DOMAIN DECOMPOSITION FOR A MIXED FINITE ELEMENT METHOD IN THREE DIMENSIONS*

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Abstract. We consider the solution of the discrete linear system resulting from a mixed finite element discretization applied to a second-order elliptic boundary value problem in three dimensions. Based on a decomposition of the velocity space, these equations can be reduced to a discrete elliptic problem by eliminating the pressure through the use of substructures of the domain. The practicality of the reduction relies on a local basis, presented here, for the divergence-free subspace of the velocity space. We consider additive and multiplicative domain decomposition methods for solving the reduced elliptic problem, and their uniform convergence is established.

Key words. divergence-free basis, domain decomposition, second-order elliptic problems, mixed finite element method

AMS subject classifications. 65F10, 65F30, 65L60

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1. Introduction. In [6], Ewing and Wang considered and analyzed a domain decomposition method for solving the discrete system of equations which result from mixed finite element approximation of second-order elliptic boundary value problems in two dimensions. The approach in [6] is first to seek a discrete velocity satisfying the discrete continuity equation through a variation of domain decomposition (static condensation), and then to apply a domain decomposition method to the reduced elliptic problem arising from elimination of the pressure in the saddle-point problem. For analogous work, see also [8], [10], and [4]. The crucial part of the approach in [6] is to characterize the divergence-free velocity subspaces. This is also the essential difference with those in [8], [10], and [4].

In this paper, we will use the domain decomposition approach in [6] for the solution of the algebraic system resulting from the mixed finite element method applied to second-order elliptic boundary value problems in three dimensions. As mentioned above, the basis of the divergence-free velocity subspace plays an essential role in the approach; hence we will construct a basis of this subspace for the lowest-order rectangular Raviart–Thomas–Nedelec velocity space [13], [12]. The construction in two dimensions (2-D) is more general and rather easier than in three dimensions (3-D) due to the fact that any divergence-free vector in 2-D can be expressed as the curl

of a scalar stream function. Extension of this work to triangular or irregular meshes and to multilevel domain decomposition will be discussed in a forthcoming paper.

This approach has several practical advantages. For an $n \times n \times n$ grid in 3-D, the number of discrete unknowns is approximately $4n^3$, essentially one pressure and three velocity components per cell. Using the divergence-free subspace, we decouple the system in such a manner that the velocity can be obtained by solving a symmetric positive definite system of order roughly $2n^3$. In contrast to some other proposed procedures, this does not require the introduction of Lagrange multipliers corresponding to pressures at cell interfaces, and it permits direct computation of the velocity, which is often the principal variable of interest, alone. If the pressure is also needed, it can be calculated inexpensively in an additional step. Furthermore, the approach deals readily with the case of full-tensor conductivity (cross-derivatives), where the mass matrix is fuller than tridiagonal and methods based on reduced integration (mass lumping) are difficult to apply. This case results, for example, from anisotropic permeabilities in flows in porous media, where highly discontinuous conductivity coefficients are also common. For such problems, mixed methods are known to produce more realistic velocities than standard techniques [11].

The outline of the remainder of this paper is as follows. In section 2, we review the mixed finite element method for elliptic problems with homogeneous Neumann boundary conditions. The domain decomposition method for the resulting algebraic system is discussed in section 3, and its uniform convergence is established in section 5. A computationally convenient, divergence-free basis with minimal support is constructed in section 4.

2. Mixed finite element method. In this section, we begin with a brief review of the mixed finite element method with lowest-order Raviart–Thomas–Nedelec [13], [12] (RTN) approximation space for second-order elliptic boundary value problems in three dimensions. For simplicity, we consider a homogeneous Neumann problem: find p such that

$$(2.1) \qquad \begin{cases} -\nabla \cdot (k\nabla p) & = & f & \text{in} & \Omega = (0,\ 1)^3, \\ (k\nabla p) \cdot \mathbf{n} & = & 0 & \text{on} & \partial\Omega, \end{cases}$$

where $f \in L^2(\Omega)$ satisfies the relation

$$(2.2) \qquad \int_\Omega f\, dx\, dy\, dz = 0$$

and $\mathbf{n}$ denotes the unit outward normal vector to $\partial\Omega$. The symbols $\nabla\cdot$ and ∇ stand for the divergence and gradient operators, respectively. Assume that $k = (k_{ij})_{3\times 3}$ is a given real-valued symmetric matrix function with bounded and measurable entries k_{ij} $(i,\ j = 1,\ 2,\ 3)$ and satisfies the ellipticity condition; i.e., there exist positive constants α_1 and α_2 such that

$$(2.3) \qquad \alpha_1 \xi^t\, \xi \le \xi^t k(x,\ y,\ z)\xi \le \alpha_2 \xi^t\, \xi$$

for all $\xi \in \mathbb{R}^3$ and almost all $(x,\ y,\ z) \in \bar{\Omega}$.

We shall use the following space to define the mixed variational problem. Let

$$H(div; \Omega) \equiv \{\mathbf{w} \in L^2(\Omega)^3 \,|\, \nabla \cdot \mathbf{w} \in L^2(\Omega)\},$$

which is a Hilbert space equipped with the norm

$$\|\mathbf{w}\|_{H(div;\Omega)} \equiv (\|\mathbf{w}\|^2_{L^2(\Omega)^3} + \|\nabla \cdot \mathbf{w}\|^2_{L^2(\Omega)})^{1/2}$$

and the associated inner product. By introducing the flux variable

$$\mathbf{v} = -k\nabla p,$$

which is of practical interest for many physical problems, we can rewrite the PDE of (2.1) as a first-order system

$$\begin{cases} k^{-1}\mathbf{v} + \nabla p &= 0, \\ \nabla \cdot \mathbf{v} &= f \end{cases}$$

and obtain the mixed formulation of (2.1): find $(\mathbf{v}, p) \in \mathbf{V} \times \Lambda$ such that

(2.4)
$$\begin{cases} a(\mathbf{v}, \mathbf{w}) - b(\mathbf{w}, p) &= 0 & \forall\, \mathbf{w} \in \mathbf{V}, \\ b(\mathbf{v}, \lambda) &= (f, \lambda) & \forall\, \lambda \in \Lambda. \end{cases}$$

Here $\mathbf{V} = H_0(div;\Omega) \equiv \{\mathbf{w} \in H(div;\Omega)\,|\,\mathbf{w} \cdot \mathbf{n} = 0 \text{ on } \partial\Omega\}$, Λ is the quotient space $L_0^2(\Omega) = L^2(\Omega)/\{\text{constants}\}$, the bilinear forms $a(\cdot\,,\cdot) : \mathbf{V} \times \mathbf{V} \to \mathbb{R}$ and $b(\cdot\,,\cdot) : \mathbf{V} \times \Lambda \to \mathbb{R}$ are defined by

$$a(\mathbf{w}, \mathbf{u}) = \int_\Omega (k^{-1}\mathbf{w}) \cdot \mathbf{u}\, dx\, dy\, dz \quad \text{and} \quad b(\mathbf{w}, \lambda) = \int_\Omega (\nabla \cdot \mathbf{w})\lambda\, dx\, dy\, dz$$

for any $\mathbf{w}, \mathbf{u} \in \mathbf{V}$ and $\lambda \in \Lambda$, respectively, and $(\cdot, \cdot)$ denotes the $L^2(\Omega)$ inner product.

