HOLOMORPHIC ONE FORMS, INTEGRAL AND RATIONAL POINTS ON COMPLEX HYPERBOLIC SURFACES

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Abstract *The first goal of this paper is to study the question of finiteness of integral points on a cofinite non-compact complex two dimensional ball quotient defined over a number field. Along the process we will also consider some negatively curved compact surfaces. Using some fundamental results of Faltings, the question is to reduce to a conjecture of Borel about existence of virtual holomorphic one forms on cofinite non-cocompact complex ball quotients, the study of which for an arbitrary dimension is the second goal of this paper.*

1. Introduction

(1.1) It has been an interesting problem to understand the relation between negativity of the curvature of a Kähler metric on a projetive algebraic manifold defined over a number field and finiteness of the set of rational points. In particular, it follows from some well-known conjectures of Lang [La] and Vojta [Vo1] that a hyperbolic projective algebraic manifold defined over a number field has only a finite number of rational points, and a hyperbolic quasi-projective manifold defined over a number field has only a finite number of integral points. Here a complex manifold is said to be hyperbolic if the Kobayashi semi-metric is non-degenerate (cf. [La] and [Vo1]). In the case that M is compact, this is equivalent to the property that there is no non-trivial entire map from $\mathbb{C}$. The results of Faltings on Mordell Conjecture, subvarieties of abelian varieties and of Vojta on semi-Abelian varieties are the most significant results in this direction, see [F1], [F2] and Vojta [Vo2].

From a complex geometric, or more precisely, metrical point of view, the simplest hyperbolic complex manifolds are given by complex ball quotients since they support a Kähler metric with constant negative holomorphic sectional curvature. These are quotients of the complex balls of radius 1 in $\mathbb{C}^n$ by a torsion free lattice in $PU(n,1)$, which is the automorphism group of the complex ball. Hence one may ask if the conjecture on finiteness of rational or integral points is valid on such smooth complex ball quotients. The purpose of this article is to study the above conjecture for such smooth complex ball quotients in complex dimension 2. In fact, there is only one other class of complex manifolds that are known to support Kähler metrics of strictly negative Riemannian sectional curvature, first constructed by Mostow and Siu [MS], see also [D]. For complex ball quotients and the examples of Mostow-Siu, first we observe that they can be defined over a number field (cf. Lemma 1 and Proposition 4). For simplicity, we call the resulted varieties defined over a number field to be an arithmetic model. The first aim of this article is to consider finiteness of rational points for an arithmetic model of some compact

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complex two ball quotients, all non-compact complex two ball quotients and the examples of Mostow and Siu.

(1.2) The method of proof is to reduce the arithmetic problem to a geometric problem on the existence of holomorphic one forms on some appropriate unramified covers of the manifolds involved. The geometric problem is by itself very interesting and is still open in general. In a real hyperbolic space form, a conjecture of Thurston states that there exists a finite unramified covering with non-trivial first Betti number. Borel conjectured that the same conjecture should be true for a complex ball quotient from cohomological study of such manifolds (cf. [B]). Hence one expects existence of non-trivial holomorphic one forms on some appropriate covers of any complex ball quotients. The relation of the conjecture to the earlier problem on finiteness of rational points is provided by the results of Faltings [F2] and Vojta [Vo2] mentioned above.

Hence we first look for examples of complex ball quotients for which the conjecture of Borel is satisfied. Such examples for compact ball quotients have been provided by Kazhdan [K] and Shimura [Sh]. For a general compact complex two ball quotient, the conjecture is still open, though quotients which arise from arithmetic lattices of first type do enjoy such properties, see §3.2 for the definition. In particular, this is the case for examples of complex ball quotients arising from geometric consideration as studied by Picard [P], Terada [T], Deligne-Mostow [DM], Mostow [Mos] and Livne [Li]. This list above contain some non-arithmetic lattices as well. Moreover, the same is true for Mostow-Siu examples.

The main geometric observation of this paper is that such a cover always exists for any non-compact complex two ball quotients.

Once the geometric result on existence of virtual holomorphic one forms is proved, we reduce the arithmetic problem of finiteness of rational points to the corresponding question on an appropriate unramified covering, where the results of Faltings [F2] are applied. During the process, we have also proved that the varieties and the mappings involved can all be defined over some number fields.

(1.3) Here is the organization of the paper. In §2, we give some preliminary discussions and collect some number theoretical tools to be used. In §3, finiteness of rational points for compact surfaces, including arithmetic complex two ball quotients of first type, and Mostow-Siu surfaces, is discussed. In §4 and §5, we study some geometric properties of non-compact complex two ball quotients which have finite volume with respect to the Bergman metric. In §4, we study the problem of Borel for cofinite complex two ball quotients. In §5, we study the growth of number of cusps and space of holomorphic one forms on cofinite complex two ball quotients. Finally, in §6, finiteness of integral points for quasi-projective complex two ball quotients is established. The arithmetic results are given in Theorem 1, 2 and 5 of §3 and §6, and the geometric results are given in Theorem 3, 4 and Proposition 3 of §4 and §5.

(1.4) The author is grateful to Matthew Stover and Pierre Py for pointing out an error in an earlier draft of the paper and to Ben McReynold for his interests.

