

CLASSIFICATION OF SURFACES OF GENERAL TYPE WITH EULER NUMBER 3

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Abstract *The smallest topological Euler-Poincaré characteristic supported on a smooth surface of general type is 3. In this paper, we show that such a surface has to be a fake projective plane unless $h^{1,0}(M) = 1$. Together with the classification of fake projective planes given by Prasad and Yeung [PY], the recent work of Cartwright and Steger [CS], and a proof of the arithmeticity of the lattices involved in the present article, this gives a classification of such surfaces.*

1. Introduction

1.1 The main purpose of this article is to prove the following result on classification of smooth surfaces of general type with the smallest possible topological Euler-Poincaré characteristic. The topological Euler-Poincaré characteristic, denoted by $e(M)$, is the same as the second Chern number $c_2(M)$ of the surface M . It is also simply called the Euler-Poincaré characteristic or the Euler number in this paper.

Theorem 1. (a). *Let M be a smooth surface of general type. Then the Euler-Poincaré characteristic $e(M)$ of M is at least 3.*

(b). *Suppose $e(M) = 3$. Then $M = B_{\mathbb{C}}^2/\Gamma$ is the quotient of a complex hyperbolic space by a torsion free lattice of $PU(2, 1)$. Furthermore, unless $h^{1,0}(M) = 1$, M is a fake projective plane.*

(c). *Up to biholomorphism, there are only two examples of M with $e(M) = 3$ and $h^{1,0}(M) = 1$. The two examples are complex conjugate of one another.*

(d). *The moduli space of minimal surfaces of general type with $e(M) = 3$ is reduced and consists of 102 points. 100 of such points correspond to fake projective planes with $h^{1,0} = 0$. Two of such points correspond to surfaces with $h^{1,0} = 1$.*

1.2 The main examples of smooth surfaces of general type with Euler-Poincaré characteristic 3 are provided by fake projective planes, which are smooth surfaces with the same Betti numbers as the projective plane but are not biholomorphic to the projective plane. An example of fake projective plane was first constructed by Mumford [Mu3], followed by constructions of Ishida-Kato [IK] and Keum [Ke]. Recently fake projective planes have been classified by Prasad and Yeung [PY] into twenty-eight classes, each of which was shown to consist of at least two fake projective planes up to biholomorphism. Subsequently, Cartwright and Steger [CS] showed that there were precisely 50 non-isometric fake projective planes among the

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twenty eight classes. It is known that for each fake projective plane as a Riemannian manifold, it supports precisely two different conjugate complex structures (cf. [KK]).

It is natural to ask whether fake projective planes exhaust all possibilities of smooth surfaces of general type with Euler-Poincaré characteristic 3. In their work [CS] to enlist all the fake projective planes in the twenty-eight classes classified in [PY], Cartwright and Steger come up with an interesting surface with Euler-Poincaré characteristic 3 and the first Betti number 2. The results of this article show that fake projective planes as classified in [PY] and [CS], and the examples of Cartwright-Steger in [CS] mentioned above, exhaust all smooth surfaces of general type with Euler-Poincaré characteristic 3. The detailed computations are located in the weblink provided in [CS].

1.3 The following is an outline of proof of Theorem 1. Part (a) follows from classical results in geometry, as explained in §1. The main part of the article is in the proof of the statement of (b) and (c). (b) and (c) are proved using combination of classical methods from algebraic geometry as well as techniques of harmonic mappings into appropriate Bruhat-Tits buildings. Finally, the classification of Prasad-Yeung [PY] and Cartwright-Steger [CS] is applied to conclude the proof.

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2. Preliminaries

2.1 Let us denote by $c_i = c_i(M)$ the Chern numbers, $b_i = b_i(M)$ the Betti numbers of M and $h^{i,j} = h^{i,j}(M) = \dim_{\mathbb{C}} H^j(M, \Omega_M^i)$ the corresponding Hodge numbers. $c_2(M)$ is just the Euler-Poincaré characteristic of M . For our convenience, let us recall some standard identities.

$$\begin{aligned}
 (1) \quad c_2 &= 2b_0 - 2b_1 + b_2 \\
 (2) \quad \frac{1}{12}(c_1^2 + c_2) &= h^{0,0} - h^{1,0} + h^{2,0} \\
 (3) \quad b_1 &= 2h^{1,0}, \\
 (4) \quad b_2 &= 2h^{2,0} + h^{1,1}, \\
 (5) \quad h^{i,j} &= h^{j,i},
 \end{aligned}$$

where the second one is the Noether formula and the third one comes from Hodge decomposition.

For Theorem 1(a), let first M be a minimal surface of general type so that $c_1^2 > 0$. It is clear from Miyaoka-Yau inequality that $c_1^2 \leq 3c_2$. Noether's Formula implies that

$$0 < \frac{1}{12}(c_1^2(M) + c_2(M)) \leq \frac{1}{3}c_2(M).$$

It follows that $e(M) = c_2(M) \geq 3$.

Suppose now that M is an arbitrary surface of general type. Let M' be a minimal surface of general type obtained by contracting some -1 curves on M . Since contracting a -1 curve decreases the Euler Poincaré characteristic by 1, we

know that $e(M) \geq e(M') \geq 3$. In particular, if $e(M) = 3$, the above discussions imply that $M = M'$. This concludes the proof of Theorem 1(a).

Since the far right hand side of the above sequence of inequalities is 1, it follows that the inequality sign \leq is actually an equality. We conclude that $c_1^2(M) = 9 = 3c_2(M)$. It is well known that a compact complex surface M with $c_1^2(M) > 0$ is projective algebraic (cf. [BHPV], page 161). The results of Aubin [A] and Yau [Ya] on the Calabi Conjecture in the case of negative scalar curvature implies the existence of Kähler-Einstein metric, see also [Mi]. This in turn implies that the metric is the standard hyperbolic metric using the fact that $c_1^2(M) = 3c_2(M)$, see for example the survey in [Y2], page 391. Hence M is a compact complex ball quotient.

We summarize the observation above as follows.

Proposition 1. *Let M be a smooth surface of general type. Then the Euler-Poincaré characteristic $e(M) \geq 3$. Moreover, the equality occurs if and only if $M = B_{\mathbb{C}}^2/\Gamma$ is the quotient of a complex hyperbolic space by a torsion free lattice of $PU(2, 1)$.*

2.2 The moduli space of such surfaces M with $e(M) = 3$ is well-known to come with a natural scheme structure. The infinitesimal deformation of any such M in a local Kuranishi family of deformation is given by an element in $H^1(M, \Theta)$, where Θ is the sheaf of holomorphic vector fields on M . Since any such M is a locally Hermitian symmetric space, according to the local rigidity of Calabi and Vesentini [CV], $H^1(M, \Theta) = 0$. It follows that the virtual dimension of any deformation space is zero, which implies that the actual deformation space is of dimension 0. As the dimension of the virtual deformation is the same as the dimension of the actual deformation, we conclude that the moduli space is reduced. As the dimension is zero, the moduli space consists of a finite number of points.

Hence to prove Theorem 1, our working assumption from this point on is that M is a compact complex two ball quotient. From Noether's formula (0.2),

$$h^{0,0}(M) - h^{1,0}(M) + h^{2,0}(M) = \frac{1}{12}(c_1^2(M) + c_2(M)) = 1.$$

We conclude that $h^{1,0}(M) = h^{2,0}(M)$. The purpose of §3-5 is to employ classical algebraic geometric method to prove that $h^{1,0}(M) \leq 2$. The case $h^{1,0}(M) = 0$ corresponds to fake projective plane and has been classified in [PY] and [CS]. For the cases $h^{1,0}(M) = 1$ and 2, we show that the arguments in [Ye1] and [Ye2] can be modified to prove the arithmeticity of the lattice involved in §6, from which the classification results in [PY] and [CS] can be applied again.

2.3 For our later discussions, let us recall the following result on the canonical line bundle of M .

Lemma 1. *The canonical line bundle K_M of a smooth surface M of general type with $e(M) = 3$ satisfies $c_1^2(K_M) = 9 \int_M \Theta(H_{\widetilde{M}}) \wedge \Theta(H_{\widetilde{M}})$, where $\Theta(H_{\widetilde{M}})$ is the curvature form of an ample line bundle $H_{\widetilde{M}}$ on \widetilde{M} . $H_{\widetilde{M}}$ may not descend to M but $\int_M \Theta(H_{\widetilde{M}}) \wedge \Theta(H_{\widetilde{M}}) \in \mathbb{Z}$.*

Proof Denote by $H_{\widetilde{M}}$ the $SU(2, 1)$ -equivariant line bundle discussed in §10.4 of [PY] as well as in [Ko]. Let $\pi : \widetilde{M} \rightarrow M = \widetilde{M}/\Pi$ be the uniformization map. On \widetilde{M} , $H_{\widetilde{M}}$ is a third root of π^*K_M as an $SU(2, 1)$ line bundle, but may not descend

to M as a holomorphic line bundle since Π in general only lives in $PU(2, 1)$. We only know that $H_{\widetilde{M}}$ descends as a multivalued line bundle with ambiguity lying in \mathbb{Z}_3 . However, the curvature form $\Theta(H_{\widetilde{M}})$ descends as a genuine $(1, 1)$ form on M . Let \mathcal{F} be a fundamental domain of M in \widetilde{M} , It follows that

$$\int_M \Theta(H_{\widetilde{M}}) \wedge \Theta(H_{\widetilde{M}}) = \int_{\mathcal{F}} \Theta(H_{\widetilde{M}}) \wedge \Theta(H_{\widetilde{M}}) = \frac{1}{9} c_1(K_M) \cdot c_1(K_M).$$