To discretize the mixed formulation (2.4), we assume that we are given two finite element subspaces

$$\mathbf{V}^h \subset \mathbf{V} \quad \text{and} \quad \Lambda^h \subset \Lambda$$

defined on a uniform rectangular mesh with elements of size $O(h)$. The mixed approximation of $(\mathbf{v}, p)$ is defined to be the pair $(\mathbf{v}^h, p^h) \in \mathbf{V}^h \times \Lambda^h$ satisfying

(2.5)
$$\begin{cases} a(\mathbf{v}^h, \mathbf{w}) - b(\mathbf{w}, p^h) &= 0 & \forall\, \mathbf{w} \in \mathbf{V}^h, \\ b(\mathbf{v}^h, \lambda) &= (f, \lambda) & \forall\, \lambda \in \Lambda^h. \end{cases}$$

We refer to [13] for the definition of a class of approximation subspaces $\mathbf{V}^h$ and Λ^h. In this paper, we shall consider only the lowest-order RTN space defined on a rectangular triangulation of Ω. Such a space for the velocity consists of vector functions whose ith component is continuous piecewise linear in the x_i variable and discontinuous piecewise constant in the x_j variable for $j \neq i$. The corresponding pressure space Λ^h consists of discontinuous piecewise constants with respect to the triangulation $\mathcal{T}^h$ with a fixed value on one element. Specifically, let $\mathcal{T}^h$ denote a uniform rectangular triangulation of Ω. Then the lowest-order RTN approximation space for the velocity on a rectangle $K \in \mathcal{T}^h$ is defined by

(2.6)
$$\mathbf{V}^h(K) = \mathcal{P}_{1,0,0} \times \mathcal{P}_{0,1,0} \times \mathcal{P}_{0,0,1},$$

and the corresponding pressure space is

(2.7)
$$\Lambda^h(K) = \mathcal{P}_{0,0,0},$$

where $\mathcal{P}_{i_1,i_2,i_3}(K)$ denotes the polynomials of degree i_j $(j = 1, 2, 3)$ with respect to x_j. It is well known that the above RTN space satisfies the Babŭska–Brezzi stability

condition (cf. [13]): there exists a positive constant β independent of the mesh size h of $\mathcal{T}^h$ such that

$$(2.8) \qquad \sup_{\mathbf{w} \in \mathbf{V}^h} \frac{b(\mathbf{w}, \lambda)}{\|\mathbf{w}\|_{H(div, \Omega)}} \geq \beta \|\lambda\|_{L^2(\Omega)} \quad \forall \lambda \in \Lambda^h.$$

Also, Raviart and Thomas in [13] demonstrated the existence of a projection operator $\mathbf{\Pi}_h : \mathbf{V} \longrightarrow \mathbf{V}^h$ such that, for any $\mathbf{v} \in \mathbf{V}$,

$$(2.9) \qquad b(\mathbf{\Pi}_h\mathbf{v}, \lambda) = b(\mathbf{v}, \lambda) \quad \forall \lambda \in \Lambda^h,$$

$$(2.10) \qquad \|\mathbf{\Pi}_h\mathbf{v} - \mathbf{v}\|_{L^2(\Omega)^3} \leq C\,h^s\|\mathbf{v}\|_{H^s(\Omega)^3}, \quad s = 0, 1.$$

3. Domain decomposition. Problem (2.5) is clearly symmetric and indefinite. To reduce it to a symmetric positive definite problem, we need a discrete velocity $\mathbf{v}_I^h \in \mathbf{V}^h$ satisfying

$$(3.1) \qquad b(\mathbf{v}_I^h, \lambda) = (f, \lambda) \quad \forall \lambda \in \Lambda^h.$$

Define the discretely (as opposed to pointwise) divergence-free subspace $\mathbf{D}^h$ of $\mathbf{V}^h$:

$$(3.2) \qquad \mathbf{D}^h = \{\mathbf{w} \in \mathbf{V}^h \,|\, b(\mathbf{w}, \lambda) = 0 \quad \forall \lambda \in \Lambda^h\},$$

and let

$$\mathbf{v}_D^h = \mathbf{v}^h - \mathbf{v}_I^h,$$

which is obviously in $\mathbf{D}^h$ by the second equation of (2.5) and which satisfies

$$(3.3) \qquad a(\mathbf{v}_D^h, \mathbf{w}) = -a(\mathbf{v}_I^h, \mathbf{w}) \quad \forall \mathbf{w} \in \mathbf{D}^h,$$

by the first equation. This problem is symmetric and positive definite.

This suggests the following procedure for obtaining $\mathbf{v}^h$, the solution of (2.5): find $\mathbf{v}_I^h \in \mathbf{V}^h$ satisfying (3.1), compute the projection $\mathbf{v}_D^h \in \mathbf{D}^h$ satisfying (3.3), then set $\mathbf{v}^h = \mathbf{v}_I^h + \mathbf{v}_D^h$. This procedure will be the basis for Algorithms 3.1 and 3.2 below. Given $\mathbf{v}_I^h$, (3.3) leads to a unique $\mathbf{v}^h$, which is independent of the choice of $\mathbf{v}_I^h$. (A term added to a given $\mathbf{v}_I^h$ must be in $\mathbf{D}^h$, and it is canceled by the resulting change in $\mathbf{v}_D^h$.) For an $n \times n \times n$ grid, computing the projection $\mathbf{v}_D^h$ involves solving a system of order approximately $2n^3$. Solving for p^h is optional; if it is desired, it can be obtained from the first equation in (2.5) once $\mathbf{v}^h$ is known.

There are many discrete velocities in $\mathbf{V}^h$ satisfying (3.1), and several approaches have been discussed in the literature for seeking such a discrete velocity (e.g., [6], [8], and [10]). All of these approaches are based on a type of domain decomposition (static condensation) method applied to problem (2.5). In this paper, we will adopt the approach discussed in [6] by Ewing and Wang. This approach requires solving only a coarse-grid problem and some local problems of the form (2.5).

To compute $\mathbf{v}_I^h$ and define the domain decomposition method for problem (3.3), we start with a coarse initial rectangular triangulation $\mathcal{T}^H = \{K_j\}_{j=1}^J$ of the domain Ω (so that $\bar{\Omega} = \cup_{j=1}^J \bar{K}_j$), and a regular fine rectangular triangulation $\mathcal{T}^h$ obtained by further partitioning all of the elements in $\mathcal{T}^H$. Associated with the coarse triangulation $\mathcal{T}^H$, we construct a set of overlapping subdomains $\{\Omega_j\}_{j=1}^J$ by extending each

element $K_j \in \mathcal{T}^H$ to a larger subdomain Ω_j, whose diameter is denoted by $H_j \leq C H$. Assume that the maximum number of subdomain overlaps is bounded, and further that the distance between the boundaries ∂K_j and $\partial \Omega_j$ is bounded below by $\zeta_1 H$ and above by $\zeta_2 H$; i.e., for all $j \in \{1, \ldots, J\}$ there exist constants $\zeta_1, \zeta_2 > 0$ such that

$$\zeta_1 H \leq \text{dist}(\partial K_j, \partial \Omega_j) \leq \zeta_2 H.$$

Also assume that the boundaries of the Ω_j do not cut through any element in $\mathcal{T}^h$, i.e., they must coincide with boundaries of elements of $\mathcal{T}^h$. Thus, the restrictions of $\mathcal{T}^h$ on Ω_j and K_j provide two uniform triangulations $\mathcal{T}_j^h$ and $\tilde{\mathcal{T}}_j^h$ for Ω_j and K_j, respectively.

