP)}_{i\bar{j}}(t) = \delta_{ij}$, and $\partial_k h^{(WP)}_{i\bar{j}}(t) = \partial_{\bar{k}} h^{(WP)}_{i\bar{j}}(t) = 0$. The notation $\mathscr{L}_v\Phi$ denotes the Lie derivative of Φ with respect to the vector field v and $H(\Phi_i \otimes \Phi_k)$ denotes the harmonic projection as a bundle-valued form of the wedge product of Φ_i and Φ_k in both the form and tangent vector directions.

The pointwise computation of the curvature formula in (5) is a beautiful formula on which all later curvature computations of Weil-Petersson type metrics built on. The deduction comes from clever grouping of terms and involved loops of integration by parts guided from geometric intuitions.

The holomorphic sectional curvature curvature corresponds to components of form $R^{(WP)}_{i\bar{i}i\bar{i}}$. The first two terms on the right hand side of (5) are negative from our assumption of effective parametrization and the third one is semi-negative. The problem is on control of the fourth term which is semi-positive. Hence beautiful as it is, the formula (5) is not sufficient to deduce hyperbolicity properties of the moduli space except under very restrictive situations corresponding to the vanishing of the fourth term.

For a long time, people have been trying to dominate the fourth term by the first three terms. This seems to be not possible in general (cf. the remark in [52]). In the next subsection, we will introduce the method of [51] to handle the difficulty. At this point, we mention that the approach of (5) has been applied to the case of families of polarized Kähler Ricci flat manifolds in Nannicini [35]. The results of (5) has also been formulated in a sometimes more efficient way in [41]. The work of [51] follows more closely the formulation in [43], but also makes use of some simplifications in [41]. In Sect. 4, we will present the results on hyperbolicity for family of Kähler-Ricci flat manifolds.

3.2 Generalized Weil-Petersson metric and curvature formula

In the next few subsections, we will summarize the results in [51], whose goal is to provide a Finsler metric with holomorphic sectional curvature bounded from above by a negative constant so that the space is Kobayashi hyperbolic. Results in this section is the first step, which is a generalization of the formula of Siu in (5).

We fix a coordinate open subset $U \subset S$ with coordinate functions $t = (t^1, \dots, t^m)$ such that the origin $t = 0$ lies in U. For each $t \in S$ and each coordinate tangent vector $\frac{\partial}{\partial t^i}$, we recall the horizontal lifting v_i and the harmonic representative Φ_i of $\rho_t(\frac{\partial}{\partial t^i})$ on M_t as given earlier. Fix an integer ℓ satisfying $1 \leqslant \ell \leqslant n$, and let $J = (j_1, \dots, j_\ell)$ be an ℓ-tuple of integers satisfying $1 \leqslant j_d \leqslant m$ for each $1 \leqslant d \leqslant \ell$. We denote by

$$\Psi_J := H(\Phi_{j_1} \owedge \cdots \owedge \Phi_{j_\ell}) \in \mathscr{A}^{0,\ell}(\wedge^\ell TM_t) \tag{6}$$

the harmonic projection of $\Phi_{j_1} \owedge \cdots \owedge \Phi_{j_\ell}$. As t varies, we still denote the resulting family of tensors by Ψ_J (suppressing its dependence on t), when no confusion arises.

Observe that from definition, the expression $\Phi_{j_1} \owedge \cdots \owedge \Phi_{j_\ell}) \in \mathscr{A}^{0,\ell}(\wedge^\ell TM_t)$ is symmetric in $j_1, \dots, j_\ell$. Hence after composing with the Kodaira-Spencer map, we may define a Hermitian metric on $S^\ell(T_S)$ with norm given by

$$\|v_{i_1} \cdots v_{i_\ell}\|_2^2 = \int_{M_t} \langle v_{i_1} \cdots v_{i_\ell}, v_{i_1} \cdots v_{i_\ell} \rangle \frac{\omega^n}{n!}. \tag{7}$$

We call such an expression a generalized Weil-Petersson metric. To compute the curvature, we need to study $\partial_i \|\Psi_J\|_2^2$. The following proposition is a direct generalization of the identity in (5).

