Positivity-preserving high order discontinuous Galerkin schemes for compressible Euler equations with source terms[1]

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Abstract

In [16, 17], we constructed uniformly high order accurate discontinuous Galerkin (DG) schemes which preserve positivity of density and pressure for the Euler equations of compressible gas dynamics with the ideal gas equation of state. The technique also applies to high order accurate finite volume schemes. For the Euler equations with various source terms (e.g., gravity and chemical reactions), it is more difficult to design high order schemes which do not produce negative density or pressure. In this paper, we first show that our framework to construct positivity-preserving high order schemes in [16, 17] can also be applied to Euler equations with a general equation of state. Then we discuss an extension to Euler equations with source terms. Numerical tests of the third order Runge-Kutta DG (RKDG) method for Euler equations with different types of source terms are reported.

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1 Introduction

The one dimensional version of the Euler equations with source terms for gas dynamics is given by

$$\mathbf{w}_t + \mathbf{f}(\mathbf{w})_x = \mathbf{s}(\mathbf{w}, x), \quad t \geq 0,\ x \in \mathbb{R}, \tag{1.1}$$

$$\mathbf{w} = \begin{pmatrix} \rho \\ m \\ E \end{pmatrix}, \quad \mathbf{f}(\mathbf{w}) = \begin{pmatrix} m \\ \rho u^2 + p \\ (E+p)u \end{pmatrix} \tag{1.2}$$

with

$$m = \rho u, \quad E = \frac{1}{2}\rho u^2 + \rho e, \tag{1.3}$$

where ρ is the density, u is the velocity, m is the momentum, E is the total energy, e is the internal energy, and the pressure p can be obtained from the equation of state. For instance, for the perfect gas,

$$p = (\gamma - 1)\rho e. \tag{1.4}$$

We consider four different types of the source term $\mathbf{s}(\mathbf{w}, x)$ in this paper. The first one is the axial symmetry, i.e., the three-dimensional axially symmetric flow is governed by the one- or two-dimensional Euler equations with spherical or cylindrical symmetry. The second type is the gravity. The third type is the chemical reaction, for which we use the non-equilibrium model in [14]. The last one that we consider is the radiative cooling in [4].

Physically, the density ρ and the pressure p should both be positive. Therefore we are interested in positivity-preserving high order schemes, which maintain the positivity of density and pressure at time level $n+1$, provided that they are positive at time level n. Failure of preserving positivity of density or pressure may cause blow-ups of the numerical algorithm, for example, for low density problems in computing blast waves, and low pressure problems in computing high Mach number astrophysical jets, see [16] for more details. Most commonly used high order numerical schemes for solving hyperbolic conservation law systems, including, among others, the Runge-Kutta discontinuous Galerkin (RKDG) method with a total variation bounded (TVB) limiter [1, 2], the essentially non-oscillatory (ENO) finite volume and finite difference schemes [5, 13], and the weighted ENO (WENO) finite volume and finite difference schemes [9, 6], do not in general satisfy the positivity property for Euler equations automatically.

Motivated by the approach in [10], we proposed a general framework in [15] to construct a maximum-principle-satisfying high order scheme for scalar conservation laws on rectangular meshes. The framework was extended to positivity-preserving high order schemes for the Euler equations for the perfect gas without the source term in [16]. We have also extended this method to unstructured triangular meshes in [17]. The main result of our previous work is, by adding a simple limiter to a high order DG scheme or a high order finite volume scheme, the numerical solution will satisfy a strict maximum principle for scalar conservation laws or will be positivity-preserving for the Euler equations under a suitable CFL condition, while maintaining uniform high order accuracy. In this paper, we extend this result to high order schemes for the Euler equations with various source terms.

The conclusion of this paper is, by adding a positivity-preserving limiter which will be specified later to a high order accurate finite volume scheme or a discontinuous Galerkin scheme solving one or multi-dimensional Euler equations with a source term, with the time evolution by a SSP Runge-Kutta or multi-step method, we obtain a uniformly high order accurate scheme preserving the positivity in the sense that the density and pressure of the cell averages are always positive if they are positive initially.

The paper is organized as follows: we show a general formulation in Section 2. In Section 3, we discuss the results and show the numerical tests of the third order DG method for different types of source terms case by case. Concluding remarks are given in Section 4.

2 A general formulation for positivity-preserving high order schemes

2.1 The Euler equations without the source term

In this subsection, we will show that the framework in [16] to construct high order schemes for Euler equations without source terms also holds for a general equation of state satisfying the following assumption:

$$\text{if} \quad \rho \geq 0, \quad \text{then} \quad e > 0 \Leftrightarrow p > 0. \tag{2.5}$$

We have the following results.

Lemma 2.1. The set of admissible states $G = \left\{ \mathbf{w} = \begin{pmatrix} \rho \\ m \\ E \end{pmatrix} \middle| \rho > 0 \quad \text{and} \quad p > 0 \right\}$ is a convex set.