Let $\mathbf{V}_j \times \Lambda_j$ and $\tilde{\mathbf{V}}_j \times \tilde{\Lambda}_j$ be the lowest-order RTN approximation spaces corresponding to the triangulations $\mathcal{T}_j^h$ and $\tilde{\mathcal{T}}_j^h$, respectively. For convenience, let $\mathbf{V}^H = \mathbf{V}_0 = \tilde{\mathbf{V}}_0$ and $\Lambda^H = \Lambda_0 = \tilde{\Lambda}_0$. As in [6], let f^h and $f_0^h \equiv f^H$ be the L^2 projection of f in Λ^h and Λ^H, respectively, and $f_j^h \in \tilde{\Lambda}_j^h$ be the restriction of $f^h - f_0^h$ on K_j. Then the discrete velocity $\mathbf{v}_I^h$ satisfying (3.1) may be determined by the sum of $\mathbf{v}_j$'s which are the solutions of the following problems: find $(\mathbf{v}_j, p_j) \in \tilde{\mathbf{V}}_j \times \tilde{\Lambda}_j$ such that

$$(3.4) \qquad \begin{cases} (\tilde{k}\mathbf{v}_j, \mathbf{w}) - b(\mathbf{w}, p_j) &= \quad 0 \qquad \quad \forall \mathbf{w} \in \tilde{\mathbf{V}}_j, \\ b(\mathbf{v}_j, \lambda) &= \quad (f_j^h, \lambda) \quad \forall \lambda \in \tilde{\Lambda}_j, \end{cases}$$

where $\tilde{k} \in \mathbb{R}^{3 \times 3}$ is an arbitrary matrix-valued function which is symmetric positive definite and defined on Ω_j for all $j \in \{0, 1, \ldots, J\}$. Note that $\mathbf{v}_0$ is the solution of problem (2.5) corresponding to the coarse triangulation $\mathcal{T}^H$, and that $\mathbf{v}_j$ for $1 \leq j \leq J$ can be obtained by solving some local problems.

We shall use additive and multiplicative domain decomposition methods for approximate computation of the solution of problem (3.3). To this end, we define the family of discretely divergence-free velocity subspaces $\{\mathbf{D}_j\}_{j=0}^J$ by $\mathbf{D}_0 = \mathbf{D}^H$, and for $j \in \{1, 2, \ldots, J\}$,

$$\mathbf{D}_j = \{\mathbf{u} \in \mathbf{V}_j \,|\, b(\mathbf{u}, \lambda) = 0 \quad \forall \lambda \in \Lambda_j\}.$$

For any $\mathbf{u} \in \mathbf{D}^h$, we define the projection operators $\mathbf{P}_j : \mathbf{D}^h \longrightarrow \mathbf{D}_j$ associated with the bilinear form $a(\cdot, \cdot)$ by

$$a(\mathbf{P}_j\mathbf{u}, \mathbf{w}) = a(\mathbf{u}, \mathbf{w}) \quad \forall \mathbf{w} \in \mathbf{D}_j$$

for $j \in \{0, 1, \ldots, J\}$.

ALGORITHM 3.1 (Additive domain decomposition).

1. *For $j = 0, 1, \ldots, J$, compute $\mathbf{v}_j \in \tilde{\mathbf{V}}_j$ by solving problems* (3.4). *Then set*

$$\mathbf{v}_I^h = \mathbf{v}_0 + \mathbf{v}_1 + \cdots + \mathbf{v}_J.$$

2. *Compute an approximation $\mathbf{v}_D$ of $\mathbf{v}_D^h \in \mathbf{D}^h$ by applying conjugate gradient iteration to*

$$(3.5) \qquad\qquad\qquad\qquad \mathbf{P}\mathbf{v}_D = \mathbf{F},$$

where $\mathbf{P} = \mathbf{P}_0 + \mathbf{P}_1 + \cdots + \mathbf{P}_J$, $\mathbf{F} = \mathbf{F}_0 + \mathbf{F}_1 + \cdots + \mathbf{F}_J$, and $\mathbf{F}_j = \mathbf{P}_j\mathbf{v}_D^h$.
3. *Set*

$$\mathbf{v}^h = \mathbf{v}_D + \mathbf{v}_I^h.$$

Remark 3.1. The right-hand side $\mathbf{F}$ in (3.5) can be computed by solving the coarse-grid problem and local subproblems. Specifically, for each $j \in \{0, 1, \ldots, J\}$, $\mathbf{F}_j$ is the solution of the following problem:

$$(3.6) \qquad a(\mathbf{F}_j, \mathbf{w}) = a(\mathbf{P}_j \mathbf{v}_D^h, \mathbf{w}) = -a(\mathbf{v}_I^h, \mathbf{w}) \quad \forall \mathbf{w} \in \mathbf{D}_j.$$

For $\mathbf{F}_j$ given in (3.6), we can see as follows that (3.5) is equivalent to problem (3.3). Given (3.3), define $\mathbf{F}_j$ as in (3.6), and $\mathbf{P}$ and $\mathbf{F}$ as above. Then $a(\mathbf{F}_j, w) = a(v_D^h, w) \; \forall w \in \mathbf{D}_j$, so that $\mathbf{F}_j = \mathbf{P}_j \mathbf{v}_D^h$, and summing on j yields $\mathbf{P} \mathbf{v}_D^h = \mathbf{F}$. To complete the equivalence, we claim that $\mathbf{v}_D^h$ is the only solution of (3.5). It suffices to show that $\mathbf{Pu} = \mathbf{0}$ implies that $\mathbf{u} = \mathbf{0}$ for $\mathbf{u} \in \mathbf{D}^h$. If $\mathbf{Pu} = \mathbf{0}$, then

$$0 = a(\mathbf{Pu}, \mathbf{u}) = \sum_j a(\mathbf{P}_j \mathbf{u}, \mathbf{u}) = \sum_j a(\mathbf{P}_j \mathbf{u}, \mathbf{P}_j \mathbf{u}),$$

so that $a(\mathbf{P}_j \mathbf{u}, \mathbf{P}_j \mathbf{u}) = 0 \; \forall j$; hence $\mathbf{P}_j \mathbf{u} = \mathbf{0} \; \forall j$. In Lemma 5.1 below, we prove that $\mathbf{u}$ has a decomposition $\mathbf{u} = \mathbf{u}_0 + \mathbf{u}_1 + \cdots + \mathbf{u}_J$, where $\mathbf{u}_j \in \mathbf{D}_j$. With this,

$$a(\mathbf{u}, \mathbf{u}) = \sum_j a(\mathbf{u}_j, \mathbf{u}) = \sum_j a(\mathbf{u}_j, \mathbf{P}_j \mathbf{u}) = 0,$$

and hence $\mathbf{u} = \mathbf{0}$, as claimed.