2. Preliminaries

(2.1) Let us recall some number theoretical results in this section. Let k be a number field and $\bar{k}$ its algebraic closure. For a projective variety $\overline{M}$ defined over a number field k, the set of rational points is well defined and is denoted by $M(k)$. The notion of integral points of M with respect to an effective Cartier divisor D is defined as follows (cf. [Vo3], page 154). Let V_k be the set of all valuations on k. A V_k-constant is a collection (c_v) of constants $c_v \in \mathbb{R}$ for each $v \in V_k$ such that $c_v = 0$ for almost all $v \in V_k$. Let S be a finite set of places containing S_∞, the set of Archimedean places of k. A set $\Sigma \subset M(\bar{k})$ is said to be a (D, S)-integral set of points if $\Sigma \cap supp(D) = \emptyset$ and there is a Weil function λ_D for D and a V_k-constant (c_v) satisfying $\lambda_{D,w}(x) \leqslant c_w$ for all $x \in \Sigma$ and for all places $w \in V_k - S$. In the case of an affine variety X with inclusion $i : X \to \mathcal{A}^n_k$, this is the same as requiring that $i(x) \in (1/a)\mathcal{O}_{k,S}$ for some $a \in k$ and all $x \in \Sigma$. We say that the set of (D, S)-integral points has finite cardinality if all such Σ has a finite cardinality. For simplicity, we would also say that X has a finite number of integral points with respect to D.

(2.2) We recall the result of Chevalley-Weil, and Hermite, related to defining number fields of an unramified covering (cf. [Vo3], page 156, [HS], page 292-293, [Vo1], page 58, or [Se]). Let k be a number field and $\pi : Y \to X$ be a finite unramified covering of normal projective varieties defined over k. Then there exists a finite extension k' of k such that $\pi^{-1}(X(k)) \subset Y(k')$. Similarly, suppose that $\pi : Y - E \to X - D$ a finite unramified covering of normal quasi-projective varieties defined over k. Then there exists a finite extension k' of k such that for any (D, S) integral set Σ, $\Sigma' = \pi^{-1}(\Sigma)$ is a (E, S) integral set on Y. The statement implies that the pull back of set of rational points (resp. integral points) in k on X are rational (resp. integral) points on Y in k', where k' is a finite extension of k.

(2.3) The following results of Faltings (cf. [F2]) provide the crucial tool for our argument.

Let A be an abelian variety defined over k and X be a k-closed subvariety of A. Then the irreducible components of the Zariski closure $\overline{X(k)}$ of $X(k)$ are translates of abelian subvarieties of A over k by elements of $X(k)$.

By considering the Jacobian of a hyperbolic projective algebraic curve defined over a number field, this provides an alternative proof of Mordell Conjecture which was solve earlier by Faltings in [F1] Mordell conjecture states that the cardinality of the se of rational points on such a curve is finite.

The analogue of the results for semi-abelian varieties have been obtained by Vojta [Vo2] in the following way. Let k be a number field, with ring of integers $\mathcal{O}_k$. Let X be a closed subvariety of a semiabelian variety A, where we assume both are dened over k. Then the Zariski closure of the set of integral points of X in $\mathcal{O}_k$ is a translation of a semi-abelian subvariety of A.

(2.4) For a variety to be defined over a number field, we need to introduce the notion of rigidity.

A complex manifold is said to be locally rigid if there is no local definition of the complex structure on M. Let M be a compact complex manifold. Then M is locally rigid if the Kodaira-Spencer class $\rho \in H^1(M, \Theta)$ vanishes, where Θ is the sheaf of holomorphic vector field on M. Suppose M is a quasi-projective variety $M \cong \overline{M} - D$, where D is a normal crossing divisor. Then M is locally rigid if the corresponding Kodaira-Spencer map $\rho \in H^1(\overline{M}, \Omega(\log D)^*)$ vanishes, where V^* denotes the dual bundle of a vector bundle V.

Proposition 1. *(cf. [Va], page 83) A projective manifold M can be defined over a number field if $H^1(M,\Theta)=0$. Similarly, a quasi-projective manifold $(\overline{M},D)$ can be defined over a number field if $H^1(\overline{M},[\Omega(\log D)]^*)=0$.*

(2.5) We refer the readers to [BHPV], [Mu] and [GH] for standard facts concerning Albanese mappings and Abelian varieties. For semi-abelian varieties and the Albanese mappings for quasi-projective varieties, we refer the readers to Itaka [I] for dicussions. Here we briefly recall the construction in terms of holomorphic one forms or holomorphic logarithmic one forms.

On a projective algebraic manifold M, Albanese map α is a mapping $\alpha: M\to \mathrm{Alb}(M)$ defined as follows. Let $\omega_i, i=1,\dots n$, be a basis of holomorphic one forms on M. Let $h_j, j=1,\dots,2n$ be a basis of $H_1(M,\mathbb{Z})$. Fix a point $x_o\in M$. For any point $x\in M$, we join x to x_o by a path ℓ and define $\alpha: M\to \mathrm{Alb}$ by

$$\alpha(x)=(\int_\ell \omega_1,\dots,\int_\ell \omega_n)/\Lambda,$$

where Λ is the lattice on $\mathbb{C}^n$ generated by $\int_{h_j}\omega_i$ for each $1\leqslant i\leqslant n$ and $1\leqslant j\leqslant 2n$. It is known the Albanese variety is dual to the Picard variety.