Proposition 1 *We have*

$$\begin{aligned} &\partial_i \overline{\partial_i} \log \|\Psi_J\|_2^2 \\ = &\frac{1}{\|\Psi_J\|_2^2} \big(-k((\square - k)^{-1}(\overline{\Phi_i} \cdot \Psi_J), \overline{\Phi_i} \cdot \Psi_J) - k((\square - k)^{-1} \langle \Phi_i, \Phi_i \rangle, \langle \Psi_J, \Psi_J \rangle) \\ &\quad -k((\square - k)^{-1}(\mathscr{L}_{v_i} \Psi_J), \mathscr{L}_{v_i} \Psi_J) - \big|(\mathscr{L}_{v_i} \Psi_J, \frac{\Psi_J}{\|\Psi_J\|_2})\big|^2 \\ &\quad -(H(\Phi_i \owedge \Psi_J), H(\Phi_i \owedge \Psi_J))\big). \end{aligned} \tag{8}$$

In the remaining part of this subsection, we give some ideas for the proof of Proposition 1. The expression that we need to compute is given by

$$\begin{aligned} \partial_i \overline{\partial_i} \log \|\Psi_J\|_2^2 &= \partial_i \Big(\frac{\partial_{\bar{i}} \|\Psi_J\|_2^2}{\|\Psi_J\|_2^2} \Big) \\ &= \frac{\partial_i \partial_{\bar{i}} \|\Psi_J\|_2^2}{\|\Psi_J\|_2^2} - \frac{(\partial_i \|\Psi_J\|_2^2)(\partial_{\bar{i}} \|\Psi_J\|_2^2)}{\|\Psi_J\|_2^4}. \end{aligned} \tag{9}$$

For this purpose, we observe that

$$\begin{aligned}\partial_i\|\Psi_J\|_2^2 &= \frac{\partial}{\partial t^i}\int_{M_t}\langle\Psi_J,\Psi_J\rangle\frac{\omega^n}{n!}\\ &= \int_{M_t}\langle\mathscr{L}_{v_i}\Psi_J,\Psi_J\rangle\frac{\omega^n}{n!}+\int_{M_t}\langle\Psi_J,\mathscr{L}_{\overline{v_i}}\Psi_J\rangle\frac{\omega^n}{n!}\\ &= \int_{M_t}\langle\mathscr{L}_{v_i}\Psi_J,\Psi_J\rangle\frac{\omega^n}{n!},\end{aligned}$$

where in the last step we have used the fact that Ψ_J is harmonic and that $(\mathscr{L}_{\overline{v_i}}\Psi_J)^{(0,\ell)}_{(\ell,0)}$ is $\overline{\partial}$-exact (cf. [51], Lemma 3), so that

$$\int_{M_t}\langle\Psi_J,\mathscr{L}_{\overline{v_i}}\Psi_J\rangle\frac{\omega^n}{n!}=0 \tag{10}$$

Differentiating the complex conjugate of above expression, we get

$$\begin{aligned}0 &= \frac{\partial}{\partial t^i}\int_{M_t}\langle\mathscr{L}_{\overline{v_i}}\Psi_J,\Psi_J\rangle\frac{\omega^n}{n!}\\ &= \int_{M_t}\langle\mathscr{L}_{v_i}\mathscr{L}_{\overline{v_i}}\Psi_J,\Psi_J\rangle\frac{\omega^n}{n!}+\int_{M_t}\langle\mathscr{L}_{\overline{v_i}}\Psi_J,\mathscr{L}_{\overline{v_i}}\Psi_J\rangle\frac{\omega^n}{n!}.\end{aligned} \tag{11}$$