At each iteration of the conjugate gradient method applied to (3.5), we need to compute the action of the projection operator $\mathbf{P}_j$ on a given $\mathbf{u} \in \mathbf{D}^h$, which may be obtained by solving the following problem:

$$(3.7) \qquad a(\mathbf{P}_j \mathbf{u}, \mathbf{w}) = a(\mathbf{u}, \mathbf{w}) \quad \forall \mathbf{w} \in \mathbf{D}_j.$$

When analyzing the preconditioned conjugate gradient method for a system of linear equations, the crucial issue is to estimate the condition number of the preconditioned operator. In section 5, we will establish a uniform estimate of the condition number for $\mathbf{P}$ and find a basis for $\mathbf{D}^h$ that allows for efficient computations.

ALGORITHM 3.2 (Multiplicative domain decomposition).
1. *Compute* $\mathbf{v}_I^h$ *as in the first step of* Algorithm 3.1.
2. *Given an approximation* $\mathbf{v}_D^l \in \mathbf{D}^h$ *to the solution* $\mathbf{v}_D^h$ *of* (3.3), *define the next approximation* $\mathbf{v}_D^{l+1} \in \mathbf{D}^h$ *as follows:*
   (a) *Set* $W_{-1} = \mathbf{v}_D^l$.
   (b) *For* $j = 0, 1, \ldots, J$ *in turn, define* W_j *by*

   $$W_j = W_{j-1} + \omega \mathbf{P}_j(\mathbf{v}_D^h - W_{j-1}),$$

   where the parameter $\omega \in (0, 2)$.
   (c) *Set* $\mathbf{v}_D^{l+1} = W_J$.
3. *Set*

$$\mathbf{v}^h = \mathbf{v}_I^h + \mathbf{v}_D^L.$$

Remark 3.2. $\mathbf{P}_j(\mathbf{v}_D^h - W_{j-1})$ can be computed by solving the following problem:

$$(3.8) \qquad a(\mathbf{P}_j(\mathbf{v}_D^h - W_{j-1}), \mathbf{w}) = -a(\mathbf{v}_I^h + W_{j-1}, \mathbf{w}) \quad \forall \mathbf{w} \in \mathbf{D}_j.$$

A simple computation implies that the error propagation operator of multiplicative domain decomposition at the second step of Algorithm 3.2 has the form of

$$(3.9) \qquad \mathbf{E} = (\mathbf{I} - \mathbf{P}_J)(\mathbf{I} - \mathbf{P}_{J-1}) \cdots (\mathbf{I} - \mathbf{P}_0).$$

Define a norm associated with the bilinear form $a(\cdot, \cdot)$ by

$$\|\mathbf{u}\|_a = a(\mathbf{u}, \mathbf{u})^{1/2} \quad \forall \mathbf{u} \in \mathbf{D}^h.$$

We shall show in the last section that $\|\mathbf{E}\|_a$ is bounded by a constant which is less than one and independent of the mesh size h and the number of subdomains.

4. Construction of a divergence-free basis. Since the technique of the mixed method leads to a saddle-point problem, which causes the final system to be indefinite, many well-established efficient linear system solvers cannot be applied. As we mentioned earlier, (2.5) could be symmetric and positive definite if we discretize it in the discrete divergence-free subspace $\mathbf{D}^h$. The construction of a basis for $\mathbf{D}^h$ is essential.

In this section, we will construct a computationally convenient basis for $\mathbf{D}^h$—the divergence-free subspace of $\mathbf{V}^h$. We will do this by first constructing a vector potential space $\mathbf{U}^h$ such that

$$(4.1) \qquad\qquad\qquad \mathbf{D}^h = \mathbf{curl}\, \mathbf{U}^h.$$

Next, we will find a basis for $\mathbf{U}^h$, and we will define a basis for $\mathbf{D}^h$ by simply taking the curls of the vector potential basis functions.

Denote the mesh on $\Omega = (0,1)^3$ by $0 = x_0 < \cdots < x_i < \cdots < x_n = 1$, and similarly with y_j and z_k, $0 \le j, k \le n$. The assumption of the same number n of intervals in each direction is merely for convenience and is not necessary for the construction to follow. Let

$$\phi^x_{i,j,k}(x, y, z) = \chi_i(x)\psi_j(y)\psi_k(z), \qquad 1 \le i \le n, \quad 1 \le j, k \le n - 1,$$

where χ_i is the characteristic function of (x_{i-1}, x_i), ψ_j is the standard hat function supported on (y_{j-1}, y_{j+1}), and similarly ψ_k is supported on (z_{k-1}, z_{k+1}). Then $\phi^x_{i,j,k}$ is the standard bilinear nodal basis function on $(y_{j-1}, y_{j+1}) \times (z_{k-1}, z_{k+1})$, extended as a constant in the x-direction in the ith slice only, zero in the other slices. For economy of notation, write $\phi_i(y, z)$ for $\phi^x_{i,j,k}(x, y, z)$, where the single index i, $1 \le i \le n(n-1)^2$, runs through the triples (i, j, k) lexicographically (k varying most rapidly). The support of a typical $\phi_i(y, z)$ consists of a $1 \times 2 \times 2$ set of 4 cells and is shown in Figure 4.1. Similarly, let

$$\phi^y_{j,i,k}(x, y, z) = \chi_j(y)\psi_i(x)\psi_k(z) = \phi_j(x, z), \quad 1 \le j \le n,\ 1 \le i, k \le n - 1,$$
$$\phi^z_{k,i,j}(x, y, z) = \chi_k(z)\psi_i(x)\psi_j(y) = \phi_k(x, y), \quad 1 \le k \le n,\ 1 \le i, j \le n - 1,$$

where j, k run lexicographically through (j, i, k) and (k, i, j), respectively. Finally, let $\mathbf{U}^h$ be defined as follows:

$$(4.2) \qquad \mathbf{U}^h = \mathrm{span}\left\{ \begin{pmatrix} \phi_i(y, z) \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \phi_j(x, z) \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ \phi_k(x, y) \end{pmatrix} \right\},$$