On a quasi-projective manifold $M=\overline{M}-D$, where $\overline{M}$ is compact and D is a normal crossing divisor, the Albanese map can be defined as above, except that now we use a set of generators for holomorphic logarithmic one forms $\omega_i\in H^0(M,\Omega(\log D))$ instead of holomorphic one forms on $\widetilde{M}$ (cf. [I]).

It is known that if M (resp. (M,D)) is defined over a number field k, the Albanese (resp. the quasi-Albanese) maps can be defined over a number field if the original manifold M is defined over the same number field. This can be found in [Mi], [I] and [Vo2].

In this paper, we would only need the fact for the Albanese map for a compact projective algebraic variety.

3. Rational points on some compact surfaces

(3.1) Let us first make the following observation for projective algebraic manifold defined over a number field.

Theorem 1. *Let M be a smooth projective algebraic surface defined over a number field F. Assume that there exists an unramified covering $M'\to M$ defined over some number field F' so that the irregularity $q(M')=dim_{F'}H^1(M',\mathcal{O}_{M'})$ is at least 3. Assume also that there is no non-constant morphism from a curve of genus 0 or 1 into M, then $M(F)$ has finite cardinality.*

Proof We are going to prove the theorem by contradiction. Hence we assume that $M(F)$ has infinite cardinality. It follows from our hypothesis that there exists an unramified covering M' of M on which the dimension of the space of holomorphic $1-$forms is at least 3, where both M' and the morphism $M'\to M$ are defined over a number field F_1. Let F_2 be the field generated by F_1 and F. According to the theorem of Chevalley-Weil as stated in §2.2, we conclude for some fixed F' with $[F':F_2]=t\leqslant d$, $M'(F')$ has infinite cardinality.

Let $x\in M'(F')$. The Albanese variety $Alb_{M'(F')}$ is the dual abelian variety of the Picard variety. The rank of the Albanese map $\alpha: M'\to Alb_{M'(F')}$ is given by the irregularity $q(M')$, which is at least 3 from our hypothesis.

The set of rational points on M', $M'(F')$, cannot be Zariski dense in M', for otherwise $\alpha(M')$ has complex dimension 2 and is an Abelian subvariety of $Alb(M')$ according to the results of Faltings in §2.3. This contradicts the fact that A as the Albanese variety of M' is generated by $\underline{f(M')}$.

Hence an irreducible components of $\overline{M'(F')}$ has dimension either 0 or 1. Let D be an irreducible component of dimension 1. From the results of Faltings in §2.3, $f(D)$ is defined over F', since it is the translation by an element over F' of an abelian subvariety defined over F'. As f is defined over F', it follows that D is defined over F' as well. The genus of D is at least 2 from our hypothesis. According to the solution of Mordell Conjecture of Faltings as stated in §2.3, we conclude that D has only a finite number of rational points over F', contradicting the assumption that $M'(F')$ has infinite cardinality. This concludes the proof of Theorem 1.

$\square$

Remark In a similar way, one can prove algebraic degeneracy of the Zariski closure of the set of rational points in higher dimensions.

(3.2) As mentioned in the introduction, it has been a consequence of some conjectures of Lang and Vojta [Vo1] that a complex hyperbolic manifold defined over a number field has at most a finite number of rational points. From a differential geometric point of view, one may consider the conjecture on a more restricted set of manifolds, namely projective algebraic manifolds equipped with a Kähler metric of strictly negative Riemannian sectional curvature which can be defined over a number field. Apart from complex hyperbolic ball quotients, the only examples with such negative sectional curvature are given by surfaces constructed by Mostow-Siu [MS], see also [D]. First we make the following observation.

Lemma 1. *A compact complex two ball quotient can be defined over a number field. Similarly, a Mostow-Siu surface can be defined over a number field.*

Proof It is known that the sectional curvature for such surfaces are strongly negative in the complexified sense and hence are rigid complex analytically (cf.[Si] and [MS]). In particular, they cannot be locally deformed. Hence these surfaces have models defined over appropriate number fields according to Proposition 1.

$\square$

We say that an arithmetic lattice of $PU(2,1)$ is of *Second Type* if it is defined by a Hermitian form over a division algebra with involution of second type, which is a non-trivial extension of a number field ℓ, which itself is a totally imaginary quadratic extension over a totally real number field k. The lattice is of *First type* if the lattice is defined by a Hermitian form over ℓ directly, in the sense that the division algebra is just ℓ, cf [Y3].

Theorem 2. *(a). Let M be a compact complex ball quotient of complex dimension 2 which has a finite unramified covering with irregularity at least 3. Assume that M has a model defined over a number field F. Then $M(F)$ has finite cardinality.*
(b). In particular, this is the case for arithmetic lattices of $PU(2,1)$ of first type, and all the examples of lattices of $PU(2,1)$ appearing in the list of [DM], [Mos] or [Li].
(c). Let M be an example of Mostow-Siu surface with negative sectional curvature. Assume that M is defined over a number field F. Then $M(F)$ has finite cardinality.

Proof (a) follows from Theorem 1 and the fact that M is hyperbolic. We may now apply Theorem A.