However, we observe from Chern number equality $c_1^2(M) = 3c_2(M)$ and the Noether Formula that $c_2(M) = \frac{1}{4}(c_1^2(M) + c_2(M)) = 3\chi(\mathcal{O}_M)$. Hence $c_1^2(M) = 9\chi(\mathcal{O}_M)$ and $\int_M \Theta(H_{\widetilde{M}}) \wedge \Theta(H_{\widetilde{M}}) \in \mathbb{Z}$. \square

3. Case of irregularity ≥ 3 .

3.1 We note that for any two linearly independent holomorphic one forms ω_1 and ω_2 on M , the wedge product $\omega_1 \wedge \omega_2$ cannot be identically zero on M . Otherwise Castelnuovo-de Franchi Theorem implies that there is a fibration $\pi : M \rightarrow S$ of M over a Riemann surface S of genus at least 2 (cf. [BHPV], page 157). Let $g(S)$ be the genus of S and $g(M_s)$ be the genus of a generic fiber M_s of π . Denote by $e(M)$ the Euler-Poincaré characteristic of a manifold M . It follows that

$$(6) \quad e(M) = e(S)e(M_s) + \sum_{s_o} n_{s_o},$$

where the sum is taken over the finite number of singular fibers M_{s_o} of π , of which each $n_{s_o} = e(M_{s_o}) - e(M_s)$ is a non-negative integer, and is positive unless M_{s_o} is a multiple fiber with $(M_{s_o})_{red}$ nonsingular elliptic (cf. [BHPV], page 118). Hence $e(M) \geq (2g(S) - 2)(2g(M_s) - 2) \geq 4$. This contradicts our assumption that $c_2(M) = 3$. Hence $\omega_1 \wedge \omega_2$ is a non-trivial holomorphic two form on M whenever ω_1 and ω_2 are linearly independent.

Since $\omega_1 \wedge \omega_2 \neq 0$, this implies that $h^{2,0} \geq 2h^{1,0} - 3$ by considering $\omega_1 \wedge \omega_i$ and $\omega_i \wedge \omega_2$, where $\{\omega_i\}_{1 \leq i \leq h^{1,0}(M)}$ is a basis of $H^0(M, \Omega)$. Since we know that $h^{1,0}(M) = h^{2,0}(M)$, we conclude that $h^{1,0}(M) = h^{2,0}(M) \leq 3$.

3.2 The case of $h^{1,0}(M) = h^{2,0}(M) = 3$ was ruled out from the classification of Hacon and Pardini ([HP], Theorem 2.2, see also [CCM]).

Denote by $A = A(M)$ the Albanese variety of M and $\alpha : M \rightarrow A$ the Albanese mapping. We are going to eliminate in the next section the case of $h^{1,0} = 2$. The case of $h^{1,0} = 1$ will be discussed in §5-6.

4. The case of irregularity 2

4.1 This section can be skipped without affecting the the rest of the paper. It has been our original hope to eliminate the case $h^{1,0} = 2$ completely by classical algebraic geometric method. It turns out that we only succeeded in obtaining a weaker statement stated as Proposition 2 below using the conventional methods. The case would finally be eliminated using methods of harmonic maps into Bruhat-Tits buildings in §6. The purpose here to record how far we can push using conventional methods.

Proposition 2. *Let M is a complex two ball quotient with $e(M) = 3$. Assume that K_M is numerically $3H_{\widetilde{M}}$ as a line bundle and that the Albanese map does not contract any divisor on M . Then $h^{1,0} \neq 2$.*

Proof We assume for the sake of proof by contradiction that $h^{1,0}(M) = 2$.

Applying equation (6) and the argument of the first paragraph of §3, we conclude that $\alpha(M)$ cannot be an algebraic curve. Hence we conclude that $\alpha(M)$ is of complex dimension two and hence $\alpha(M) = A$. We claim that A is a simple Abelian surface. Otherwise from Poincaré Complete Reducibility Theorem, there exists an elliptic curve C in A and the quotient subvariety A/C is another an elliptic curve. Let $p : A(M) \rightarrow A/C$ be the holomorphic projection. $p \circ \alpha$ gives rise to a fibration of M over an elliptic curve. From hyperbolicity of M again, we know that the fibers of the fibration are hyperbolic. Instead of analyzing the problem as in previous section, we refer to the result of Zucconi [Z] on the classification of surface of Albanese general type admitting an irrational pencil of curves to rule out this situation. Note that the pencil cannot be isotrivial from the argument in the first paragraph of §3 again.

4.2 Let $R = \sum_{i=1}^n a_i R_i$ be the ramification divisor of α , where R_i are reduced and irreducible curves and a_i are positive integers. In particular, $a_i + 1$ is the local branching order at a generic point of R_i . We have

$$(7) \quad 9 = K_M \cdot K_M = K_M \cdot (\alpha^* K_A + R) = \sum_{j=1}^n a_j K_M \cdot R_j.$$

Each R_i is cohomologous to $b_i H_M + T_i$ as a rational linear combination of elements in the Neron-Severi group modulo torsion, where T_i is a $(1, 1)$ -class orthogonal to H_M with respect to the quadratic form obtained from intersection pairing. From Hodge Index Theorem (cf. [BHPV], Cor 2.14, page 142), we conclude that T_i has negative self-intersection unless T_i is cohomologous to 0. Since both K_M and H_M represent classes in $H^2(M, \mathbb{Z})$ and hence lie in the Neron-Severi group of M , the same is true for T_i . It follows that T_i is an element in the Neron-Severi group. Clearly by taking intersections with H , we see that b_i is a rational number. As $(3H_M) \cdot (3H_M) = K_M \cdot K_M = 9$, we conclude that $H_M \cdot H_M = 1$ and hence $b_i = R_i \cdot H_M / H_M \cdot H_M$ are positive integers.

Computing $K_M \cdot H_M$, we conclude that

$$(8) \quad 3 = K_M \cdot H_M = \sum_{j=1}^n b_j a_j.$$

Since a_i and b_i are both positive integers, the followings are all the possibilities for n and (a_i, b_i) , $i = 1, \dots, n$.

Cases	n	(a_i, b_i)
<i>I</i>	3	$(1, 1), (1, 1), (1, 1)$
<i>II</i>	2	$(1, 1), (2, 1)$
<i>III</i>	2	$(1, 1), (1, 2),$
<i>IV</i>	1	$(3, 1)$
<i>V</i>	1	$(1, 3)$

Furthermore, since

$$9 = K_M \cdot K_M = \left(\sum_i a_i b_i H_i + a_i T_i \right)^2 = \left(\sum_i a_i b_i \right)^2 + \left(\sum_i a_i T_i \right) \cdot \left(\sum_i a_i T_i \right).$$

It follows from equation (8) that $(\sum_i a_i T_i) \cdot (\sum_i a_i T_i) = 0$. Again, according to Hodge Index Theorem it follows that the following divisor is trivial,

$$(9) \quad T := \sum_i a_i T_i = 0.$$

We introduce some notations. For a curve C , denote by $e(C)$ and $\chi(C)$ the topological Euler number and the Euler number of C as an embedded curve respectively. The two notions are the same if C is smooth. Let $\nu : \tilde{C} \rightarrow C$ be the normalization of C , we denote (cf. [Be], page 96)

$$(10) \quad \begin{aligned} \epsilon_C &:= e(\tilde{C}) - e(C) = h^0(\nu_* \mathbb{C}_{\tilde{C}} / \mathbb{C}_C) = \sum_{x \in C} \dim_{\mathbb{C}}(\nu_* \mathbb{C}_{\tilde{C}} / \mathbb{C}_C)_x, \\ \delta_C &:= \sum_{x \in C} \dim_{\mathbb{C}}(\nu_* \mathcal{O}_{\tilde{C}} / \mathcal{O}_C)_x. \end{aligned}$$

For each R_i , it follows from Adjunction formula that

$$(11) \quad 3b_i + b_i^2 + T_i \cdot T_i = K_M \cdot R_i + R_i \cdot R_i = -\chi(R_i) = 2(g_{R_i} - 1) = 2(g_{\tilde{R}_i} + \delta_{R_i} - 1).$$

Denote $S_i = \alpha(R_i)$. From earlier discussions, we may assume that S_i is an hyperbolic curve and hence genus of its normalization is at least 2. Proposition 2 now follows from Proposition 3 below.

Proposition 3. *With the assumption of Proposition 2, none of the cases I-V can occur.*

Proof Since the proof is given by detailed case by case analysis, let us first explain briefly the idea behind our argument. The argument is based on the following three facts, (i) from Lemma 2 and 3, none of the components of the ramification divisor for the albanese map $\alpha : M \rightarrow A(M)$ is collapsed to a point by α , (ii) $c_2(M) = 3, c_1^2(M) = 9, c_2(A(M)) = c_1^2(A(M)) = 0$ are all small non-negative integers, and (iii) all divisors on M have genus at least 2. As a result, the possible set of branching divisors is very restricted. The situation will be simpler if the divisors involved are smooth. However, a singularity in a divisor contributes to the genus formula as in identity (11) below, which leads to further constraints on the numerical data. In the end, the numerical conditions, some detailed analysis near branching divisors in the favor of Weierstrass Preparation Theorem, and Hurwitz Formula finally lead to contradiction for each of the above cases. Though it may take some work to go through all the above five cases, the basic idea of proof is already presented in the proof of Case IV below. The proof of the other cases are some technical improvements.

4.3 We need the following lemma for local analysis of the mappings around the branching locus. Parts (a) and (b) deal with branching behavior near points which are either smooth or in simple normal crossings. As we will see, the numerical constraints on the Chern numbers impose restrictions on singularities, and (a) and (b) are sufficient for several of the above cases. For the remaining cases, the conclusion in case (c) is sufficient after applying more detailed analysis.