where $1 \le i \le (n-1)^2$ (thus, only the first yz-slice is included) and $1 \le j, k \le n(n-1)^2$ (all xz- and xy-slices are included). Note that the number of excluded ϕ_i's is $(n-1)^3$. If the number of intervals in the x-, y-, and z-directions were ℓ, m, and n, respectively, the number excluded would be $(\ell - 1)(m - 1)(n - 1)$ and would be the same if all but one xz- or xy-slice were excluded instead of all but one yz-slice.
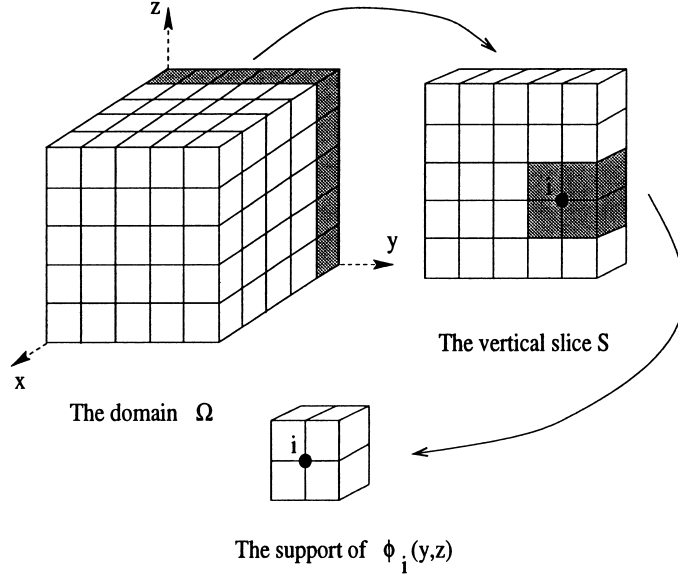
FIG. 4.1. *The support of a typical potential basis function.*

Next, we list some properties of $\mathbf{U}^h$ which follow directly from the definition of the potential space.

Remark 4.1. $\mathbf{U}^h \not\subset H(div; \Omega)$ (because, e.g., $\phi_i(y, z)$ is discontinuous in x), and hence, $\mathbf{U}^h \not\subset H^1(\Omega)^3$.

Remark 4.2. Every $\mathbf{\Phi} \in \mathbf{U}^h$ satisfies $\mathbf{\Phi} \times \mathbf{n} = \mathbf{0}$ on $\partial\Omega$ (because, e.g., $\phi_j(0, z)$ and $\phi_k(0, y)$ are identically zero).

Remark 4.3. $\mathbf{U}^h$ is locally divergence-free, i.e., $\nabla \cdot \mathbf{\Phi} = 0$ on each element $K \in \mathcal{T}^h$ for every $\mathbf{\Phi} \in \mathbf{U}^h$.

Remark 4.4. $\mathbf{U}^h \subset H(curl; \Omega)$, and hence $\mathbf{curl}\, \mathbf{U}^h \subset \mathbf{V}^h$. To see this, consider as a typical case the vector function $(\phi_i(y, z), 0, 0) \in \mathbf{U}^h$, whose curl is $(0, \partial\phi_i/\partial z, -\partial\phi_i/\partial y)$. Because ϕ_i is discontinuous only in the x-direction and no x-derivatives appear in the curl, we have $(\phi_i, 0, 0) \in H(curl; \Omega)$. Further, the y-component of $\mathbf{curl}(\phi_i, 0, 0)$ is $\partial\phi_i/\partial z = \chi_i(x)\psi_j(y)\psi'_k(z)$, which is continuous piecewise linear in y and discontinuous piecewise constant in x and z; similarly, the other components have the correct form to yield $\mathbf{curl}(\phi_i, 0, 0) \in \mathbf{V}^h$.

Since div $\mathbf{curl} \equiv 0$, we have $\mathbf{curl}\, \mathbf{U}^h \subset \mathbf{D}^h$. Counting dimensions,

$$\dim \mathbf{U}^h = (2n + 1)(n - 1)^2 = 2n^3 - 3n^2 + 1.$$

Also, div $\mathbf{V}^h$ consists of those piecewise constants with integral zero over Ω, and hence has dimension $n^3 - 1$, and we obtain

$$\dim \mathbf{D}^h = \dim \mathbf{V}^h - \dim \text{div}\, \mathbf{V}^h = 3(n - 1)n^2 - (n^3 - 1) = 2n^3 - 3n^2 + 1.$$

We claim that the curls of the vectors in (4.2) are linearly independent, so that

$$\dim \mathbf{D}^h = \dim \mathbf{curl}\, \mathbf{U}^h = \dim \mathbf{U}^h = 2n^3 - 3n^2 + 1,$$

which implies that for every divergence-free vector $\mathbf{v} \in \mathbf{D}^h$ there exists a unique

potential vector $\mathbf{\Phi} \in \mathbf{U}^h$ such that

$$\mathbf{v} = \mathbf{curl}\,\mathbf{\Phi}.$$

To prove linear independence, first note that vectors in $\mathbf{V}^h$ can be characterized in terms of normal fluxes across the $3(n-1)n^2$ interior faces between elements. For example, some calculations will show that $\mathbf{curl}\,(\phi_1(y,z),0,0) = \mathbf{curl}\,(\phi^x_{1,1,1}(x,y,z),0,0)$ has y-component 1 on face $(1,3/2,1) = (x_0,x_1) \times \{y_1\} \times (z_0,z_1)$ and -1 on face $(1,3/2,2) = (x_0,x_1) \times \{y_1\} \times (z_1,z_2)$, and z-component 1 on $(1,2,3/2)$ and -1 on $(1,1,3/2)$, where the four fluxes have been scaled to unit magnitude. This is shown in Figure 4.2. We denote this particular curl by $+1(1,3/2,1) - 1(1,3/2,2) + 1(1,2,3/2) - 1(1,1,3/2)$.
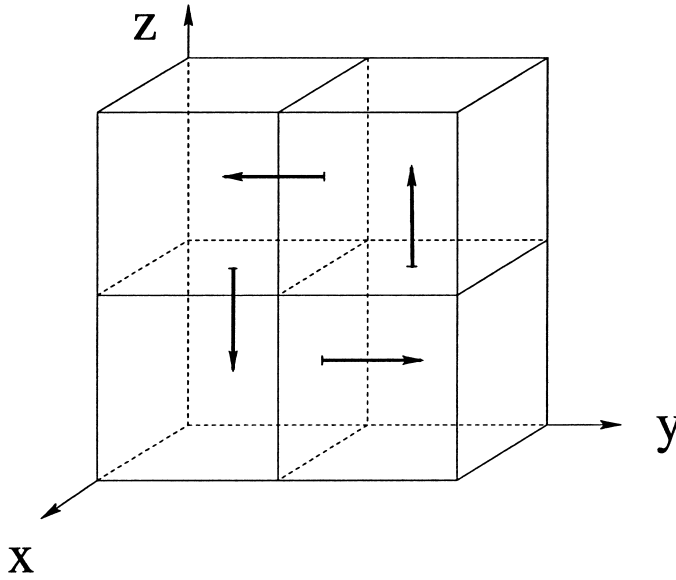

FIG. 4.2. *The curl of a typical potential basis function.*

Now consider $\mathbf{curl}\,(0,\phi_j(x,z),0)$, $1 \le j \le n(n-1)^2$. Put $\phi_j(x,z) = \phi^y_{j,i,k}(x,y,z)$ in lexicographic order, noting that $\mathbf{curl}\,(0,\phi^y_{j,i,k},0) = +1(i+1/2,j,k+1) - 1(i+1/2,j,k) + 1(i,j,k+1/2) - 1(i+1,j,k+1/2)$ and that face $(i+1/2,j,k+1)$ appears for the first time in $\mathbf{curl}\,(0,\phi^y_{j,i,k},0)$. Since each curl introduces a nonzero flux on a new face, the curls of $(0,\phi_j(x,z),0)$ are linearly independent. Next, we have $\mathbf{curl}\,(0,0,\phi_k(x,y)) = \mathbf{curl}\,(0,0,\phi^z_{k,i,j}) = +1(i+1,j+1/2,k) - 1(i,j+1/2,k) + 1(i+1/2,j,k) - 1(i+1/2,j+1,k)$, and face $(i+1,j+1/2,k)$ appears for the first time in $\mathbf{curl}\,(0,0,\phi^z_{k,i,j})$. Thus, the curls of $(0,\phi_j(x,z),0)$ and $(0,0,\phi_k(x,y))$, $1 \le j,k \le n(n-1)^2$, are all linearly independent.