For (b), we know that for an arithmetic lattices Γ of first type, there is a congruence subgroup of Γ_1 of finite index such that first Betti number of $B^2_{\mathbb{C}}/\Gamma_1$ is at least 5, following the results of Kazhdan, Shimura or more generally Borel-Wallach in [BW]. All arithmetic lattices in the list of Picard-Terada-Deligne-Mostow as in [DM] are of first type. [DM] also listed some non-arithmetic lattices in $PU(2,1)$. For non-arithmetic examples in [DM], one can show that after going to a finite unramified covering of sufficiently large order, there exists a holomorphic map to a complex one ball quotient, and the pull-back by the map to a one-ball quotient actually supports at least three holomorphic one forms, cf. [DM] or [D]. It is also known that the list of Livne was included in the list of [DM].

For (c), we first observe that according to Proposition 1, a Mostow-Siu example is analytically rigid, since it has strictly negative sectional curvature, and the Strong Rigidity of Siu [Si] is applicable. A Mostow-Siu surface can be considered as a branch cover of a smooth Deligne-Mostow surface N over a totally geodesic curve (cf. [D]). By taking a finite unramified covering of both M and N if necessary, we may assume that N has irregularity at least 3. This immediately implies that M has irregularity at least 3 as well, after pulling back the holomorphic one forms from M. We may now apply Theorem 1 to conclude the proof.

□

(3.3) In the next few sections, we will consider the geometric problem of existence of holomorphic one forms on non-compact complex ball quotients.

4. A conjecture of Borel in the case of cofinite complex hyperbolic space forms

(4.1) It is a conjecture of Thurston that any real hyperbolic space supports some unramified covering which has non-trivial first Betti number. The conjecture has been extended by Borel to complex hyperbolic spaces in [B]. For compact complex ball quotients, the first such example has been obtained by Kazhdan and Shimura, cf. [BW]. However, the question is still open in general. For such quotients coming from arithmetic lattices of the first type, that is, those defined by Hermitian forms over a number field, the conjecture was known. On the other hand, for arithmetic lattices of the second type, that is, defined over a non-trivial division algebra, it is proved by Rogawski [R] and Clozel [C] that towers defined by congruence subgroups all have vanishing first Betti number. However, there may still be unramified coverings which do not arise from congruence subgroups, since the Congruence Subgroup Property does not hold for such cases.

The section deals with the conjecture of Borel mentioned above for non-compact complex ball quotients of finite volume. Since the author does not know of any reference in this aspect in the literature, the details of the proof, though elementary, are presented here. The result will be used in the next section to produce finiteness of integral points on some models of such a manifold defined over a number field.

(4.2) Since we are considering non-compact manifolds, we need to consider various notions of cohomology. Let M be a non-compact complex manifold of complex dimension n equipped with a complete Kähler metric g. The usual k-th de Rham cohomology, denoted by $H^k_{dR}(M)$, is the quotient of the space of d-closed smooth

k-forms by the space of d-exact smooth k-forms. The k-th de Rham cohomology with compact support, denoted by $H^k_c(M)$, is the quotient of the space of d-closed smooth k-forms with compact support by the space of d-exact smooth k-forms with compact support on M. $H^k_c(M)$ is the same as the relative cohomology $H^k(\overline{M}, \partial\overline{M})$, which is quotient of the space of smooth d-closed k forms vanishing on the boundary $\partial\overline{M}$ by the space of exact k forms given by $d\beta$, where β vanishes at the boundary. The k-th L^2 cohomology, denoted by $\widetilde{H}^k_2(M)$, is the quotient of the space of d-closed L^2 k-forms by the space of L^2-exact k-forms. The k-th reduced L^2 cohomology, denoted by $H^k_2(M)$, is the quotient of the space of d-closed L^2 k-forms by closure of the space of L^2-exact k-forms with respect to the L^2-topology. For complex manifolds, we similarly define Dolbeault cohomology and reduced Dolbeault cohomology in terms of $\overline{\partial}$-operator instead of d-operator. We denote the corresponding cohomology groups by $\widetilde{H}^{p,q}(M)$ and $H^{p,q}_2(M)$ respectively, in analogous to the compact situation.

From de Rham Theorem, $H^k_{dR}(M)$ is isomorphic to the singular k-th cohomology $H^1(M, \mathbb{R})$. From Poincaré Duality, there is an isomorphism between $H^k_{dR}(M)$ and $H^{2n-k}_c(M)$. The calculus for L^2-cohomology on a complete manifold is essentially the same as on a compact manifold, thanks to the use of cut-off functions as given by Gaffney [G]. From Hodge Theory, the reduced cohomology $H^k_2(M)$ is isomorphic to the space of L^2 harmonic k forms on M with respect to a given Kähler metric. The usual Hodge Decomposition allows us to decompose $H^k_2(M) = \sum_{p+q=k} H^{p,q}_2(M)$.

(4.3) Before we concentrate on the complex ball quotients, let us recall the corresponding picture for a general locally symmetric spaces, which shows that the picture is somewhat different in the cases that the symmetric spaces involved are neither real nor complex balls as considered in the conjecture of Thurston and Borel.

Let $M = \Gamma\backslash G/K$ be a locally symmetric space, where G is a semi-simple Lie group, K is a maximal compact subgroup, and Γ is a lattice in G. We have the following vanishing theorem following essentially the work of Matsushima [Ma].