Lemma 2. (a). *Suppose R is a local irreducible component of the ramification divisor of α . Let m be the ramification index of α along R . Suppose $p \in R$ is a smooth point of R at which the local index of α is greater than m . Then locally $p \in R_1$, where R_1 is another local ramification divisor intersecting R .*

(b) Suppose R_1 and R_2 are the only two branches in simple normal crossing at a point p and the branching order of α around R_1 and R_2 are m_1 and m_2 respectively. Then the local degree d_1 of $\alpha|_{R_1}$ at p is at least m_2 , the local degree d_2 of $\alpha|_{R_2}$ is at least m_1 , and the local index of p is at least $m_1 m_2$.

(c). In general, if p is the intersection of at least two local components R_1, \dots, R_l of the ramification divisor. Then the degree of $\alpha|_R$ is at least 2.

Proof Note that R_1 and R may belong to the same global irreducible component of the ramification locus. The whole problem is local. We choose an appropriate local holomorphic coordinate system (z, w) in a neighborhood U_p of p , and a local holomorphic coordinate system (u, v) of $\alpha(p)$, such that R is defined locally by $z = 0$, $\alpha(R)$ is locally defined by $u = 0$, and the mapping $\alpha|_{U_p}$ is given by

$$\begin{aligned} u &= z^m f(z, w) \\ v &= g(z, w) \end{aligned}$$

for some local holomorphic functions f and g defined on U_p . A direct computation shows that

$$\begin{aligned} du &= (mz^{m-1}f + z^m f_z)dz + z^m f_w dw, \\ dv &= g_z dz + g_w dw, \\ du \wedge dv &= z^{m-1}[(mf + zf_z)g_w - zf_w g_z]dz \wedge dw \end{aligned}$$

Letting $h(z, w) := (mf + zf_z)g_w - zf_w g_z$, the above shows that the Jacobian of α has constant vanishing order along $R \cap U_p$ if and only if $h(z, w)$ is a non-zero constant c on U_p . In particular, if the local index at p is larger than m , $h(z, w) = 0$ gives rise to another local component R_1 of the ramification locus. This leads to the conclusion in (a).

For (b), in the above discussion, R_2 has to be given locally by $h(z, w) = 0$ from assumption. $h(z, w)$ is not identically zero on $z = 0$. Along $z = 0$, $h(0, w) = mf(0, w)g_w(0, w)$. Hence the vanishing order of $h(z, w)$ is bounded from below by the vanishing order of $g_w(0, w)$. We may expand h by Weierstrass Preparation Theorem (or by Taylor expansion) (cf. [GH])

$$h(z, w) = \psi(z, w)(w^d + a_1(z)w^{d-1} + \dots + a_d(z))$$

where $\psi(0, 0) \neq 0$ and $a_i(0) = 0$ for $i = 1, \dots, d$.

Since R_2 is defined by $h(z, w) = 0$ and at generic point of $h(z, w) = 0$, the Jacobian $J(\alpha)$ is precisely given by the vanishing of $h(z, w)$ at $z \neq 0$. Note that α does not contract in the direction tangential to R_2 at the intersection $\{z = z_0\} \cap R_2$ for a generic z_0 . $h(z, w)$ has vanishing order m_2 along a generic point of R_2 . Hence we may write $h(z, w) = (l(z, w))^{m_2}$ at a generic point of U_p , so that R_2 is defined by $l(z, w) = 0$ set theoretically. At $z = 0$, the set R_2 is defined by $l(0, w) = 0$ (which coincides with $w = 0$ when $z = 0$.) Hence by semi-continuity, the vanishing order of $\alpha|_{U_p \cap \{z=0\}}$ is at least m_2 .

Reversing the roles of R_1 and R_2 , we see that the same conclusion holds for R_2 . Moreover, the vanishing order of the Jacobian $J(\alpha)$ at $(z, w) = (0, 0)$ is bounded from below by the corresponding one from $z^{m_1}w^{m_2}$.

For (c), the mapping around p can be described in general by $u = f(z, w), v = g(z, w)$ for some holomorphic coordinates z and w . Then $du \wedge dv = (f_z g_w - f_w g_z)dz \wedge dw$. Suppose we can factor $H := (f_z g_w - f_w g_z)$ into irreducible polynomials $H_i, 1 \leq i \leq l$, where $l \geq 2$, such that $H_1 = 0$ (respectively $H_2 = 0$) gives rise

to R_1 (respectively R_2). Changing the coordinates if necessary, we may assume that H_1, H_2 do not vanish identically along w -axis. From Weierstrass Preparation Theorem, we may consider without loss of generality that H_1, H_2 are Weierstrass polynomials in w in a neighborhood of $p = (0, 0)$. Using Weierstrass Division theorem, we may write near $(0, 0)$, $H_2 = A(z, w)H_1 + B(z, w)$, where $B(z, w)$ is a Weierstraass polynomials of degree less than that of H_1 . (cf. [GH]). $B(z, w)$ is not a constant, for otherwise the constant is 0 and there is only one irreducible component through p . Hence restricting to $H_1 = 0$, $d\alpha$ has extra vanishing order at p coming from the contribution of H_2 .

4.4 We are now going to eliminate the cases I-V one by one.

Case IV: $a = 3$ and $b = 1$. Since there is only one component $R = R_1$ in $\sum_i a_i T_i$, equation (9) implies that $T = 0$ and $R = H$. It follows from equation (11) that $g(\tilde{R}) + \delta(R) - 1 = 2$. From hyperbolicity of R , we have the following two subcases, (i), $g(\tilde{R}) = 3$, $\delta(R) = 0$ and hence R is smooth with $\tilde{R} \cong R$; and (ii), $g(\tilde{R}) = 2$, $\delta(R) = 1$

In subcase (i), R is a smooth curve and has no self-intersection. According to the previous lemma, the local index of α is equal to 4 everywhere along R . The mapping $\alpha|_{M-R}$ hence has local index 4 everywhere on $M - R$. Considering Euler number by cell decomposition according to S and $A - S$, we conclude that

$$3 = e(M) = 4(e(A) - e(S)) + e(R) = -4ce(S) - (3 + b) \cdot b.$$

The right hand side is even and we reach a contradiction. Note that if there is an irreducible component R_o of $\alpha^{-1}(S)$, R_o should have trivial intersection with R , otherwise there will be extra local index of α at the intersection point, contradicting Lemma 2a.

Consider now subcase (ii). In such case, there is a simple normal crossing of R at say, $q \in R$. Let R_a and R_b be local branches of $R \cap U_q$ for a small neighborhood of q . Since S is hyperbolic from earlier discussions, it follows that the genus $g(\tilde{S})$ of the normalization of S is at least 2. As S is covered by R , it follows that there is a holomorphic covering $\tilde{\alpha} : \tilde{R} \rightarrow \tilde{S}$. Hence we conclude that $g(\tilde{S}) = 2$ from Riemann-Hurwitz Formula, and that $\tilde{\alpha}$ is actually a biholomorphism. In particular, $\alpha|_{U_q \cap R - \{q\}}$ is one to one in a punctured neighborhood of $U_q - \{q\}$ of q . On the other hand, in the notation of Lemma 2, we see that the mapping $\alpha|_R : R \rightarrow S$ has degree $d \geq 2$ on U_q , ramifying at q . This clearly contradicts the fact that $\alpha|_{U_q \cap R - \{q\}}$ is injective. The contradiction rules out Case IV.

Case III: there are two components R_1 and R_2 in the ramification locus. Let $S_i = \alpha(R_i)$ for $i = 1, 2$. For R_1 , $a_1 = 1$ and $b_1 = 1$. For R_2 , $a_2 = 1$ and $b_2 = 2$. From equation (11), we conclude that

$$(12) \quad g(\tilde{R}_1) + \delta(R_1) = 3 + \frac{1}{2}T_1 \cdot T_1$$

$$(13) \quad g(\tilde{R}_2) + \delta(R_2) = 6 + \frac{1}{2}T_2 \cdot T_2.$$

We claim that $T_1 \cdot T_1 = 0$. Assume on the contrary that $T_1 \cdot T_1 \neq 0$. From equation (12), as $g(\tilde{R}_1) \geq 2$ from hyperbolicity, and $\delta(R_1) \geq 0$, the left hand side is at least 2. Hence as $T_1 \cdot T_1 \neq 0$, it is a negative number and the only possibility is $T_1 \cdot T_1 = -2$ to make sure that the left hand side is at least 2, which also leads to

$g(\tilde{R}_1) = 2$ and $\delta(R_1) = 0$. In such case $\tilde{R}_1 = R_1$. From equation (9), we know that $a_1T_1 + a_2T_2 = 0$ and hence $T_2 \cdot T_2 = T_1 \cdot T_1 = -2$. Hence equation (10) implies that $g(\tilde{R}_2) + \delta(R_2) = 5$ and $\chi(R_2) = 8$. Furthermore, $T_1 \cdot T_2 = -T_1 \cdot T_1 = 2$. It follows that $R_1 \cdot R_2 = b_1 \cdot b_2 + T_1 \cdot T_2 = 4$. S_i is a curve on A . Since we assume that there is no elliptic curve on A , it follows that the genus $g(\tilde{S}_i) \geq 2$ and $\chi(S_i) \leq -2$. Since S_1 is covered by R_1 , for which the normalization \tilde{R}_1 has genus 2, it follows from Riemann-Hurwitz Formula that $g(\tilde{S}_1) = 2$ as well, and $\tilde{R}_1 \cong \tilde{S}_1$ following from the argument of Case V. In particular, $\alpha|_{R_1}$ induces a map $\tilde{\alpha} : R_1 \rightarrow \tilde{S}_1$ which has to be a biholomorphism since \tilde{R}_1 and \tilde{S}_1 have the same genus. Note that $R_1 \cdot R_2 > 0$ from earlier discussions and hence they do intersect. On the other hand, as R_1 is smooth, we know from Lemma 2 that at $p \in R_1 \cap R_2$, the vanishing order of $p \in R_1 \cap R_2$ is d , which is at least 2, the local index of α at a generic point of R_2 . This implies that $\alpha|_{R_1}$ has local degree $d \geq 2$ in a neighborhood of p , contradicting the earlier conclusion that $\tilde{\alpha}|_{R_1} : R_1 \rightarrow \tilde{S}_1$ is a biholomorphism. Hence we conclude that $T_1 \cdot T_1 = 0$. The claim is proved.