We obtain

$$\begin{aligned}\partial_i\partial_{\bar{i}}\|\Psi_J\|_2^2 = \partial_{\bar{i}}\partial_i\|\Psi_J\|_2^2 &= \frac{\partial}{\partial\bar{t}^i}\int_{M_t}\langle\mathscr{L}_{v_i}\Psi_J,\Psi_J\rangle\frac{\omega^n}{n!}\\ &= \int_{M_t}\langle\mathscr{L}_{\overline{v_i}}\mathscr{L}_{v_i}\Psi_J,\Psi_J\rangle\frac{\omega^n}{n!}+\int_{M_t}\langle\mathscr{L}_{v_i}\Psi_J,\mathscr{L}_{v_i}\Psi_J\rangle\frac{\omega^n}{n!}.\\ &= I+II+III,\end{aligned} \tag{12}$$

where

$$\begin{aligned}I &:= -\int_{M_t}\langle\mathscr{L}_{\overline{v_i}}\Psi_J,\mathscr{L}_{\overline{v_i}}\Psi_J\rangle\frac{\omega^n}{n!}, \qquad (13)\\ II &:= \int_{M_t}\langle\mathscr{L}_{[\overline{v_i},v_i]}\Psi_J,\Psi_J\rangle\frac{\omega^n}{n!}=(\mathscr{L}_{[\overline{v_i},v_i]}\Psi_J,\Psi_J),\\ III &:= \int_{M_t}\langle\mathscr{L}_{v_i}\Psi_J,\mathscr{L}_{v_i}\Psi_J\rangle\frac{\omega^n}{n!}=(\mathscr{L}_{v_i}\Psi_J,\mathscr{L}_{v_i}\Psi_J).\end{aligned}$$

after applying the identity $\mathscr{L}_{\overline{v_i}}\mathscr{L}_{v_i}=\mathscr{L}_{v_i}\mathscr{L}_{\overline{v_i}}+\mathscr{L}_{[\overline{v_i},v_i]}$. It remains to compute the expressions I, II and III. After some careful manipulation of the terms similar to [43], we find that

$$\begin{aligned}I &= -k((\Box-k)^{-1}(\overline{\Phi_i}\cdot\Psi_J),\overline{\Phi_i}\cdot\Psi_J)-(\overline{\Phi_i}\cdot\Psi_J,\overline{\Phi_i}\cdot\Psi_J) \qquad (14)\\ &\quad+(\overline{\Phi_i}\searrow\Psi_J,\overline{\Phi_i}\searrow\Psi_J)+(\overline{\Phi_i}\nearrow\Psi_J,\overline{\Phi_i}\nearrow\Psi_J),\\ II &= -(\langle\Phi_i,\Phi_i\rangle,\langle\Psi_J,\Psi_J\rangle)-k((\Box-k)^{-1}\langle\Phi_i,\Phi_i\rangle,\langle\Psi_J,\Psi_J\rangle),\end{aligned}$$

$$III = (\Phi_i \oslash \Psi_J, \Phi_i \oslash \Psi_J) - (H_1(\Phi_i \oslash \Psi_J), H_1(\Phi_i \oslash \Psi_J))$$
$$-k((\Box - k)^{-1}(\mathscr{L}_{v_i}\Psi_J)\mathscr{L}_{v_i}\Psi_J),$$

where $\overline{\Phi_i} \searrow \Psi_J \in \mathscr{A}^{1,\ell-1}(\wedge^\ell TM_t)$ and $\overline{\Phi_i} \nearrow \Psi_J \in \mathscr{A}^{0,\ell}(\wedge^{\ell-1}TM_t \wedge \overline{TM_t})$ are defined by

$$(\overline{\Phi_i} \searrow \Psi_J)^{\alpha_1\cdots\alpha_\ell}_{\delta\overline{\beta_1}\cdots\overline{\beta_{\ell-1}}} := \overline{(\Phi_i)^{\sigma}_{\overline{\delta}}}(\Psi_J)^{\alpha_1\cdots\alpha_\ell}_{\overline{\sigma}\overline{\beta_1}\cdots\overline{\beta_{\ell-1}}} \quad \text{and} \tag{15}$$
$$(\overline{\Phi_i} \nearrow \Psi_J)^{\alpha_1\cdots\alpha_{\ell-1}\overline{\gamma}}_{\overline{\beta_1}\cdots\overline{\beta_\ell}} := \overline{(\Phi_i)^{\gamma}_{\overline{\sigma}}}(\Psi_J)^{\alpha_1\cdots\alpha_{\ell-1}\sigma}_{\overline{\beta_1}\cdots\overline{\beta_\ell}}$$

respectively. Proposition 1 follows by putting the above information together.