Proposition 2. *Assume that G/K is a symmetric space of non-compact type which is neither a complex nor real hyperbolic space.*
(a). Suppose Γ is cocompact. Then the first Betti number of $M = \Gamma\backslash G/K$ vanishes.
(b). Suppose Γ is cofinite. Then the reduced L^2 first Betti number vanishes.

Proof For compact Γ, this follows from the original work of Matsushima [Ma], Kaneyuki-Nagano [KN 1-2] and Kazhdan [K] for the quaternionic and Kähler hyperbolic cases. A uniform geometric proof in terms of Bochner formula can be given as in [MSY], see also [Y1] for related ideas. In terms of the Bochner formula, the arguments are readily applicable to cofinite lattices. The reason is as follows. As mentioned in **4.1**, a class in the reduced L^2-cohomology can be represented by a L^2-harmonic forms on M by Hodge Theory. Now the Bochner formula in [MSY] still applies to a locally symmetric space of finite volume. The reason is that integration by part still makes sense for L^2-harmonic forms, noting that the curvature terms involved in the Bochner formula in [MSY] are all bounded in locally symmetric spaces. The argument of cut-off functions as given by Gaffney [G] can be readily applied to complete the proof.

□

(4.4) From this point on, we focus on cofinite complex ball quotients. The following is the main result of this section.

Theorem 3. *Let* $M = \Gamma\backslash PU(n,1)/(U(n)\times U(1))$ *be a cofinite (non-compact) complex ball quotient. Then there exists non-trivial holomorphic one forms on a Toroidal compactification* $\overline{M}$ *of* M*. Moreover, the reduced* L^2 *first cohomology of* M *is non-trivial.*

Proof Let us first recall the structure of a cusp in a complex ball quotient of finite volume. Consider first the case that Γ is an arithmetic lattice. From Toroidal Compactification of Mumford et al [AMRT], we know that each cusp C_i can be compactified by a torus T_i. It was shown by Mok in [Mok], utilizing a compactification of Siu-Yau [SY], that the same local behavior appears in a cusp for a non-arithmetic complex ball quotient as well. A tubular neighborhood of T_i in M is diffeomorphic to $A_i = \Delta_i \times T_i$, where Δ_i is diffeomorphic to the unit disk Δ in $\mathbb{C}$. M is decomposed as $M = M_o \cup \bigcup_{i=1}^N A_i$, where A_i, $i = 1, \cdots, N$ are ends as described above, M_o is a relative compact set on M.

We first consider $n = 2$ for simplicity in notation. Let $A = A_1$ be an end in M. Denote by $\overline{A}$ be the compactified end in $\overline{M}$, which can be considered as the unit disk bundle U of the normal bundle L of the compactifying elliptic curve $T = \mathbb{C}/\Lambda$ in Mumford compactification, where Λ is a lattice in $\mathbb{C}$. A neighborhood of T pulled back to the universal covering $\widetilde{T} \cong \mathbb{C}$ of T is of the form

$$\widehat{G} = \{(z', z) \in \widetilde{T} \times \mathbb{C} : |z|^2 < e^{\frac{-4\pi}{\tau}} \cdot e^{-\frac{4\pi}{\tau}|z'|^2}\}$$

where z is the fiber coordinate and τ is a fixed real number, cf. [Mok], page 336. A lattice element $a' \in \Lambda$ acts on $\widehat{G}$ by

$$a' \circ (z', z) = (z' + a', e^{-\frac{4\pi}{\tau}\overline{a'}\cdot z' - \frac{2\pi}{\tau}|a'|^2} \cdot z).$$

Hence a norm on a vector in L can be defined by $\|(z', z)\| = e^{-\frac{2\pi}{\tau}|z'|^2} \cdot |z|$, which satisfies $\|a' \circ (z', z)\| = \|(z', z)\|$. We identify U with the set of vectors of length $< e^{-2\pi/\tau}$.

From Mayer-Vietoris sequence of forms with compact support (cf. [BT], p. 26),

$$\to H_c^2(\overline{M}) \to H_c^3(A \cap B) \xrightarrow{\alpha} H_c^3(A) \oplus H_c^3(B) \xrightarrow{\beta} H_c^3(\overline{M}) \to .$$

We claim that there exists $\psi \in H_c^3(A)$ that does not lie in the image of α.

First of all, we note that $A \cap B \cong I \times N$, where I is the unit interval and N is a nilmanifold. Hence $H_c^3(A \cap B) \cong H_c^1(I) \otimes H^2(N)$. The de Rham cohomology $H^2(N)$ is dual to $H^1(N)$ from Poincaré Duality, which is isomorphic to $\mathbb{R}^2$, corresponding to pulling back harmonic 1-forms from the elliptic curve T by the projection map $\pi : U \to T$. Let a and b be two simple loops corresponding to the generating cycle of $H_1(T)$ on T, so that T is parametrized by real coordinates $(x, y) \in a \times b$. Let $\chi \geqslant 0$ be a cut-off function with compact support on $[0, 1)$. Let $v \in U$, which can be parametrized by (x, y, z), where z is the fiber coordinate as before. Define $f(v) = \int_{b_{x,y}} \chi(\|v\|)dy$, where $b_{x,y}$ is the circle passing through (x, y) parallel to b. Define a three form on U by

$$\psi = f\partial\|v\| \wedge \overline{\partial}\|v\| \wedge dx.$$

Note that from construction, $\frac{\partial}{\partial y}f = 0$. As the real and imaginary parts of $\partial\|v\|$ (or $\overline{\partial}\|v\|$) together with dx, dy form a local basis of the cotangent bundle of A, we conclude that $d\psi = 0$ on A. Hence $\psi \in H_c^3(A)$ as it has compact support.