It follows from the claim and the Signature Theorem that $T_1 = 0 = T_2$.

Lemma 3. *In the case of $g(\tilde{R}_1) + \delta(R_1) = 3$, we conclude that $\delta(R_1) = 0$ and R_1 is smooth with $g(\tilde{R}_1) = 3$.*

Proof Assume on the contrary that $\delta(R_1) > 0$. The only possibility is $\delta(R_1) = 1$ and $g(\tilde{R}_1) = 2$. In particular, let q be the singularity, where since $\delta = 1$, there is simple normal crossing of R_1 at q . The same reason as above implies that $\tilde{\alpha} : \tilde{R}_1 \rightarrow \tilde{S}$ is a biholomorphism and hence $\alpha|_{R_1}$ is of degree 1 on R_1 . The above argument making use of Lemma 2 implies that $R_1 \cap R_2$ occurs at the single point q . Recall that there are two local branches R_{1a} and R_{1b} of R_1 at q . Lemma 2 applied to R_{1a} and R_2 again implies that the local degree of $\alpha|_{R_{1a}}$ in a neighborhood of q is $d \geq 2$, contradicting again that $\tilde{\alpha}|_{\tilde{R}_1} : \tilde{R}_1 \rightarrow \tilde{S}_1$ is a biholomorphism. Hence $\delta(R_1) = 0$, concluding the proof of the lemma.

Since $H \cdot H = 1$ on M , we know that $R_1 \cdot R_2 = 2$. Hence R_1 and R_2 have positive intersection. Let $q \in R_1 \cap R_2$. As $\alpha|_{R_1} : R_1 \rightarrow S_1$ is a covering map and $g(\tilde{R}_1) = 3$ from Lemma 2, we conclude easily that the followings are the only possibilities,

- (i) $g(\tilde{S}_1) = 3$, and
- (ii) $g(\tilde{S}_1) = 2$.

In Subcase (i), $\alpha|_{R_1}$ has degree 1 at a generic point of R_1 and the induced map $\tilde{\alpha} : \tilde{R} \rightarrow \tilde{S}$ is a biholomorphism. However, according to Lemma 2(c), at $q \in R_1 \cap R_2$, $\alpha|_{R_1}$ has local degree at least 2 at a generic point of a neighborhood U_q of q . This contradicts the conclusion above that $\tilde{\alpha}$ is an biholomorphism.

In Subcase (ii), $\tilde{\alpha} : \tilde{R}_1 \rightarrow \tilde{S}_1$ is a holomorphic map. Since \tilde{S} has genus 2 and $\tilde{\alpha}$ has genus 3, the Riemann-Hurwitz Formula implies that $\alpha|_{\tilde{R}_1}$ is an unramified holomorphic covering of degree 2. On the other hand, recall that there exists $q \in R_1 \cap R_2$, and $\alpha|_{U_q \cap R_1}$ has local degree at least 2 at a generic point of $U_q \cap R_1$ according to Lemma 2(c). It follows that $d = 2$. In fact, if we choose the neighborhood U_q of q sufficiently small, $\alpha|_{U_q \cap R_1 - \{q\}}$ is a two to one mapping everywhere, but $\alpha|_{U_q \cap R_1}$ is ramified at q . The mapping $\alpha|_{U_q \cap R_1}$ is given by $x \rightarrow x^2$ in terms of suitable holomorphic coordinates. This clearly contradicts the fact that $\tilde{\alpha} : \tilde{R}_1 \rightarrow \tilde{S}$

is an unramified mapping. Again, as R_1 is smooth, $\tilde{R}_1 = R_1$. The contradiction eliminates Case III.

Case II: In this case, $a_1 = 1$, $b_1 = 1$ for R_1 , and $a_2 = 2$ and $b_2 = 1$ for R_2 . From equation (9), we conclude that

$$g(\tilde{R}_i) + \delta(R_i) = 3 + \frac{1}{2}T_i \cdot T_i$$

for $i = 1, 2$.

Assume that $T_1 \cdot T_1 \neq 0$. Then $a_1T_1 + a_2T_2 = 0$ implies that $T_2 \cdot T_2 \neq 0$ as well. Hyperbolicity implies that the left hand side of the above identity is at least 2. Hence we conclude for $i = 1, 2$ that $T_i \cdot T_i = -2$, $g(R_i) = 2$ and $\delta(R_i) = 0$. This however contradicts $T_1 \cdot T_1 = (\frac{a_2}{a_1})^2 T_2 \cdot T_2 = 4T_2 \cdot T_2$.

We conclude that $T_1 \cdot T_1 = 0$ and hence $T_2 \cdot T_2 = 0$ as well. Hence $T_i = 0$ from Signature Theorem for each $i = 1, 2$. From Adjunction Formula, $g(\tilde{R}_i) + \delta(R_i) = \frac{1}{2}(3 + b_i)b_i + 1$. Hence $g(\tilde{R}_i) + \delta(R_i) = 3$ for $i = 1, 2$. Lemma 3 allows us to conclude that $\delta(R_i) = 0$, R_i is smooth and $g(R_i) = g(\tilde{R}_i) = 3$ for $i = 1, 2$.

Let us now apply the argument in the last paragraph of Case III to R_2 . In particular, we have two subcases (i) $g(\tilde{S}_2) = 3$, and (ii) $g(\tilde{S}_2) = 2$. The same argument as in the proof for Case III implies that none of the subcases can occur. We conclude that Case II cannot occur.

Case I: there are three irreducible components R_1, R_2, R_3 in the ramification locus with $a_i = 1 = b_i$ for $i = 1, 2, 3$. We write $R_i = H_M + T_i$, which satisfies

$$(14) \quad T_1 + T_2 + T_3 = 0$$

from Signature Theorem.

Assume that one of the $T_i \neq 0$, say, $T_1 \neq 0$ so that $T_1 \cdot T_1 < 0$ according to the Signature Theorem. Then equation (12) is still valid, which implies that $T_1 \cdot T_1 = -2$, $g(\tilde{R}_1) = 2$ and $\delta(R_1) = 0$. Equation (14) shows that at least one of T_2 and T_3 is non-trivial. Suppose $T_2 \neq 0$. The same argument as above implies that $T_2 \cdot T_2 = -2$, $g(\tilde{R}_2) = 2$ and $\delta(R_2) = 0$. Hence R_1 and R_2 are both smooth.

Note that $R_1 \cdot R_2 = 1 + T_1 \cdot T_2 \geq 1 > 0$. Hence $R_1 \cap R_2 \neq \emptyset$. Let $q \in R_1 \cap R_2$. As R_2 is smooth, Lemma 2(c) implies that the local degree of $\alpha|_{R_2 \cap U_q}$ is $d \geq 2$. This clearly contradicts the earlier conclusion that $\deg(\alpha|_{R_1}) = 1$.

Hence we conclude that $T_i = 0$ for each i . In such case, arguments in the previous cases imply that $g(\tilde{R}_i) + \delta(R_i) = 3$. Lemma 3 implies immediately that $\delta(R_i) = 0$ for each i . As in Case II, there are two subcases to consider again, depending on whether $g(\tilde{S}_i) = 3$ or $g(\tilde{S}_i) = 2$. In either subcase, the argument as in Case II leads to a contradiction. Hence Case V can be eliminated as well.

Case V: In this case, $a = 1$ and $b = 3$. Since there is only one irreducible component $R = R_1$ in $\sum_i a_i T_i$, equation (8) implies that $T = 0$ and $R = bH = 3H$. As $a = 1$, suppose U is a sufficiently small neighborhood of a generic point $x \in R$, $\alpha|_U : U \rightarrow \alpha(U)$ is locally a two fold ramified covering branching along a single divisor $S \cap \alpha(U)$, where $S := \alpha(R)$, though we do not have control on the degree of α .

Consider first the case that R is smooth. Then from Lemma 2(a), the ramification index of α along R is constantly 2.

We claim that if R_o is a component in $\alpha^{-1}(S)$ different from R , $R \cap R_o = \emptyset$. Suppose $p \in R \cap R_o$. Note that the degree of α along a generic point of R_o is 1. Assume first that R and R_o meet in normal crossing at p . Let E be a generic curve on A intersecting S transversally at $\alpha(p)$, such that tangent vectors to S and E at $\alpha(p)$ are linearly independent, and there is a component D of $\alpha^{-1}(E)$ such that $\alpha|_D$ is non-degenerate at p . $v := T_p D$ is a linear combination of R and R_o , and $\alpha_*(v)$ is thus mapped to a vector in $T_p S$ as α is smooth and $\alpha(R) = \alpha(R_o) = S$, which contradicts the fact that $\alpha_* v$ is a non-trivial tangent vector in $T_{\alpha(p)} E$ transversal to R . In case that R and R_o meet but do not meet transversally, we consider embedded resolution of $R \cap R_o$ by repeated blowing ups, and similar argument leads to a contradiction. The claim is proved.

From the claim the restriction of α to such R_o is unramified everywhere. Since $e(A) = 0$ and an unramified cover of A does not contribute to change in Euler number, the only non-trivial contributions to change in Euler number in applying topological Hurwitz formula occur only along the ramification divisor, which gives rise to

$$(15) \quad 3 = e(M) = -(2c)e(S) + e(R),$$

where c is the degree of $\alpha|_R : R \rightarrow S$. Assume that R is smooth. Then the right hand side is $2ce(S) - (3+b) \cdot b$, an even number and we reach a contradiction.