We remark that the above calculations follows closely the one in [43], where the case of $\ell = 1$ was treated. Note that the fourth term on the right hand side is controlled by the third term there from spectral decomposition. Hence the first four terms gives rise to a non-negative sign, but the fourth one is of non-positive sign and is the one to be controlled.

After the completion of the paper [51], we noticed that an analogous formula in dual formulation had been obtained independently by [42]. Here by dual formulation, we refer to computation of curvature for the dual bundle in the sense of Kodaira-Serre, cf. [23].

3.3 A telescopic formulation

The second step in the construction of the Finsler metric in [51] is to formulate estimates in identity (8) in a way that we may apply a telescopic argument.

Fix $v_i \in TS$ and let Φ_i the corresponding harmonic representative in the Kodaira-Spencer class. For a positive integer ℓ, we define the relative tensor

$$H^{(\ell)} := H(\underbrace{\Phi_i \oslash \cdots \oslash \Phi_i}_{\ell\text{–times}}), \tag{16}$$

so that $H^{(\ell)} = \Psi_J$ with J given by the ℓ-tuple $(i, i, \cdots, i)$, here $H(\cdot)$ refers to the projection to the harmonic component. The second main step of our argument is the following.

Proposition 2 *Suppose $\|H^{(\ell)}\|_2 > 0$ (which automatically implies that $\|H^{(\ell-1)}\|_2 > 0$. Then we have*

$$\partial_i \overline{\partial_i} \log \|H^{(\ell)}\|_2^2 \geq \frac{\|H^{(\ell)}\|_2^2}{\|H^{(\ell-1)}\|_2^2} - \frac{\|H^{(\ell+1)}\|_2^2}{\|H^{(\ell)}\|_2^2}. \tag{17}$$

Here is the outline of proof. From Proposition 1 and the remark there, we conclude that

$$\partial_i \overline{\partial_i} \log \|H^{(\ell)}\|_2^2 \geqslant \frac{1}{\|H^{(\ell)}\|_2^2}\big(-k((\Box - k)^{-1}(\overline{\Phi_i} \cdot H^{(\ell)}), \overline{\Phi_i} \cdot H^{(\ell)}) \\ -k((\Box - k)^{-1}\langle \Phi_i, \Phi_i\rangle, \langle H^{(\ell)}, H^{(\ell)}\rangle) \\ -(H(\Phi_i \oslash H^{(\ell)}), H(\Phi_i \oslash H^{(\ell)}))\big). \tag{18}$$

The key point of our argument in this step is to neglect the second term on the right hand side, and observe that numerator of the first term satisfies

$$\begin{aligned}(-k((\Box - k)^{-1}(\overline{\Phi_i} \cdot H^{(\ell)}), \overline{\Phi_i} \cdot H^{(\ell)}) &\geqslant (H(\overline{\Phi_i} \cdot H^{(\ell)}), \overline{\Phi_i} \cdot H^{(\ell)}) \\ &\geq \big|\big(\overline{\Phi_i} \cdot H^{(\ell)}, \frac{H^{(\ell-1)}}{\|H^{(\ell-1)}\|_2}\big)\big|^2 \\ &= \frac{\|H^{(\ell)}\|_2^4}{\|H^{(\ell-1)}\|_2^2}.\end{aligned}$$

In the above the first inequality follows from spectral decomposition. The key observation is the second identity which follows from linear algebra. The proposition follows directly from combining the above estimates.