To conclude the proof of the claim, we let $y_o \in b$ and denote $U_{y_o} := \pi_1^{-1}(y_o)$, where $\pi_1 : U \to T \to b$ is the projection map. Then clearly, $\int_{U_{y_o}} \psi > 0$. On the other hand, consider any element $\varphi \in H_c^1(I) \otimes H^2(N)$. Recall that $H^2(N)$ is generated by two-forms dual to dx and dy respectively. Hence if there exists φ such that ψ represents the same cohomology class as $\alpha(\varphi)$, φ could be written as $\varphi = gd\|v\| \wedge \xi$, where ξ is dual to the pull-back of dy. Hence on U_{y_o}, we may write $\varphi = gd\|v\| \wedge d\theta \wedge dx$, where $d\theta$ is angular coordinate given by $\|v\| = c$, a constant, on the set $U_{y_o} \cap (A \cap B) \cong (\Delta_{r_2} - \Delta_{r_1}) \times a$, where Δ_r represents a disk of radius r in $\mathbb{C}$, measured with respect to $\|\cdot\|$ mentioned earlier. However, we observe that $\int_{U_{y_o}} \varphi = 0$, since the integral of $d\theta$ takes values along circles $\|v\| = c$, a constant, on $U_{y_o} \cap (A \cap B)$, and such circles are contractible on U_{y_o}. Hence ψ cannot be represented by $\alpha(\varphi)$ cohomologically for some φ and the claim is proved.

The case of $n > 2$ follows by considering T as a higher dimensional torus. Analogously, we have

$$\to H_c^{2n-2}(\overline{M}) \to H_c^{2n-1}(A \cap B) \xrightarrow{\alpha} H_c^{2n-1}(A) \oplus H_c^{2n-1}(B) \xrightarrow{\beta} H_c^{2n-1}(\overline{M}) \to$$

for which there exists non-trivial $\psi \in H_c^{2n-1}(A) \oplus H_c^{2n-1}(B)$ such that $\beta(\psi)$ represents a non-trivial cohomology class in $H^{2n-1}(\overline{M})$. From Poincaré Duality, there exists a non-trivial class in $H^1(\overline{M})$.

From Hodge decomposition, there exists a non-trivial holomorphic one form τ on $\overline{M}$. Clearly the restriction $\tau|_M$ gives rise to a non-trivial holomorphic one form on M, which is a Zariski-open subset of $\overline{M}$. τ is d-closed and defines an element in $H^1(M)$. Moreover, since M is a complex ball quotient, it follows from a result of Zucker ([Z], page 210), that $H^1_{(2)}(M) \cong H^1(M)$ is non-trivial.

□

5. Cusps, holomorphic one forms and rigidity on a tower

(5.1) We recall some notations. Let $M \cong B_{\mathbb{C}}^2/\Gamma$ be a cofinite complex two ball quotient. It is well known that the lattice Γ is residually finite. Hence there exists a tower of normal covering in the following sense. There exists a tower of normal subgroups $\Gamma_0 > \Gamma_1 > \cdots > \{1\}$ of $\Gamma_0 = \Gamma$ corresponding to an infinite sequence of normal coverings of M, so that $\cap_{i=0}^{\infty}\Gamma_i = \{1\}$. Denote by N_i the number of cusps of M_i.

Proposition 3. *Let $M \cong B_{\mathbb{C}}^2/\Gamma$ be a smooth cofinite complex two ball quotient. Let $M_i = B_{\mathbb{C}}^2/\Gamma_i$ be a tower of coverings as before. Then there exists $i_k > 0$ such that for $i \geqslant i_k$, the number of cusps $N_i > k$ and $b^1(M_i) \geqslant k$.*

We are going to prove the proposition in several steps.

(5.2)

Lemma 2. *Let $M \cong B_{\mathbb{C}}^2/\Gamma$ be a smooth cofinite complex two ball quotient with $N \geqslant 2$ cusps. Then $b^1(M) \geqslant N$.*

Proof This follows essentially from the proof of Theorem 3, where we have shown that there exists a non-trivial class ψ_i in $H^1(M, \mathbb{R})$ coming from each end A_i and coming from cohomology with compact support in A_i. From the proof, we know that the restriction of ψ_i to $A_i \cap B_i$, where B_i is a neighborhood of $\overline{M} - A_i$, does not come from cohomology with compact support on $A_i \cap B_i$. It follows that each end of M_i contribute to at least one cohomology class which are linearly independent on $\overline{M}$.

□

(5.3)

Lemma 3. *Let $M \cong B^2_{\mathbb{C}}/\Gamma$ be a cofinite complex two ball quotient. Then given any positive integer N, there exists a finite unramified covering M_1 of M such that the number of cusps of M is at least N.*

Proof Since M is cofinite non-cocompact, there exists at least one cusp on M. The goal is to show that there exists a normal covering $M_1 \to M$ such that the number of cusps on M_1 can be chosen to be arbitrarily large as the index of the covering $[M_1 : M]$ is getting large.