Hence the case that R is smooth can be ruled out and we may assume that R is singular. Let $\mathcal{S}(R)$ be the set of singular points of R . We are going to consider the following subcases,

- (i), there exists $q \in \mathcal{S}(R)$ such that $R \cap U_q$ has more than two irreducible components for a sufficiently small neighborhood U_q of q , and
- (ii), for all $q \in \mathcal{S}_q$, $R \cap U_q$ consists of only one irreducible component.

Let us consider first subcase (ii). In this case, since there is only one local component of $R \cap U_q$ for each singularity q of R , Lemma 2(a) still implies that the ramification index of α along R is constantly 2 and hence formula (15) applies. Moreover, the normalization replaces each singular point by another point on \tilde{R} , as explained in [G], Chapter 2. It means that $\dim_{\mathbb{C}}(\nu_* \mathbb{C}_{\tilde{R}} / \mathbb{C}_R) = 0$ and hence $e(\tilde{R}) = e(R)$ according to equation (11). Hence equation (15) still leads to a contradiction as the polarities of the left and right hand sides are different.

Let us now consider subcase (i). Let S_1 is a local component of S at a singular point $p \in \mathcal{S}(R)$. Let R_1 be a local component of $R \cap \alpha^{-1}(S_1)$ on which α ramifies, and R_{1o} a local component of $R \cap \alpha^{-1}(S_1)$ on which α does not ramify. Then arguments of the claim above still implies that $R_1 \cap R_{1o} = \emptyset$. Hence equation (15) still holds.

According to equations (9), (11), Riemann-Hurwitz formula and (15), we have

$$(16) \quad 10 = g(\tilde{R}) + \delta(R)$$

$$(17) \quad e(R) = 2 - 2g(\tilde{R}) - \epsilon(R)$$

$$(18) \quad e(S) = 2 - 2g(\tilde{S}) - \epsilon(S)$$

$$(19) \quad 2 - 2g(\tilde{R}) = c(2 - 2g(\tilde{S})) - r$$

$$(20) \quad 3 = -2c(2 - 2g(\tilde{S}) - \epsilon(S)) + 2 - 2g(\tilde{R}) - \epsilon(R)$$

where $r := \sum_{i \in \tilde{R}} (r_i - 1)$, and r_i is the local index of $\tilde{\alpha} : \tilde{R} \rightarrow \tilde{S}$ induced from $\alpha|_R : R \rightarrow S$. Since $\epsilon(\tilde{R}) \leq \delta(\tilde{R})$ (cf. [Be], page 98), we conclude from equations (20) and (16) that

$$\begin{aligned} 2c(2g(\tilde{S}) - 2 + \epsilon(S)) &= 1 + 2g(\tilde{R}) + \epsilon(R) \\ &\leq 1 + 2g(\tilde{R}) + \delta(R) \\ &= 11 + g(\tilde{R}) \end{aligned}$$

Assume first that $2g(S) - 2 + \epsilon \geq 3$. The above inequality then implies that

$$(21) \quad c \leq \frac{1}{6}(11 + g(\tilde{R})) = \frac{1}{6}(11 + 10 - \delta(\tilde{R})) \leq \frac{10}{3}.$$

Hence $c \leq 3$. On the other hand, we are in subcase (ii) so that there exists at least a $q \in \mathcal{S}(R)$ at which there are two local branches R_a and R_b of $R \cap U_q$ passing through q . From Lemma 2, we know that the branching along R_2 gives rise to an extra vanishing order to q compared to $x \in U_q - \{q\}$. This implies that $\alpha|_{R_a \cap U_q} : R_a \cap U_q \rightarrow S$ is ramified at q , to order of at least two. Hence the local degree of $\alpha|_{R_a \cap U_q}$ is at least 2. The same argument applied to R_b implies that the local degree of $\alpha|_{R_b \cap U_q}$ is at least 2 as well. Hence for a generic point of $y \in S \cap \alpha(U_q) - \{\alpha(q)\}$, the cardinality of $\alpha^{-1}(y)$ is at least 4. This contradicts the earlier conclusion that the degree of $\tilde{\alpha} : \tilde{R} \rightarrow \tilde{S}$ is $c \leq 3$.

Hence we need only to consider the case that $\epsilon(S) = 0$ so that S is smooth, and $g(\tilde{S}) = 2$. In this case, equation (20) becomes

$$(22) \quad c \leq \frac{1}{4}(11 + g(\tilde{R})) = \frac{1}{4}(11 + 10 - \delta(\tilde{R})) \leq 5.$$

From equation (19), we conclude that $g(\tilde{R}) \leq 6$. Substituting to equation (22), we conclude that $c \leq \frac{1}{4}(17)$ and hence $c \leq 4$. On the other hand, since we are in subcase (ii), there exists $q \in \mathcal{S}(R)$ lying in the intersection of local branches of R , and the discussions in the last paragraph implies that $c \geq 4$. Hence we conclude that $c = 4$. Equation (19) with $r = 0$ implies that $g(\tilde{R}) \leq 5$. On the other hand, the first inequality in (22) then implies that $g(\tilde{R}) \geq 5$. Hence $g(\tilde{R}) = 5$, and $\tilde{\alpha} : \tilde{R} \rightarrow \tilde{S} \cong S$ is an unramified covering of order 4. As the mapping $\tilde{\alpha} : \tilde{R} \rightarrow S$ is unramified, there exists $\tau > 0$ such that for every point $y \in S$, the mutual distance between any of the four points of $\tilde{\alpha}^{-1}(y)$ is at least τ with respect to Bergman metric on \tilde{R} , from compactness of \tilde{S} and \tilde{R} . However for points $y \in \alpha(U_q) - \{q\}$, the two points x_{y1}, x_{y2} in the set $(\alpha|_{U_q \cap R_a})^{-1}(y)$ get arbitrarily close to each other as $y \rightarrow q$. As the normalization map $\nu : \tilde{R} \rightarrow R$ is continuous and proper without collapsing any subvarieties, it is clear that the distance between $\nu^{-1}(x_{y1})$ and $\nu^{-1}(x_{y2})$ gets arbitrarily close to 0 as $y \rightarrow q$. This contradicts the earlier conclusion that the distance has to be at least τ . Hence subcase (ii) does not occur as well and Case V is ruled out.

In conclusion, none of the cases I to V can occur. Proposition 3 and hence Proposition 2 are proved.

5. Example of M with irregularity 1 and $e(M) = 3$

5.1 In trying to enumerate the set of all fake projective planes in the class \mathcal{C}_{11} according to [PY], Cartwright and Steger [CS] came across a torsion free lattice

of Euler-Poincaré characteristic 3 and $h^{1,0}(M) = 1$. This is surprising since before their work, it was generally expected that smooth surfaces of general type with $c_2 = 3$ were fake projective planes. To describe the example, we need to explain the scheme of classification in [PY] briefly. We will refer the readers to [PY] for all the unexplained notations in the following discussions.

An arithmetic lattice Λ for $PU(2, 1)$ is described as follows (cf. [PY]). We refer the readers to [PY] for all the unexplained notations. Let k be a totally real number field. Let ℓ be a totally imaginary quadratic extension of k . Let \mathcal{D} be a division algebra with center ℓ of degree 3 equipped with an involution σ of second kind, such that for the hermitian form h_0 on \mathcal{D} defined by $h_0(x, y) = \sigma(x)y$, the group $SU(h_0)$ is isotropic at v_o , and is anisotropic at every other real place of k . For $x \in \mathcal{D}^\times$, let $\text{Int}(x)$ denote the automorphism $z \mapsto xzx^{-1}$ of \mathcal{D} . Let $\mathcal{D}^\sigma = \{z \in \mathcal{D} \mid \sigma(z) = z\}$. Observe that for all $x \in \mathcal{D}^\sigma$, $\text{Int}(x) \cdot \sigma$ is again an involution of \mathcal{D} of the second kind, and any involution of \mathcal{D} of the second kind is of this form. Now for $x \in \mathcal{D}^\sigma$, given an hermitian form h' on \mathcal{D} with respect to the involution $\text{Int}(x) \cdot \sigma$, the form $h = x^{-1}h'$ is a hermitian form on \mathcal{D} with respect to σ , and $SU(h') = SU(h)$. Therefore it suffices to work just with the involution σ , and to consider all hermitian forms h on \mathcal{D} , with respect to σ , of determinant 1, such that the group $SU(h)$ is isotropic at v_o , and is anisotropic at all other real places of k . Let h be such a hermitian form. Then $h(x, y) = \sigma(x)ay$, for some $a \in \mathcal{D}^\sigma$. The determinant of h is $\text{Nrd}(a)$ modulo $N_{\ell/k}(\ell^\times)$. As the elements of $N_{\ell/k}(\ell^\times)$ are positive at all real places of k , we see that the signatures of h and h_0 are equal at every real place of k , which leads to the isometry between the hermitian forms h and h_0 . Hence, $SU(h)$ is k -isomorphic to $SU(h_0)$. Thus \mathcal{D} determines a unique k -form G of $SU(2, 1)$, up to a k -isomorphism, namely $SU(h_0)$, with the desired behavior at the real places of k . The group $G(k)$ of k -rational points of this G is

$$G(k) = \{z \in \mathcal{D}^\times \mid z\sigma(z) = 1 \text{ and } \text{Nrd}(z) = 1\}.$$

Let $P = (P_v)_{v \in V_f}$ be a coherent collection of parahoric subgroups P_v for each place $v \in V_f$, the set of all finite places of k , chosen as in [PY] (see also the Addendum). In [PY], the set of all possible arithmetic lattices Γ with $e(B_{\mathbb{C}}^2/\Gamma) = 3$ was classified into a small number of classes. Each of these classes determines a unique principal arithmetic subgroup $\Lambda (= G(k) \cap \prod_{v \in V_f} P_v)$, whose normalizer in $\overline{G}(k_{v_o})$ is denoted by $\overline{\Gamma}$. Each Λ determines a class of fake projective planes with fundamental group given by a lattice Π of $PU(2, 1)$, where Π is an element in

$$A_\Lambda = \{\Pi < \overline{\Gamma} : [\overline{\Gamma} : \Pi] = \frac{3}{\chi(\overline{\Gamma})}, |\Pi/[\Pi, \Pi]| < \infty, \text{ and } \Pi \text{ is torsion-free}\}$$