Finally, consider a linear combination

$$\sum_i \alpha_i \mathbf{curl}\,(\phi_i(y,z),0,0) = \sum_{j=1}^{n-1}\sum_{k=1}^{n-1} \alpha_{jk} \mathbf{curl}\,(\phi^x_{1,j,k},0,0)$$

(terms from first yz-slice only). Because the curls of $(\phi_i,0,0)$ are linearly independent by the argument applied above to the curls of $(0,\phi_j,0)$, it suffices to show

that this linear combination is independent of the curls of $(0, \phi_j, 0)$ and $(0, 0, \phi_k)$. We have $\mathbf{curl}\,(\phi^x_{1,j,k}, 0, 0) = +1(1, j + 1/2, k) - 1(1, j + 1/2, k + 1) + 1(1, j + 1, k + 1/2) - 1(1, j, k + 1/2)$. Each of these four terms occurs exactly once in the curls of $(0, \phi_j(x, z), 0)$ and $(0, 0, \phi_k(x, y))$, namely in $-\mathbf{curl}\,(0, 0, \phi^z_{k,1,j})$, $+\mathbf{curl}\,(0, 0, \phi^z_{k+1,1,j})$, $+\mathbf{curl}\,(0, \phi^y_{j+1,1,k}, 0)$, $-\mathbf{curl}\,(0, \phi^y_{j,1,k}, 0)$, respectively. Hence, a dependency relationship for $\mathbf{curl}\,(\phi^x_{1,j,k}, 0, 0)$ in terms of the preceding curls must involve these four curls, and when they are combined we get $\mathbf{curl}\,(\phi^x_{1,j,k}, 0, 0) - \mathbf{curl}\,(\phi^x_{2,j,k}, 0, 0)$. Applying this fact to each term of the linear combination, we have $\sum_{j=1}^{n-1} \sum_{k=1}^{n-1} \alpha_{jk}(\mathbf{curl}\,(\phi^x_{1,j,k}, 0, 0) - \mathbf{curl}\,(\phi^x_{2,j,k}, 0, 0))$. To cancel $\sum_{j,k} \alpha_{jk}(-\mathbf{curl}\,(\phi^x_{2,j,k}, 0, 0))$ with the preceding curls, the forced combination yields $\sum_{j,k} \alpha_{jk}(\mathbf{curl}\,(\phi^x_{2,j,k}, 0, 0) - \mathbf{curl}\,(\phi^x_{3,j,k}, 0, 0))$, and so on until $\sum_{j,k} \alpha_{jk}(-\mathbf{curl}\,(\phi^x_{n,j,k}, 0, 0))$ remains, and it is not possible to cancel it. It follows that no dependency relationship exists, so that the curls of the vectors in (4.2) are indeed linearly independent.

The vector functions in (4.2) constitute one choice of a basis for $\mathbf{U}^h$. As noted above, this choice includes all $2n(n - 1)^2$ vectors of the forms $(0, \phi_j, 0)$ and $(0, 0, \phi_k)$, but only $(n - 1)^2$ vectors of the type $(\phi_i, 0, 0)$ with support contained in one vertical slice S of Ω (say, the shaded one in Figure 4.1).

Remark 4.5. The above-defined basis for $\mathbf{U}^h$ (and hence for $\mathbf{D}^h$) consists of vector functions with minimal possible support. (A moment's reflection shows that a nontrivial divergence-free vector function must be supported on at least four elements, as in the pattern in Figure 4.2.)

Now we need to prove the following Poincaré-type inequality.

LEMMA 4.1. *There exists a constant $C(\Omega) > 0$, independent of the quasi-uniform mesh size h, such that for all $\mathbf{\Phi} \in \mathbf{U}^h$ we have*

$$(4.3) \qquad \|\mathbf{\Phi}\|_{L^2(\Omega)^3} \leq C(\Omega)\,\|\mathbf{curl}\,\mathbf{\Phi}\|_{L^2(\Omega)^3}.$$

(Since the vector potential space $\mathbf{U}^h \not\subset H^1(\Omega)^3$, inequality (4.3) does not follow from the standard Poincaré inequality.)

Proof. Keeping in mind our choice for a basis in $\mathbf{U}^h$, we have

$$\mathbf{\Phi} = (\Phi_x, \Phi_y, \Phi_z)^T,$$

and since Φ_x vanishes outside the vertical slice S we have

$$\|\mathbf{\Phi}\|^2_{L^2(\Omega)^3} = \|\Phi_x\|^2_{L^2(S)} + \|\Phi_y\|^2_{L^2(S)} + \|\Phi_z\|^2_{L^2(S)}$$
$$+ \|\Phi_y\|^2_{L^2(\Omega \backslash S)} + \|\Phi_z\|^2_{L^2(\Omega \backslash S)}.$$

Let us estimate the last term first. Noting that $\Phi_z(x, y, z)$ is a continuous function of x and vanishes for $x = 0$ and $x = 1$, we can write

$$\Phi_z(x, y, z) = \int_1^x \frac{\partial \Phi_z(x', y, z)}{\partial x} dx'.$$

After squaring both sides of the above identity, then using the Cauchy–Schwarz inequality on the right-hand side, and finally integrating both sides over $\Omega \backslash S$, we obtain

$$\|\Phi_z\|_{L^2(\Omega \backslash S)} \leq C(\Omega)\,\|\mathbf{curl}\,\mathbf{\Phi}\|_{L^2(\Omega)^3},$$

where we have used the fact that on $\Omega \backslash S$ we have $\mathbf{curl}\,\mathbf{\Phi} = (*, -\frac{\partial \Phi_z}{\partial x}, \frac{\partial \Phi_y}{\partial x})^T$. Exactly in the same manner we get $\|\Phi_y\|_{L^2(\Omega \backslash S)} \leq C(\Omega)\,\|\mathbf{curl}\,\mathbf{\Phi}\|_{L^2(\Omega)^3}$.