Let $\Gamma_o > \Gamma_1 > \cdots > \{1\}$ be a tower of normal subgroups of $\Gamma_0 = \Gamma$ corresponding to an infinite sequence of normal coverings of M as mentioned in **5.1**. Let D_i be a fundamental domain of Γ_i. Since we are taking a tower of normal coverings, we may assume that the fundamental domains D_i of Γ_i are nested in the sense of $D_i \subset D_{i+1}$ after translating by Γ_i/Γ_{i+1} if necessary. As $\cap_i \Gamma_i = 1$, $D_k = \cup^k_i D_i \to B^2_{\mathbb{C}}$ as a set as $k \to \infty$.

Let N_i be the number of cusps on M_i. Given any $N > 0$, we are going to show that $N_i > N$ for i sufficiently large through proof by contradiction. Hence assume that $\sup_i N_i = N < \infty$. Each cusp of D_i occurs as the intersection of the closure of D_i with $\partial B^2_{\mathbb{C}}$. Hence if we denote by S_i the set of cusps of D_i, it is well-known that D_i consists of a finite number of points (cf. [Mok], [SY]), each corresponding to an end of M_i. Since we have chosen D_i to be a nested sequence of fundamental domains of $\Gamma_,$ we conclude that $S_i \subset S_{i+1}$ for $i \geqslant 0$. Hence from our assumption, there exists $S = \{p_1, \ldots, p_N\}$ and an integer i_o such that $S_i = S$ for $i \geqslant i_o$.

We claim that the set $\Gamma S \backslash S \neq \emptyset$. Suppose that this is not the case, we know that Γ leaves the set S invariant. Since S is a finite set, by going to a subgroup of Γ is necessary, we may assume that Γ leaves S pointwise fixed. This however contradicts the fact that Γ is Zariski dense and cannot leave a point at the infinity fixed.

Hence we can find a $\gamma \in \Gamma_{i_o}$ such that $\gamma(S)$ contains a point $q \notin S$. From the construction, $q = \gamma(p_i)$ for some $p_i \in S$. Hence $\gamma(D_{i_o})$ is another fundamental domain of Γ_{i_o} on which $q \in \overline{\gamma(D_{i_o})} \cap \partial B^2_{\mathbb{C}}$.

Let $x_o \in D_{i_o}$ so that $\gamma(x_o) \in \gamma(D_{i_o})$. Since $\cap^\infty_{i=0}\Gamma_i = \emptyset$, the union $\cup^\infty_{i=1} D_i = B^2_{\mathbb{C}}$ by our choice of D_i. We know that $\gamma(x_o) \in D_j$ for some $j = i_1$ and hence for all $j \geqslant i_1$. Hence $D_j \cap \gamma(D_{i_o}) \neq \emptyset$. As D_j is a tessellation of translations of domains D_{i_o} corresponding to translations given by Γ_0/Γ_j on M, we conclude that we may assume that $\gamma(D_{i_o}) \subset D_j$, except possibly a measure zero set corresponding to the boundary of D_{i_o} in $B^2_{\mathbb{C}}$. This however implies that

$$q \in \overline{\gamma(D_{i_o})} \cap \partial B^2_{\mathbb{C}} \subset \overline{D_j} \cap \partial B^2_{\mathbb{C}}.$$

It follows that q is also a cusp of D_j and hence M_j has at least $N+1$ cusp points, contradictory to our assumption. This concludes the proof of the lemma.

□

Proof of Proposition 3 This follows from Lemma 2 and Lemma 3.

□

(5.4) Let now $p : M' \to M$ be a finite unramified covering between two non-compact complex two ball quotients. Let $\overline{M}' = M' \cup D'$ and $\overline{M} \cup D$ be the respective Toroidal compactification as in [AMRT] for arithmetic lattices and as in [Mok] for non-arithmetic lattices. We denote by V^* the dual of a holomorphic vector bundle V on M. Then we have the following conclusions.

Proposition 4. *(a). The covering map $p : M' \to M$ extends to a holomorphic map $p : \overline{M}' \to \overline{M}$.*
(b). $(\overline{M}, D)$ is rigid in the sense that $H^1(\overline{M}, [\Omega(\log D)]^*) = 0$. Similarly for $(\overline{M}', D')$.
(c). $p : \overline{M}' \to \overline{M}$ is rigid.

Proof (a) follows from the description of Toroidal compactification (cf. [AMLT], [Mu2] or [Mok]). It also follows from the more geometric observation below. The Bergman metric g_M on M has holomorphic sectional curvature bounded from above by a negative constant, and the Bergman metric $g_{M'}$ on M' has sectional curvature bounded from below by a constant. Hence we know from Schwarz Lemma (cf. [CCL]) that $p : M \to M'$ satisfies $p^* g_{M'} \leqslant g_M$. Consider a polydisk neighbourhood U of a point on D in $\overline{M}$ such that D is given by $z = 0$ for some coordinate function z on U, where $|z| < 1$. Then as given in [Mok], the Bergman metric g_M is given explicitly as

$$c_1 \frac{\sqrt{-1} dz \wedge d\overline{z}}{|z|^2 (\log |z|)^2} + c_2 \frac{\theta}{(-\log |z|)}, \tag{1}$$

where θ is a positive $(1,1)$ form on the tangent space of D, and c_1, c_2 are smooth positive functions on U. The same asymptotic growth of metric applies to an end of M' as well. From the conclusion of the above Schwarz Lemma, it follows that $p|_M$ has uniformly bounded derivative on some neighbourhood of a point $x' \in D' \subset \overline{M}'$. It follows from boundedness of p and Riemann Extension Theorem that the map can be extended to D'. The claim follows.