It follows that from the work of [PY] that there are twenty-eight distinct set $\{k, \ell, G, (P_v)_{v \in V_f}\}$ which can support fake projective planes. Five more classes which may contain smooth surfaces of Euler number 3 are listed in [PY] but are not expected to support fake projective planes. The latter was confirmed by [CS] which also shows that there is precisely one class containing Γ with $e(B_{\mathbb{C}}^2/\Gamma) = 3$ and $h^{1,0}(B_{\mathbb{C}}^2/\Gamma) = 1$, and there is only one such Γ . The defining number fields (k, ℓ) for the example of $h^{1,0}(M) = 1$ found in [CS] are given by $k = \mathbb{Q}(\sqrt{3})$ and $\ell = \mathbb{Q}(\zeta_{12})$, the cyclotomic field associated to the 12th root of unity. The pair of number fields is denoted by \mathcal{C}_{11} in [PY]. The division algebra in the definition of the lattice is chosen to be $\mathcal{D} = \ell$. There is a maximal arithmetic lattice $\overline{\Gamma}$ defined over \mathcal{C}_{11} which may contain a torsion free lattice of Euler number 3 as explained in

page §8.2 of [PY]. It follows from the volume formula of Prasad [P], (see the table on page 354 of [PY]), that orbifold characteristic $e(B_{\mathbb{C}}^2/\bar{\Gamma}) = 1/288$. Cartwright and Steger showed in [CS] that indeed a torsion-free subgroup Γ of index 864 existed in $\bar{\Gamma}$. Moreover, Γ is in fact a congruence subgroup of $\bar{\Gamma}$. This is obtained by writing down explicitly a set of generators for $\bar{\Gamma}$, from which a torsion-free subgroup of index 864 is found.

From explicit computation, Cartwright and Steger verify that the first Betti number of $B_{\mathbb{C}}^2/\Gamma$ is 2.

In the next section, we will sketch a proof that the example constructed by Cartwright and Steger is unique in the sense that the fundamental group of any such example has to be conjugate to the one constructed by Cartwright and Steger.

6. Classification of M with $e(M) = 3$

6.1 In this section, we approach the remaining cases, $h^{1,0} = 1$ and 2, by a method very different from §4. It is a modification of the approach developed in [Kl], [Ye1], [PY] and [CS] for the classification of fake projective planes. Before we go to the actual proof, we would like to outline the principle involved and the idea of proof.

Here are the main steps, working under the assumption that the smooth surface M satisfies the conditions that $K_M \cdot K_M = 9$.

Step 1: To show that M is a complex two ball quotient $B_{\mathbb{C}}^2/\Gamma$, where Γ is a torsion-free lattice in $PU(2, 1)$.

Step 2: To show that the lattice Γ is an arithmetic lattice in $PU(2, 1)$.

Step 3: To classify all possible torsion-free arithmetic lattices Γ for which the corresponding M satisfies the topological condition above.

Step 1 was already achieved in §2. Step 2 is the key step in this section and is a modification of the argument used in [Ye1]. Step 3 has already been achieved by the results in [PY] and [CS], since the setting of [PY] actually aims at classification of all torsion-free arithmetic lattices in $PU(2, 1)$ with Euler number 3. As explained in §5, it is shown in [PY] and [CS] that among all such arithmetic lattices, there is only one lattice with $h^{1,0} = 1$. All the others have $h^{1,0} = 0$ and are fake projective planes.

6.2 The following is the result for Step 2.

Proposition 4. *Let $M = B_{\mathbb{C}}^2/\Gamma$ be a smooth compact complex two ball quotient with $e(M) = 3$. Then Γ is an arithmetic lattice.*

Proof The structure of proof will be similar to the approach we took in [Ye1], which was originally designed for a lattice corresponding to a fake projective plane. In [Ye1], we need the assumptions that the Picard number is 1 and $h^{1,0} = 0$. The argument of the original article of [Ye1] and its corrections in the erratum was presented in a self-contained and coherent way in [Ye2], stated as Theorem 7 in [Ye2]. For Proposition 4, $M = B_{\mathbb{C}}^2/\Gamma$ is a torsion free compact complex ball quotient with $e(M) = 3$. Comparing to the conditions required for the results in [Ye1] or [Ye2], the only modification needed is that the Picard number may not be equal to 1.

Before we go to the actual places of [Ye2] where modification is needed, we would like to outline the main idea of proof.

A result of Weil tells us that any cocompact lattice Γ of $PU(2, 1)$ is locally rigid, from which it follows that Γ can be defined over a number field, that is, there exists an injective homomorphism $\rho : \Gamma \rightarrow G(k)$, G an algebraic group defined over a number field k with a Archimedean place v_o such that $G(k_{v_o}) \cong PU(2, 1)$. We say that Γ is integral if there exists a subgroup Γ' of finite index in Γ so that $\rho(\Gamma') \subset G(\mathcal{O}_k)$. The details of the above can be found in §1 of [Ye1].

6.3 As in the proof of [Ye1], [Ye2], there are two main steps for Step 2 above.

Step 2A, to prove that Γ is integral, and

Step 2B, to prove an analogue of archimedean superrigidity as in §4.7 of [Ye2], in the sense that there is ρ as above satisfying the condition that $G(k_v)$ is compact for all $v \neq v_o$.

Let us first give an overview of the proofs of Step 2A and Step 2B before we work on the details.

Step 2A is a modification of the proof of Integrality in §4.4 and §4.5 of [Ye2] (§2-4 of [Ye1]). For the sake of proof by contradiction, assume that Γ is not integral so that there exists for a finite place v a non-trivial unbounded representation $\rho_v : \Gamma \rightarrow G(k_v)$. Then there exists a faithful energy minimizing ρ_v -equivariant harmonic map $f : \tilde{M} \rightarrow X$, the Bruhat-Tits building associated to $G(k_v)$, as explained in the first two paragraphs of §4.4 in [Ye2], using a result of Gromov and Schoen. The Bruhat-Tits building is either of rank 1 or 2, from which the pull back of the differentials of the affine coordinate functions on an apartment by the harmonic map leads to harmonic forms on a finite sheeted cover M_1 , namely a spectral covering, of M . The covering group is given by \bar{W}_1 , a subgroup of the Weyl group associated to the root system of $G(k_v)$. Bochner formula implies the existence of non-trivial holomorphic one forms ω on M_1 , from which we construct a non-trivial Albanese map associated to ω 's from A_1 to an abelian variety $Alb_{\bar{W}_1, \omega}$, to be explained in more details below. The key point of the argument is to use properties of the Albanese map, its relation to the original harmonic map into the buildings and the geometry of the Bruhat-Tits building to prove that M_1 is an unramified covering of M , from which we conclude a contradiction as explained in [Ye1] or page 406 of [Ye2].

For Step 2B, the idea is a modification of §4.7 of [Ye2] (§5 of [Ye1]). Let $v_i, i = 2, \dots, n$ be the other Archimedean places of k so that we may consider $R_{k/\mathbb{Q}}(G)(\mathbb{R}) = G(k_{v_o}) \times G(k_{v_2}) \times \dots \times G(k_{v_n})$. From the type of Lie algebra and a result of Simpson on complex variation of Hodge structure, we conclude that $G(k_{v_i})$ for $i \geq 2$ is either $PU(2, 1)$ or $PU(3)$. The key point is to rule out the possibility that $G(k_{v_i}) = PU(2, 1)$ for some $i \geq 2$. Assume that such an i exists. It follows that there exists a Γ -equivariant harmonic map Φ from $\tilde{M} = B_{\mathbb{C}}^2$ to $B_{\mathbb{C}}^2$, the latter corresponding to the symmetric space associated to $G(k_{v_i})$, $i \geq 2$. Bochner type argument implies that Φ gives rise to a holomorphic mapping. The idea then is to show that Φ has no ramification divisor and hence is biholomorphic, making use of $c_1(M)^2 = 9$. This would then lead to a contradiction as in §4.7 of [Ye2], using the fact that the homomorphism induced by conjugation corresponding to different embeddings of k in \mathbb{C} is not even continuous.

To carry out the scheme of proof of [Ye2] in our situation, we need to get rid of the use of Picard number 1 in both steps 2A and 2B above. For Step 2A, this is achieved by Lemma 4 below and the two paragraphs near the end of **6.4**, which

will be used in the proof of Step 2A in [Ye2] to replace the restriction on the Picard number 1. For Step 2B, we get rid of Picard number 1 by a more skillful use of the fact that $K_M \cdot K_M = 9$, using an argument related to Hurwitz Formula.

6.4 In this subsection, we will provide a more detailed exposition of Step 2A parallel to the discussions in [Ye1] and [Ye2]. We will also provide the details of the modification required for our case when it is needed.

Lemma 4. *Let V be a proper algebraic subvariety in a compact complex ball quotient M . Let $i : V \rightarrow M$ be the embedding. Let $\pi_1(V)$ be the fundamental group of V . Then $i_*(\pi_1(V)) \subset \pi_1(M)$ is non-trivial.*

Proof Suppose on the contrary that $i_*(\pi_1(V)) \subset \pi_1(M)$ is trivial. Let $p : \widetilde{M} \rightarrow M$ be the universal covering map. Let V_o be a connected component of $p^{-1}(V)$. The restriction $p : V_o \rightarrow V$ is an unramified covering. Let ℓ be any closed loop on V based at $p \in V$. Since $i_*(\pi_1(V))$ is trivial, the lift $\tilde{\ell}$ of ℓ to V_o has to be a contractible loop on V_o . Hence $p : V_o \rightarrow V$ is a one-sheeted cover of V and therefore is a diffeomorphism since it is a covering map. Hence V_o is a compact subvariety of \widetilde{M} . However on $\widetilde{M} = B_{\mathbb{C}}^n$, there exists a strictly plurisubharmonic function given by $|z|^2 = \sum_{i=1}^n |z_i|^2$, the restriction of which to V_o is still plurisubharmonic. As V_o is compact, the plurisubharmonic function has to attain a maximum. This however contradicts the Submean Value Inequality for a plurisubharmonic function, thereby concludes the proof of the Lemma.