3.4 Construction of the Finsler metric

Given the above Proposition, we may hope to absorb the bad of term of the right hand side of estimates (17) for ℓ by the good term for $\ell+1$. Observe that the bad term $\frac{\|H^{(\ell+1)}\|_2^2}{\|H^{(\ell)}\|_2^2} = 0$ for $\ell = n$, since $H^{n+1}(M_t, \wedge^{n+1}\Theta) = 0$. However, the term may vanish for some $\ell < n$ and the greatest such ℓ may be different for different base point t. As a result, the search for Finsler metric with negative upper bound in curvature is rather challenging. This is the third step of the proof in [51]. Our result is as follows.

Let $N \geq n$ be a fixed positive integer. Let

$$A := \frac{(2\pi)^n K_{M_t}^n}{k^n n!}. \tag{19}$$

Let $C_1 := \min\big\{1, \frac{1}{A}\big\}$ and $C_\ell = \frac{C_{\ell-1}}{3} = \frac{C_1}{3^{\ell-1}}$ for $2 \leq \ell \leq n$. Let $a_1 = 1$ and $a_\ell = \big(\frac{3a_{\ell-1}}{C_1}\big)^N = \big(\frac{3}{C_1}\big)^{\frac{N(N^{\ell-1}-1)}{N-1}}$ for $2 \leq \ell \leq n$. Define for $u \in T_tS$ and $t \in S$ a function $h : TS \to \mathbb{R}$ given by

$$h(u) = \Big(\sum_{\ell=1}^n a_\ell \|u\|_{WP,\ell}^{2N}\Big)^{\frac{1}{2N}} \tag{20}$$

Then

$$\partial_t \partial_{\bar{t}} \log((h(\frac{\partial}{\partial t}))^2) \geq \frac{C_n}{n^{\frac{1}{N}} a_n^{1+\frac{1}{N}}} \cdot (h(\frac{\partial}{\partial t}))^2. \tag{21}$$

This implies that the holomorphic sectional curvature is bounded by a negative constant. Hence we conclude the following theorem after applying Ahlfors Schwarz Lemma.

Theorem 1. *Let $\pi : \mathscr{X} \to S$ be an effectively parametrized holomorphic family of compact canonically polarized complex manifolds over a complex manifold S. Then S admits a C^∞ $Aut(\pi)$-invariant Finsler metric whose holomorphic sectional curvature is bounded above by a negative constant. Hence S is Kobayashi hyperbolic.*

Here we say that a Finsler metric h on S is *$Aut(\pi)$-invariant* if $f^*h = h$ for any pair of automorphisms $(F, f) \in \mathrm{Aut}(\mathscr{X}) \times \mathrm{Aut}(\mathscr{S})$ satisfying $f \circ \pi = \pi \circ F$.

We remark that the upper bound of the holomorphic sectional curvature in (21) depends only on the degree $K^n_{M_t}$ of the fibers. In complex dimension one, the result is essentially the same as (2), the formula of Wolpert for moduli space of Riemann surfaces.

4 Moduli of polarized Kähler Ricci-flat manifolds

Recall that in complex dimension 1, the space $P^1_{\mathbf{C}} - \{0, 1, \infty\}$ parametrizes a family of elliptic curves, and the base space is Kobayashi hyperbolic. Elliptic curves have trivial canonical line bundle. A higher dimensional analogue of the result is to consider the same problem for a holomorphic family of polarized Kähler Ricci-flat manifolds. Hence a natural question is whether such a family is Kobayashi hyperbolic or not. In particular, one asks if it is possible to study such problems from the point of view of Weil-Petersson metric. The question is answered in [52]

Theorem 2. *Let $\pi : \mathscr{X} \to S$ be an effectively parametrized holomorphic family of compact polarized Ricci-flat Kähler manifolds over a complex manifold S. Then S admits a C^∞ $Aut(\pi)$-invariant Finsler metric whose holomorphic sectional curvature is bounded above by a negative constant coming from generalized Weil-Petersson metrics. As a consequence, S is Kobayashi hyperbolic.*