For (b), we observe that $\in H^1(\overline{M}, [\Omega(\log D)]^*) \subset H^1_{(2)}(M, \Omega^*_M)$. The follows by considering local basis of $\Omega(\log D)$ near D in terms of $a(z,w)\frac{dz}{z} + b(z,w)dw$. Here (z,w) is a local coordinate at a point on $z = 0$. The local form near the compactifying divisor is L^2 integrable in view of the asymptotic formula in (1) for the metric near the compactifying divisor. The vanishing of $H^1_{(2)}(M, \Omega^*_M)$ follows from computations involving Bochner formula as discussed in [CV].

For (c), we observe that the restriction $p|_{M'} : M' \to M$ is unramified. We claim that $p|_{M'}$ is rigid. Otherwise the lift of the mapping to the corresponding cover leads to deformation $f_t, |t| < \epsilon$ of the identity map on M'. However, from Siu's Strong Rigidity Theorem in [Si], we know that M' cannot be deformed and each f_t is just the identity map. Hence p is rigid.

□

6. Integral points on a cofinite complex hyperbolic space form

(6.1) We are going to apply the results of Faltings, and the geometric results of the last section to deduce finiteness of integral points for cofinite ball quotients. Recall that a quasi-projective manifold $M = \overline{M} - D$, where D is a normal crossing divisor, is defined over a number field F if both $\overline{M}$ and D can be defined over F.

We say that M has a finite unramified cover M' defined over F if both M and M' as quasi-projective manifolds are defined over F and the mapping $M' \to M$ is a finite mapping defined over F and is unramified on M'.

Here is our main result concerning finiteness of integral points concerning quasi-projective manifolds.

Theorem 4. *Let M be the quotient of the complex two ball by a torsion free cofinite lattice in $PU(2,1)$. Assume that M, its toroidal compactification $\overline{M}$ and the compactifying divisor $D = \overline{M} - M$ are all defined over a number field F. Then the number of integral points on M with respect to the compactifying divisor in F is finite.*

Proof We assume for the sake of proof by contradiction that there are infinite number of integral points on M with respect to the compactifying divisor D in the number field F.

Proposition 3 implies that after going to a unramified covering M' of M of finite index of sufficiently large index, the dimension of the space of holomorphic one forms on a toroidal compactificaition $\overline{M}'$ of M' is at least 3. M' is constructed by considering a proper normal subgroup Γ of finite index in the lattice Γ associated to M. Since M' is still a complex ball quotient with finite volume, M' itself can be compactified according to Toroidal compactification on each end of M'.

We know from Proposition 4 that $p : (\overline{M}', D') \to (\overline{M}, D)$ is a rigid morphism between the two projective algebraic varieties. Hence according to Proposition 1, p can be defined over some number field. Hence we may assume that M', M and p are all defined over a number field K_1. Letting K_2 be the compositum of K_1 and F. Hence for the sake of proof by contradiction we assume that there are infinite number of integral points in F on M with respect to D. Then M', M and p are all defined over K. Moreover, according to the result of Chevalley-Weil and Hermite as stated in §2.2, by considering a finite extension K_3 of K_2 if necessary, we conclude that there are infinitely many integral points in K_3 on M' with respect to D'.

According to Proposition 3, there are at least 3 linearly independent holomorphic one forms on $\overline{M}'$, where $M' \subset \overline{M}'$ as a Zariski open set. Hence the Albanese map $\alpha_{\overline{M}'}$ gives a morphism of $\overline{M}'$ into an Abelian variety $A = \alpha(\overline{M}')$. According to the result of Faltings in [F2], there are only a finite number of rational points on A apart from a finite number of translates of abelian subvarieties. Clearly $\alpha(\overline{M}')$ cannot be an Abelian subvariety of complex dimention 2, as the irregularity of $\alpha(M)$ is at least 3. Hence apart from a finite number of elliptic curves E_i, $i = 1, \ldots, N$, the number of rational points is finite. Let C be an irreducible curve on $\overline{M}$ defined over K_3 such that $\alpha(C)$ is one of those E_i. Since C dominates an elliptic curve, the genus $g(C)$ of C is at least 1. If $g(C) \geqslant 2$, Mordell Conjecture as proved by Faltings in [F1] implies that the number of rational points on C is finite. Hence we may assume that $g(C) = 1$. From the fact that M is complex ball quotient and hence hyperbolic, C cannot be contained in M. Hence C must intersect the compactifying divisor D. If $g(C) = 1$, it follows from Siegel's Theorem, or the result of Vojta in **(2.3)**, that the number of integral points with respect to D' is finite. Since there are only a finite number of E_i and hence of such C, we conclude that the number of integral points of M' with respect to the compactifying divisor D' is finite. This contradicts the earlier deduction that there are infinite number of integral points on

M' with respect to D' coming from our assumption. The contradiction concludes the proof of Theorem 4.

□

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