In the following we will go through the structure of proof in [Ye2], explain in details places that need to be modified under our new weakened assumption, while refer the readers to [Ye2] for details that were already written there.

As mentioned in the brief overview above, if the lattice Γ involved is not integral in k , there exists a finite place v and a Γ equivariant harmonic map from \widetilde{M} , the universal covering of M , to X , the Bruhat-Tits building associated to the induced representation of Γ in the corresponding group $G(k_v)$. Pulling back the coordinate differentials on X by the harmonic map, we get some multivalued harmonic one forms on \widetilde{M} , which after descending to M and considering the $(1,0)$ part provide multivalued holomorphic one forms. The fact that the one forms are multivalued follows from the construction, since there is an action of the affine Weyl group on X . The multivalued one forms become a set of single valued holomorphic one forms denoted by $\{\omega_i\}$ after going to a finite spectral covering M_1 of M , with the covering group \overline{W}_1 being a subgroup of the Weyl group of the root system of X involved. The details are given in the first seventh paragraphs in §4.4 of [Ye2].

The dimension of X may be 1 or 2 depending on the rank of G over k_v . Suppose first that $\text{rank}_{k_v}(G) = 1$ so that X is a tree. This corresponds to the argument given in page 400 of [Ye2]. In this case, there is a mapping $\alpha : M_1 \rightarrow \text{Alb}_{\overline{W}_1, \{\omega_i\}}(M_1)$, which is the quotient of the Albanese variety by the \overline{W}_1 -invariant abelian subvariety annihilated by all the ω_i obtained earlier. We also know that $\text{Alb}_{\overline{W}_1, \{\omega_i\}}(M_1)$ has complex dimension 1 in this case. Let x_o be a fixed point and x be an arbitrary point on M_1 . As in the proof of Lemma 3 in §4.6 of [Ye2], a generic fiber of the mapping $h_{\mathbb{R}} : \widetilde{M}_1 \rightarrow \mathbb{R}$ given by $x \rightarrow \int_{x_o}^x \text{Re}(\omega)$ is a generic fiber of $\tilde{f}_1 : \widetilde{M}_1 \rightarrow \widetilde{M} \xrightarrow{\alpha} X$. Since $h_{\mathbb{R}}$ can be considered as the real part of the universal covering of α , it follows that a generic fiber V_x of $\tilde{\alpha}$, where x is generic point in $\text{Alb}_{\overline{W}_1, \{\omega_i\}}(M_1)$, is mapped to a point by \tilde{f}_1 . Hence for a generic x , $\rho_v(i_*(\pi_1(V_x)))$ is acting trivially at $f \circ \pi(V_x)$,

where i is the inclusion mapping. Since f is ρ -equivariant, we know that for each $y \in \widetilde{M}$, $(\rho_v(\gamma))(f(y)) = f(\gamma(y))$ for each $y \in \widetilde{M}$ and $\gamma \in \pi_1(M)$. It follows that $\rho_v(i_*(\pi_1(V_x)))$ acts trivially on X . Hence $\rho_v(i_*(\pi_1(V_x)))$ is trivial. As ρ_v is one to one, $i_*(\pi_1(V_x))$ is trivial for a generic $x \in \text{Alb}_{\overline{W}_1, \{\omega_i\}}(M_1)$. This however contradicts Lemma 4. The claim is proved. The above is the first modification needed in Step 2A.

The more difficult case is that $\text{rank}_{k_v}(G) = 2$ so that the dimension of X is 2. In this case, an apartment in X can be written as $\Sigma = \{(x_1, x_2, x_3) \in \mathbb{R}^3 | x_1 + x_2 + x_3 = 0\} \cong \mathbb{R}^2$, and there are three holomorphic one forms $\omega_i, i = 1, 2, 3$ on M_1 coming from pulling back of coordinate differentials by the harmonic map f into X as mentioned earlier. The Weyl group of the root system is the symmetric group of three elements S_3 and the spectral covering group \overline{W}_1 is a subgroup of the Weyl group S_3 . In this case, the corresponding Albanese map α as defined earlier may have dimension one or two image in $\text{Alb}_{\overline{W}_1, \{\omega_i\}}(M_1)$. First we claim that $\alpha(M_1)$ cannot be a dimension one subvariety in $\text{Alb}_{\overline{W}_1, \{\omega_i\}}(M_1)$. Assume on the contrary that $\alpha(M_1)$ is of complex dimension one. In such case, for a generic point on $\alpha(M_1)$, the inverse image in M_1 is a curve. Consider the mapping $\tilde{h}_R : \widetilde{M}_1 \rightarrow \mathbb{R}^2$ defined by

$$\begin{aligned} \tilde{h}_R(z) &= \left(\int_{z_o}^z (f \circ \pi)^* dx_1, \int_{z_o}^z (f \circ \pi)^* dx_2, \int_{z_o}^z (f \circ \pi)^* dx_3 \right) \\ &\in \{(w_1, w_2, w_3) \in \mathbb{R}^3 | \sum_{i=1}^3 w_i = 0\} \cong \mathbb{R}^2, \end{aligned}$$

where f is the harmonic map into the building, and x_i 's are the affine functions defining an apartment of X , cf. [Ye2], §4.4. Clearly $\tilde{h}_R(z)$ is just the projection of $\tilde{\alpha}$ onto the real part of \mathbb{C}^2 . Again, fibers of \tilde{h}_R correspond to fibers of \tilde{f} . Similar to the argument in the last paragraph, $\rho_v(i_*(\pi_1(V_x)))$ is trivial. As ρ_v is one to one, $i_*(\pi_1(V_x))$ is trivial for a generic $x \in \text{Alb}_{\overline{W}_1, \omega}(M_1)$. This again contradicts Lemma 4. The claim is proved. The argument of this paragraph is a replacement of a similar argument on page 400 of [Ye2] and is the second modification required.

Hence we know that $\alpha(M_1)$ has complex dimension 2. The key point of the argument is to show that in such a case, the spectral mapping $\pi : M_1 \rightarrow M$ is unramified. This is achieved via proof by contradiction. Hence assume that π is ramified. Let D be an irreducible codimension one component of the ramification divisor on M_1 corresponding to $\omega_i - \sigma\omega_i = 0$ for some ω_i and $\sigma \in \overline{W}_1$ in the construction of the spectral covering. There is at least one such component whose image on M is a branching divisor, otherwise π would be unramified. Since α is the Albanese map defined by the one forms $\{\omega_i\}$, the image $\alpha(D)$ is an algebraic curve in $\text{Alb}_{\overline{W}_1, \{\omega_i\}}(M_1)$ and is an Abelian subvariety in $\text{Alb}_{\overline{W}_1, \{\omega_i\}}(M_1)$, following from the same argument as in §4.5 of [Ye2]. To derive a contradiction, we relate α to the harmonic map f as given in §4.5 of [Ye2] so that the geometry of X comes into play. Consider a generic fiber C_a of the projection $M_1 \xrightarrow{\alpha} \text{Alb}_{\overline{W}_1, \{\omega_i\}}((M_1) \rightarrow \text{Alb}_{\overline{W}_1, \{\omega_i\}}(M_1)/\alpha(D))$. The contradiction is achieved by proving the following two statements. On the one hand, $\overline{p_*(\pi(C_a))}$, the Zariski closure of $p_*(\pi(C_a))$, is a non-trivial normal subgroup of the group G . Since G as a real Lie group is isomorphic to $PU(2, 1)$, it is simple. We conclude that $\overline{p_*(\pi(C_a))} = G$. On the other hand, $f(\pi(C_a))$ is a tree lying in

X , of which the stabilizer is given by a proper subgroup of G . The two statements contradict each other. Hence we conclude that $\pi : M_1 \rightarrow M$ is unramified once the above two statements are proved.

The proof of the two statements is given in details in §4.5 (page 400-405) of [Ye2], where no further assumption is used. However it is drawn to the attention of the author that line 13 on page 403 of [Ye2], which corresponds to Sublemma on page 284 of the Erratum of [Ye1], needs to be explained. In the case that the Picard number of M is one in the setting of [Ye1] and [Ye2], Sublemma follows from the following observation. Suppose that D is a divisor contracted by α to a point on $\text{Alb}_{\overline{W}_1, \{\omega_i\}}(M_1)$. It follows that D lies in the kernel of ω_i for $i = 1, 2, 3$ from definition of the Albanese map, and hence in the kernel of $\sigma^* \omega_i$ for each $\sigma \in \overline{W}$. Since \overline{W} induces an action on $\text{Alb}_{\overline{W}_1, \{\omega_i\}}(M_1)$ and α induces a map $\beta : M \rightarrow \text{Alb}_{\overline{W}_1, \{\omega_i\}}/\overline{W}$, this implies that $\pi(D)$ is contracted by $\beta : M \rightarrow \text{Alb}_{\overline{W}_1, \{\omega_i\}}/\overline{W}$, which contradicts the assumption that the Picard number of M is 1. In our current situation we do not know if the Picard number of M is 1. However the Sublemma is only used to make sure of the assertion on line 20 of page 284 in Erratum of [Ye1] that $\alpha^* E_Q$ does not contain a contracted curve, where $Q \in \text{Alb}_{\overline{W}_1, \{\omega_i\}}(M_1)/\alpha(D)$. In the setting, Q is parametrized by $\alpha(\mathcal{S})$, where \mathcal{S} is a component of the singularity set of f at which C_a meet on M_1 . Note that as mentioned in [Ye1], [Ye2], there is nothing to be proved if C_a does not meet any singularity set of f on M_1 , and when such a singularity set \mathcal{S} is present, it is defined as $\omega_j - \sigma_k \omega_j = 0$ for some $\sigma_k \in \overline{W}_1$, and $\text{Alb}_{\overline{W}_1, \{\omega_i\}}(M_1)$ is isogeneous to $\alpha(D) \times \alpha(\mathcal{S})$. With a slight abuse of notation, we may consider $Q \in \alpha(\mathcal{S})$. It cannot be true that $\alpha^* E_Q$ contains a contracted divisor for all $Q \in \alpha(\mathcal{S})$, for otherwise $\alpha(M_1)$ would be of complex dimension 1. Hence for a generic choice of C_a corresponding to a generic choice of Q , $\alpha^* E_Q$ does not contain a contracted curve. We choose such a Q and the associated C_a . The rest of the arguments in [Ye1] as well as §4.5 of [Ye2] can then be applied to conclude the proof of the statements. This is the third modification required.