The next step of the proof will be estimating $\|\Phi_z\|_{L^2(S)}$ in terms of $\|\mathbf{curl\,}\Phi\|_{L^2(\Omega)^3}$. Since Φ_z is a piecewise constant function in the z-direction, let us denote by $\Phi_z^k(x, y)$ the restriction of Φ_z to the kth horizontal slice of Ω. Note that Φ_z^k is linear in x within S and vanishes when $x = 0$. Then

$$
\begin{aligned}
\|\Phi_z\|_{L^2(S)}^2 &= \sum_{k=1}^{n} h \int_0^1 \int_0^h \left[\Phi_z^k(x, y)\right]^2 dx\, dy \\
&\leq \sum_{k=1}^{n} h \int_0^1 \int_0^h \left[\Phi_z^k(h, y)\right]^2 dx\, dy \\
&= h^2 \sum_{k=1}^{n} \int_0^1 \left[\Phi_z^k(h, y)\right]^2 dy \leq h^2\, C(\Omega)\, \|\mathbf{curl\,}\Phi\|_{L^2(\Omega)^3}^2,
\end{aligned}
$$

where to obtain the last inequality we have again integrated $\frac{\partial \Phi_z}{\partial x}$ over $\Omega\backslash S$. The term $\|\Phi_y\|_{L^2(S)}$ is estimated in an analogous manner.

Finally, consider the identity on S,

$$
\Phi_x(y, z) = \int_0^z \left[\frac{\partial \Phi_x(y, z')}{\partial z} - \frac{\partial \Phi_z(x, y, z')}{\partial x}\right] dz' + \int_0^z \frac{\partial \Phi_z(x, y, z')}{\partial x} dz'.
$$

Again, after we square both sides, apply the Cauchy–Schwarz inequality on the right-hand side, integrate both sides over S, and note that $\frac{\partial \Phi_x}{\partial z} - \frac{\partial \Phi_z}{\partial x}$ is a component of $\mathbf{curl\,}\Phi$, we get

$$
\|\Phi_x\|_{L^2(S)}^2 \leq C(\Omega) \left\{ \|\mathbf{curl\,}\Phi\|_{L^2(S)^3}^2 + \left\|\frac{\partial \Phi_z}{\partial x}\right\|_{L^2(S)}^2 \right\}.
$$

Now we complete the proof by applying an inverse inequality on the last term and using the estimate for $\|\Phi_z\|_{L^2(S)}^2$ that was obtained earlier. $\qquad\square$

COROLLARY 4.2. *The linear system* (3.3) *to be solved in* $\mathbf{D}^h$ *has a symmetric and positive definite matrix with condition number of order* $O(h^{-2})$.

5. Convergence analysis. In this section, we provide a uniform upper bound for the condition number of the preconditioned operator $\mathbf{P}$ which indicates that the conjugate gradient iteration for problem (3.5) converges uniformly with respect to the mesh size h and the number of subdomains J. We also establish the uniform convergence of the multiplicative domain decomposition proposed in the second step of Algorithm 3.2. These convergence rates do depend on the factor ζ_1 in the minimum overlap $\zeta_1 H$, where H is the coarse-grid mesh size.

Here and henceforth, we shall use C with or without a subscript to denote a generic positive constant independent of the mesh size h and the number of subdomains J. The next lemma plays an essential role in estimating the minimum eigenvalue of the preconditioned operator $\mathbf{P}$.

LEMMA 5.1. *For any* $\mathbf{v} \in \mathbf{D}^h$, *there exists a decomposition of the form*

$$
\mathbf{v} = \mathbf{v}_0 + \mathbf{v}_1 + \cdots + \mathbf{v}_J \quad \text{with} \quad \mathbf{v}_j \in \mathbf{D}_j
$$

and

$$
(5.1) \qquad\qquad \sum_{j=0}^{J} a(\mathbf{v}_j, \mathbf{v}_j) \leq C\, a(\mathbf{v}, \mathbf{v}),
$$

where the positive constant C is independent of the mesh size h and the number of subdomains J (but depends on the factor ζ_1 in the minimum overlap $\zeta_1 H$).

Proof. For any $\mathbf{v} \in \mathbf{D}^h$, there exists a vector potential (cf. [7]) $\boldsymbol{\Phi} \in H^1(\Omega)^3$ such that $\mathbf{v} = \mathbf{curl}\,\boldsymbol{\Phi}$, $\boldsymbol{\Phi} \times \mathbf{n} = \mathbf{0}$ on $\partial\Omega$, and

$$(5.2) \quad \|\boldsymbol{\Phi}\|_{L^2(\Omega)^3} \le C\,\|\mathbf{curl}\,\boldsymbol{\Phi}\|_{L^2(\Omega)^3} \quad \text{and} \quad \|\nabla\boldsymbol{\Phi}\|_{L^2(\Omega)^3} \le C\,\|\mathbf{curl}\,\boldsymbol{\Phi}\|_{L^2(\Omega)^3}.$$

Let $\mathbf{U}^H$, associated with the coarse triangulation $\mathcal{T}^H$, be defined similarly as in the previous section and $\mathbf{Q}^H$ be the standard L^2 projection operator onto $\mathbf{U}^H$. Let $\boldsymbol{\Psi} = \boldsymbol{\Phi} - \mathbf{Q}^H\boldsymbol{\Phi}$; then it is easy to check (see [2]) that

$$(5.3) \quad \|\boldsymbol{\Psi}\|_{L^2(\Omega)^3} \le C\,H\|\nabla\boldsymbol{\Phi}\|_{L^2(\Omega)^3} \quad \text{and} \quad \|\mathbf{curl}\,(\mathbf{Q}^H\boldsymbol{\Phi})\|_{L^2(\Omega)^3} \le C\,\|\nabla\boldsymbol{\Phi}\|_{L^2(\Omega)^3}.$$

Define $\mathbf{v}_0 = \mathbf{curl}\,(\mathbf{Q}^H\boldsymbol{\Phi})$; then $\mathbf{v}_0 \in \mathbf{D}_0$. By using inequalities (5.3) and (5.2), we have that

$$\|\mathbf{v}_0\|_{L^2(\Omega)^3} = \|\mathbf{curl}\,(\mathbf{Q}^H\boldsymbol{\Phi})\|_{L^2(\Omega)^3} \le C\,\|\nabla\boldsymbol{\Phi}\|_{L^2(\Omega)^3}$$

$$(5.4) \qquad\qquad \le C\,\|\mathbf{curl}\,\boldsymbol{\Phi}\|_{L^2(\Omega)^3} = C\,\|\mathbf{v}\|_{L^2(\Omega)^3}.$$

Now, let $\theta_j \in C_0^\infty(\Omega_j)$, $j = 1, \dots, J$, be a partition of unity such that

$$(5.5) \qquad\qquad\qquad |\nabla\theta_j| \le C\,\zeta_1^{-1}\,H^{-1},$$

and let

$$\mathbf{v}_j = \boldsymbol{\Pi}_h\mathbf{curl}(\theta_j\boldsymbol{\Psi}) \in \mathbf{D}_j.$$