A holomorphic family of compact complex manifolds $\pi : \mathscr{X} \to S$ over a complex manifold S is said to be a family of polarized Ricci-flat Kähler manifold if it satisfies the following properties. The mapping $\pi : \mathscr{X} \to S$ is a surjective holomorphic map of maximal rank between two complex manifolds $\mathscr{X}$ and S, and each fiber (M_t, ω_t) is a Ricci-flat Kähler manifold polarized by ω_t, where $M_t := \pi^{-1}(t)$, $t \in S$. Moreover, we require that the cohomology class $[\phi_t^* \omega_t] \in H^2(M_0, \mathbf{C})$ is a constant class for all t, where $\phi_t : M_0 \to M_t$ is the restriction of ϕ to $M_0 \times \{t\}$ for a smooth trivialization $\phi : M_0 \times I \to \mathscr{X}$.

Historically, a result analogous to the work of [43] for a family of polarized Kähler Ricci-flat manifolds was obtained by Nannicini [35] as follows.

$$R^{(WP)}_{i\bar{j}k\bar{\ell}}(t) = -\frac{1}{4V}(h_{i\bar{j}}h_{l\bar{k}} + h_{i\bar{k}}h_{l\bar{j}}) - \int_{M_t}\langle(\mathscr{L}_{v_i}\Phi_k, \mathscr{L}_{v_j}\Phi_\ell\rangle\frac{\omega^n}{n!}$$
$$+\int_{M_t}\langle H(\Phi_i \owedge \Phi_k), H(\Phi_j \owedge \Phi_\ell)\rangle\frac{\omega^n}{n!},$$

here V is the volume of M_o.

Modifying the argument of the last section, Wing-Keung To and myself obtained in [52] first the following generalization to the higher dimensional cases.

$$\begin{aligned}&\partial_i\overline{\partial_i}\log\|\Psi_J\|_2^2\\ = &\frac{1}{\|\Psi_J\|_2^2}\big(H(\overline{\Phi_i}\cdot\Psi_J), \overline{\Phi_i}\cdot\Psi_J) + (H(\langle\Phi_i,\Phi_i\rangle), \langle\Psi_J,\Psi_J\rangle)\\ &+((H(\mathscr{L}_{v_i}\Psi_J), \mathscr{L}_{v_i}\Psi_J) - \big|(\mathscr{L}_{v_i}\Psi_J, \frac{\Psi_J}{\|\Psi_J\|_2})\big|^2 - (H(\Phi_i \owedge \Psi_J), H(\Phi_i \owedge \Psi_J))\big).\end{aligned}$$

The rest of the argument is then a modification of the arguments in Sect.3. In particular, in place of A chosen in (19), we define

$$A := \frac{(2\pi)^n\omega_t^n}{k^n n!} = \frac{(2\pi)^n\omega_0^n}{k^n n!}. \tag{22}$$

Similar to the case of a family of canonically polarized manifolds, the upper bound on the holomorphic sectional curvature depends only on A.

5 Some higher dimensional examples of varieties with finite number of rational points

In this final section, we remark on a few observations related to the arithmetic aspects of complex hyperbolic manifolds. A basic conjecture of Lang states that a smooth projective algebraic manifold defined over a number field k is complex hyperbolic if and only if it has a at most a finite number of rational points over k, and a similar statement for integral points with respect to the divisor given by a compactifying divisor, cf. [27].

The conjecture in complex dimension one for hyperbolic compact Riemann surfaces is verified by the solution of Mordell Conjecture by Faltings [18]. The quasi-projective case in complex dimension one is known earlier in the results of Siegel. An alternative proof of the Mordell Conjecture is given by Vojta [55]. The problem in higher dimensions is wide open. An interesting class of examples known in higher dimension is the result of Faltings [19], [20] on subvarieties X of Abelian varieties A defined over a number field k, which states that the Zariski closure of the set of rational points on X is a the translate of an Abelian subvariety in A. A similar statement for integral points on the complement of an ample divisor is proved in the papers as well. Note that the corresponding results for complex hyperbolicity were proved by Bloch [6] and Siu-Yeung [48] mentioned in Sect. 3. The arithmetic results on semi-abelian varieties was given by Vojta [56].