Hence the spectral mapping $\pi : M_1 \rightarrow M$ is unramified. However, unless \overline{W}_1 is trivial, this will contradict topological consideration arising from the action of \overline{W}_1 on M_1 as well as $\alpha(M_1)$ induced from the action of the affine Weyl group on an apartment of X . We refer the readers to §4.6 of [Ye2] for details of the argument. Hence \overline{W}_1 is trivial and therefore $M = M_1$. In such case, $h^{1,0} \geq 2$ as $\alpha(M) = \alpha(M_1)$ is of complex dimension 2, our working assumption. Hence the case of $h^{1,0} = 1$ is eliminated. The only remaining case is $h^{1,0} = 2$. Here we are going to utilize more the structure of the Bruhat-Tits building involved (cf. [Br]). Since \overline{W}_1 is trivial, $\rho(\Gamma)$ can only act on each fixed apartment Σ by translation. We observe that the only subgroup G_1 of $G(\mathbb{Q}_p)$ which acts by translation on each apartment of the Bruhat-Tits building X is trivial. To prove the claim, we note that the stabilizer G_Σ of an apartment Σ in $G(k_v)$ is precisely the normalizer of the split torus T associated to Σ . Moreover, $g \in G_\Sigma$ acts by translation if and only if $g \in Z_G(T)$, the centralizer of T . Hence $G_1 \subset H := \cap_T Z_G(T)$, where T ranges over the set of all maximal split torus T . Now H is normalized by $G(K_p)$. Since G is simple, H is trivial. The observation is proved. It follows from the observation that $\rho_* \gamma(\tilde{f}(\Sigma_i))$ is trivial, contradict to our assumption from the very beginning of proof of Theorem. This is the fourth modification required.

The contradiction above completes the proof for Step 2A.

6.5 In this subsection, we will give necessary details for the outline of proof mentioned in subsection §6.3 for Step 2B. As mentioned in §6.3, the key point is to rule out the possibility that $G(k_{v_i}) = PU(2, 1)$ for some $i \geq 2$, where v_i is another embedding of k in \mathbb{C} different from v_o . Suppose such an i exists, there exists a Γ -equivariant harmonic map $\Phi : \widetilde{M} \rightarrow \widetilde{M}_2$, where \widetilde{M}_2 is also biholomorphic to $B_{\mathbb{C}}^2$ corresponding to the symmetric space associated to $G(k_{v_i})$.

We claim that Φ is holomorphic or conjugate-holomorphic. This follows again from the Strong Rigidity of Siu [Siu] if the real rank of f_i is at least 3. Consider now the case that the real rank of Φ is less than or equal to 2. Since Φ is a ρ_{σ_i} -equivariant map from \widetilde{M} to another complex two ball N^σ , the argument of Carlson and Toledo [CT], page 196-198, can be applied to conclude that one of the following two cases happens,

- (i) $\Phi(\widetilde{M})$ is a totally geodesic curve ℓ in $N^\sigma \cong B_{\mathbb{C}}^2$, or
- (ii) f_i can be written as $\Phi = \psi \circ \varphi$, where $\varphi : \widetilde{M} \rightarrow S$ is a Γ -equivariant holomorphic map from M to a Riemann surface S , and $\psi : S \rightarrow N$ is a $\varphi_*\Gamma$ -equivariant harmonic mapping from S to N . The only modification compared to the original setting of [CT] is that we are considering Γ -equivariant harmonic map for cocompact Γ instead of considering the map on a quotient manifold.

Case (i) implies that the induced action of ρ_{σ_i} on the boundary ∂N^σ of N^σ fixes the point on ∂N^σ corresponding to the line ℓ . This contradicts the fact that ρ and hence ρ_{σ_i} gives rise to a Zariski dense action on $B_{\mathbb{C}}^2$. For case (ii), we argue as in the proof of Lemma 8.1 in [CT]. As ρ_{σ_i} is faithful, we conclude that φ_* is injective and $\varphi_*\Gamma$ acts freely on the universal cover of S . Hence the cohomology dimension of Γ is at most 2. This contradicts the fact that Γ has cohomology dimension 4 since it is cocompact in \widetilde{M} , of which the real dimension is 4.

We conclude that neither case (i) nor (ii) can occur, thereby conclude the proof of the claim. It follows from the claim that Φ is holomorphic or conjugate holomorphic. In the latter case, we replace Φ by its complex conjugate so that we may assume that all Φ is holomorphic.

The canonical line bundle $K_{\widetilde{M}}$ on \widetilde{M} is identified as the pull back of K_M by the universal covering. According to Lemma 1, we may write $K_M \cdot K_M = 9 \int_M \Theta(H_{\widetilde{M}} \wedge \Theta(H_{\widetilde{M}}))$, which by abuse of notation we write $K_M \cdot K_M = 9H_{\widetilde{M}} \cdot H_{\widetilde{M}}$.

Note also that since Φ is holomorphic as an Γ -equivariant mapping $\widetilde{M}_1 \rightarrow \widetilde{M}_2$, we conclude that Φ is open and that the image of a fundamental domain of \mathcal{F} is relatively compact in \widetilde{M}_2 . Hence the image of Γ is a cocompact lattice of $\widetilde{M}_2 \cong B_{\mathbb{C}}^2$ as well.

It follows that with our abuse of notation,

$$K_M^2 = 9nH_2 \cdot H_2 + 2\Phi^*K_{\widetilde{M}_2} \cdot R + R \cdot R.$$

Since n is a positive integer and $\Phi^*K_{\widetilde{M}_2} \cdot R \geq 0$ as $K_{\widetilde{M}_2}$ is ample, it follows that $R \cdot R \leq 0$ since the left hand side of the above identity is precisely 9.

On the other hand, we may write $\Phi^*K_{\widetilde{M}_2} = K_M - R$ and hence $\pi^*K_M \cdot R = K_M \cdot R - R \cdot R$. Hence we may rewrite the above identity as

$$K_M^2 = 9nH_2 \cdot H_2 + 2K_M \cdot R - R \cdot R$$

Again, since R is effective, $K_M \cdot R \geq 0$. The same argument as above shows that $R \cdot R \geq 0$. Hence we conclude that $R \cdot R = 0$.

As the left hand side of the above identity is 9 and each term on the right hand side is non-negative, we conclude that $n = 1$, and $K_M \cdot R = 0$. As K_M is ample and R is effective, this implies that R is trivial as well. Hence we conclude that Φ is a biholomorphism. This however leads to a contradiction, since the correspondence between the images of two different embeddings of k into \mathbb{C} is not even continuous with respect to the standard topology on \mathbb{C} .

The above discussions complete the proof of Proposition 4.

6.6 We can now state the uniqueness of the examples with $h^{1,0} = 1$.

Proposition 5. *Let $M = B_{\mathbb{C}}^2/\Gamma$ be an arithmetic complex two ball quotient with $e(M) = 3$ and $h^{1,0}(M) = 1$. Then M is holomorphic or conjugate holomorphic to the surface constructed by Cartwright and Steger mentioned in §6.*

Proof The classification of Prasad and Yeung [PY] covers all arithmetic complex two ball quotients of $e(M) = 3$. In particular, the set of all arithmetic lattices of $e(M) = 3$ consists of 28 classes with $[\mathcal{D}, \ell] > 1$, each of which contains fake projective planes, and five more classes, $\mathcal{C}_1, \mathcal{C}_8, \mathcal{C}_{11}, \mathcal{C}_{18}$, or \mathcal{C}_{21} with $\mathcal{D} = \ell$ in the notation of [PY] that may contain examples. Finally in [CS], Cartwright and Steger show that there are precisely 100 fake projective planes within the first twenty-eight classes, and there exists precisely one arithmetic lattice Γ with $e(M) = 3$ in the remaining five classes above, lying within the class with number field given by \mathcal{C}_{11} and has $h^{1,0}(B_{\mathbb{C}}^2/\Gamma) = 1$. We conclude that the fundamental group of a torsion free ball quotient M with $e(M) = 3$ and $h^{1,0}(M) = 1$ has to be isomorphic to the example of Cartwright and Steger mentioned in §6. It is easy to show that the complex conjugate of such a ball quotient does not give the same complex structure, which has been shown in [KK] for fake projective planes. For the case at hand, it is known from the work of Cartwright Steger that the automorphism group of the surface has order 3. Suppose on the contrary that there is a complex conjugate diffeomorphism h on the surface. It follows that either h or h^3 is a conjugate involution, which has a totally real manifold as the fixed point set. Hence the fixed point set is either the real sphere or the real projective plane. Neither of the cases is possible from the proof in Lemma 4.

Hence there are precisely two such surfaces up to biholomorphism. This concludes the proof of part (b).

6.6

Proof of Theorem 1 This follows by combining the results of Proposition 1, Proposition 4 and Proposition 5.

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