Note that $\mathbf{v} = \mathbf{curl}\,\boldsymbol{\Phi} = \mathbf{v}_0 + \mathbf{curl}\,\boldsymbol{\Psi}$ and $\boldsymbol{\Pi}_h\mathbf{v} = \mathbf{v}$. Then linearity of $\boldsymbol{\Pi}_h$ and $\mathbf{curl}$ imply that $\mathbf{v}$ has a decomposition of the form

$$\mathbf{v} = \mathbf{v}_0 + \mathbf{v}_1 + \cdots + \mathbf{v}_J.$$

Since

$$\mathbf{curl}(\theta_j\boldsymbol{\Psi}) = \boldsymbol{\Psi} \times \nabla\theta_j + \theta_j\mathbf{curl}\,\boldsymbol{\Psi},$$

it follows from inequalities (2.3) and (2.10), the Cauchy–Schwarz inequality, and inequality (5.5) that for $j \in \{1, 2, \dots, J\}$

$$a(\mathbf{v}_j,\,\mathbf{v}_j) \le C\,\|\mathbf{v}_j\|_{L^2(\Omega)^3}^2$$

$$\le C\,\|\mathbf{curl}(\theta_j\boldsymbol{\Psi})\|_{L^2(\Omega)^3}^2$$

$$\le C\int_{\Omega_j} \left(|\nabla\theta_j|^2|\boldsymbol{\Psi}|^2 + \theta_j^2|\mathbf{curl}\,\boldsymbol{\Psi}|^2\right)$$

$$\le C\,\zeta_1^{-2}\,H^{-2}\int_{\Omega_j}|\boldsymbol{\Psi}|^2 + C\int_{\Omega_j}|\mathbf{curl}\,\boldsymbol{\Psi}|^2.$$

By summing the above inequality over j, it follows from the fact that the maximum number of subdomain overlaps is bounded, and from inequalities (5.2), (5.3), (5.4), and (2.3), that

$$\sum_{j=0}^{J} a(\mathbf{v}_j,\,\mathbf{v}_j) \le C\,\|\mathbf{v}_0\|_{L^2(\Omega)^3}^2 + C\,\zeta_1^{-2}\,H^{-2}\int_{\Omega}|\boldsymbol{\Psi}|^2 + C\int_{\Omega}|\mathbf{curl}\,\boldsymbol{\Psi}|^2$$

$$\le C\,\|\mathbf{v}_0\|_{L^2(\Omega)^3}^2 + C\,\zeta_1^{-2}\,\|\mathbf{v}\|_{L^2(\Omega)^3}^2$$

$$\le C\,(1 + \zeta_1^{-2})\,\|\mathbf{v}\|_{L^2(\Omega)^3}^2$$

$$\le C\,a(\mathbf{v},\,\mathbf{v}).$$

This completes the proof of the lemma.   □

Now, the standard argument provides the condition number estimate for $\mathbf{P}$.

THEOREM 5.1. *For any vector* $\mathbf{v} \in \mathbf{D}^h$, *we have*

(5.6) $$C_1 a(\mathbf{v},\, \mathbf{v}) \le a(\mathbf{P}\mathbf{v},\, \mathbf{v}) \le C_2 a(\mathbf{v},\, \mathbf{v}),$$

where the constants C_1 *and* C_2 *are independent of h and J.* *(C_1 contains the factor* $(1 + \zeta_1^{-2})^{-1}$.*)*

Proof. The proof of the right-hand inequality follows from the boundedness of $\mathbf{P}_j$ and the maximum number of subdomain overlaps. The left-hand inequality follows from Lemma 5.1 and Lions' lemma [9].   □

Remark 5.1. In 2-D, a special Poincaré-type lemma (see [5, Lemma 3.1]), together with a bound of $\|\nabla\phi\|$ in terms of $\|\mathbf{curl}\,\phi\|$, allows an argument from Chapter 5 of [14] to prove a condition-number bound involving $1 + \zeta_1^{-1}$ instead of $1 + \zeta_1^{-2}$. It is not clear whether the analogous bound holds in 3-D.

To analyze the convergence of the multiplicative domain decomposition method defined at the second step in Algorithm 3.2, we note that for any $\mathbf{w} \in \mathbf{D}$ we have by the definition of the projection operators $\mathbf{P}_j$

(5.7) $$a(\omega\mathbf{P}_j\mathbf{w},\, \omega\mathbf{P}_j\mathbf{w}) = \omega\, a(\omega\mathbf{P}_j\mathbf{w},\, \mathbf{w}).$$

And Lemma 5.1, the Cauchy–Schwarz inequality, and the bound on the number of subdomain overlaps give that

$$
\begin{aligned}
a(\mathbf{v},\, \mathbf{v}) = \sum_{j=0}^{J} a(\mathbf{v},\, \mathbf{v}_j) &= \sum_{j=0}^{J} a(\mathbf{P}_j\mathbf{v},\, \mathbf{v}_j) \\
&\le \left(\sum_{j=0}^{J} a(\mathbf{P}_j\mathbf{v},\, \mathbf{P}_j\mathbf{v}) \right)^{1/2} \left(\sum_{j=0}^{J} a(\mathbf{v}_j,\, \mathbf{v}_j) \right)^{1/2} \\
&\le C \left(\sum_{j=0}^{J} a(\omega\mathbf{P}_j\mathbf{v},\, \omega\mathbf{P}_j\mathbf{v}) \right)^{1/2} a(\mathbf{v},\, \mathbf{v})^{1/2},
\end{aligned}
$$

which implies that

(5.8) $$a(\mathbf{v},\, \mathbf{v}) \le C \sum_{j=0}^{J} a(\omega\mathbf{P}_j\mathbf{v},\, \omega\mathbf{P}_j\mathbf{v}).$$

Hence, a straightforward consequence of [1] (see also Remark 2.2 in [3]) gives the following result.

THEOREM 5.2. *The iterative method defined at the second step in Algorithm 3.2 is uniformly convergent; i.e.,*

(5.9) $$\|\mathbf{E}\|_a \le \gamma < 1,$$

where γ *is a constant that does not depend on the number of subdomains and the mesh size.* *(γ does depend on ζ_1.)*

6. Numerical results. We briefly summarize some computations [15] that will be presented in more detail elsewhere. The additive preconditioner has been implemented and run on a variety of test problems. Corollary 4.2 was confirmed, as the smallest and largest eigenvalues of the system matrix varied as $O(h)$ and $O(h^{-1})$, respectively. For coarse grids ranging from $H = 1/4$ (thus $4 \times 4 \times 4$) to $H = 1/32$, with fine grids $h = H/4$ and overlaps $\zeta_1 H = h$, the iteration counts needed to reduce the preconditioned residual by 10 orders of magnitude were 31 to 32 for constant k (Poisson's equation), 31 for $k = 10^{-5}$ in $(1/4, 3/4)^3$ and $k = 1$ elsewhere, and 32 to 36 for $k = 10^{-5}$ in randomly-distributed coarse-grid blocks and $k = 1$ elsewhere. These results correspond to norm reductions of 0.47 (31 iterations) to 0.52 (36 iterations) per iteration. When random heterogeneity was combined with random anisotropy (k a diagonal tensor, three random entries of 10^{-5}, 10^{-4}, ..., 10^0 in each coarse-grid block), so that there was an increasing number of random blocks on finer grids, norm reductions were significantly worse (0.79 to 0.91) and worsened on finer grids. The theory of this paper does not address the dependence of iteration counts on jumps in coefficients, but it appears that this dependence is substantial only when heterogeneity and anisotropy are intertwined.

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