From Riemann Uniformization Theorem, the universal covering of a compact hyperbolic Riemann surface is just the complex ball of dimension one, $B^1_{\mathbf{C}} \cong \Delta^1$, and similarly for a non-compact Riemann surface of finite volume. From differential geometric point of view, the simplest complex hyperbolic manifolds are provided by complex hyperbolic spaces $B^n_{\mathbf{C}}/\Gamma$ for some discrete group Γ. The hyperbolicity follows from Ahlfors Schwarz Lemma and the existence of a Kähler metric with negative Riemannian sectional curvature. The only other compact complex manifolds known to possess a Kähler metric of negative Riemannian sectional curvature are the examples known as Mostow-Siu surfaces, see [34] and [15]. In the following we describe a few examples studied in [60] for which results in complex geometry and the results of Faltings above allow us to deduce Mordellic properties. We will only consider complex dimension two. First we make the following observation in [60].

Proposition 3 *Let M be a smooth projective algebraic surface defined over a number field F. Assume that there exists an unramified covering $M' \to M$ defined over some number field F' so that the irregularity $q(M') = \dim H^1(M', \mathscr{O}_{M'})$ is at least* 3 *and that there is no non-constant morphism from a curve of genus* 0 *or* 1 *into M, then $M(F)$ has finite cardinality.*

The idea of the proof is to relate rational points $M(F)$ on M to rational points $M'(F')$ for some finite extension F' of F by classical results of Hermite and Chevalley-Weil. Then one considers the Albanese map on M' and apply the results of Faltings in [19] mentioned above.

We have the following immediately corollary, for which conditions in the proposition above can be verified.

Corollary 1. *The number of rational points on a smooth Picard Modular Surface defined over a number field is finite. Similarly the result holds for the number of rational points on a Mostow-Siu surface defined over a number field.*

Proposition 3 shows that Mordellic properties would follow from virtual positivity of the first Betti number for surfaces, at least for complex two ball quotients. In this way, the problem is related to the following problem in cohomology of Lie groups and geometric topology. It has been conjectured by Borel [7], parallel to a corresponding conjecture of Thurston for real hyperbolic spaces, that the first Betti number of of a complex ball quotient is virtually positive. Recall that a property is virtually true on a manifold if it holds after passing to a finite unramified covering if necessary. Hence Mordellic properties of complex two ball quotients can be established if the first Betti number is shown to be virtually at least 5. The conjecture of Borel is open for a general compact complex ball quotient at this point. It is proved in [60] that the conjecture is true for a non-compact complex ball quotient of finite volume.

Theorem 3. *Let $M \cong B^2_{\mathbf{C}}/\Gamma$ be a smooth cofinite complex two ball quotient. Then given any $N > 0$, there exists a finite unramified covering of M' with $b^1(M') \geqslant N$. In particular, M has at most a finite number of integral points with respect to some compactifying divisor.*

The idea of proof is to observe that the Betti number of such a complex two ball quotient increases with the number of cusps, which increases when one goes to some unramified coverings. In the case of an arithmetic quotient of a complex two ball, a compactification can be given by Baily-Borel compactification [8] which adds a point to a cusp and is singular, or by toroidal compactification developed by Ash-Mumford-Rapoport-Tai [4] which adds a torus to an end and is smooth. In the case of non-arithmetic quotients, a differential geometric construction to each of the above two cases has been developed by Siu-Yau [47] and Mok [33]. To find some non-trivial class in H_1, we consider the structure near the toroidal compactification of an end, and in a sense show that some 1-cycle from the compactifying torus lifts to M. Further discussions can be found in [61].

At this point, the situation for other complex ball quotients is still not completely understood.

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