Proof: Denote $\widetilde{e} = \rho e$, then the assumption (2.5) implies $G = \left\{ \mathbf{w} = (\rho, m, E)^T \middle| \rho > 0, \widetilde{e} > 0 \right\}$. By (1.3) we have $\widetilde{e} = E - \frac{1}{2}\frac{m^2}{\rho}$, which is a concave function of $\mathbf{w}$. For $\mathbf{w}_1 = (\rho_1, m_1, E_1)^T$ and $\mathbf{w}_2 = (\rho_2, m_2, E_2)^T$, Jensen's inequality implies, for $0 \leq s \leq 1$,

$$\widetilde{e}\left(s\mathbf{w}_1 + (1-s)\mathbf{w}_2\right) \geq s\widetilde{e}\left(\mathbf{w}_1\right) + (1-s)\widetilde{e}\left(\mathbf{w}_2\right), \quad \text{if} \quad \rho_1 \geq 0, \quad \rho_2 \geq 0. \tag{2.6}$$

Thanks to the Jensen's inequality, G is a convex set. ■

Lemma 2.2. Consider the first order Lax-Friedrichs scheme for (1.1) with $\mathbf{s}(\mathbf{w}, x) = \mathbf{0}$,

$$\mathbf{w}_j^{n+1} = \mathbf{w}_j^n - \lambda[\mathbf{h}(\mathbf{w}_j^n, \mathbf{w}_{j+1}^n) - \mathbf{h}(\mathbf{w}_{j-1}^n, \mathbf{w}_j^n)], \tag{2.7}$$

where $\lambda = \Delta t / \Delta x$ and

$$\mathbf{h}(\mathbf{u}, \mathbf{v}) = \frac{1}{2}\left[\mathbf{f}(\mathbf{u}) + \mathbf{f}(\mathbf{v}) - a_0(\mathbf{v} - \mathbf{u})\right], \tag{2.8}$$

where

$$a_0 \geq \max \left\| |u| + \frac{p}{\rho\sqrt{2e}} \right\|_\infty . \tag{2.9}$$

Under the CFL condition $\lambda a_0 \le 1$, the scheme is positivity-preserving, namely $\mathbf{w}_j^{n+1} \in G$ if $\mathbf{w}_j^n \in G$ for all j.

Remark: For the equation of state of the ideal gas (1.4), if we take $a_0 = \max ||(|u| + c)||_\infty$ where $c = \sqrt{\gamma \frac{p}{\rho}}$, then (2.9) is automatically satisfied.

Proof: The scheme can be written as

$$
\begin{aligned}
\mathbf{w}_j^{n+1} &= \mathbf{w}_j^n - \lambda[\mathbf{h}(\mathbf{w}_j^n, \mathbf{w}_{j+1}^n) - \mathbf{h}(\mathbf{w}_{j-1}^n, \mathbf{w}_j^n)] \\
&= (1 - \lambda a_0)\mathbf{w}_j^n + \frac{\lambda a_0}{2}[\mathbf{w}_{j+1}^n - \frac{1}{a_0}\mathbf{f}(\mathbf{w}_{j+1}^n)] + \frac{\lambda a_0}{2}[\mathbf{w}_{j-1}^n + \frac{1}{a_0}\mathbf{f}(\mathbf{w}_{j-1}^n)]
\end{aligned}
$$

Notice that $\mathbf{w}_j^{n+1}$ is a convex combination of the three vectors $\mathbf{w}_j^n$, $\mathbf{w}_{j+1}^n - \frac{1}{a_0}\mathbf{f}(\mathbf{w}_{j+1}^n)$ and $\mathbf{w}_{j-1}^n + \frac{1}{a_0}\mathbf{f}(\mathbf{w}_{j-1}^n)$, we only need to show $\mathbf{w}_{j-1}^n + \frac{1}{a_0}\mathbf{f}(\mathbf{w}_{j-1}^n)$ and $\mathbf{w}_{j+1}^n - \frac{1}{a_0}\mathbf{f}(\mathbf{w}_{j+1}^n)$ are in the set G. It is easy to check that the first components of the both vectors are positive. The only nontrivial part is to check the positivity of the "pressure". For simplicity, we drop the subscripts and superscripts, i.e., we prove $\mathbf{w} \pm \frac{1}{a_0}\mathbf{f}(\mathbf{w}) \in G$ if $\mathbf{w} \in G$. Let $\widetilde{e} = E - \frac{1}{2}\frac{m^2}{\rho}$ and $u = m/\rho$. By a direct calculation, we have

$$
\begin{aligned}
\widetilde{e}\left(\mathbf{w} \pm \frac{1}{a_0}\mathbf{f}(\mathbf{w})\right) &= \widetilde{e}\left[\left((1 \pm \frac{u}{a_0})\rho, (1 \pm \frac{u}{a_0})m \pm \frac{p}{a_0}, (1 \pm \frac{u}{a_0})E \pm \frac{up}{a_0}\right)^T\right] \\
&= (1 \pm \frac{u}{a_0})E \pm \frac{up}{a_0} - \frac{1}{2}\frac{\left[(1 \pm \frac{u}{a_0})m \pm \frac{p}{a_0}\right]^2}{(1 \pm \frac{u}{a_0})\rho} \\
&= \left(1 - \frac{p^2}{2(a_0 \pm u)^2\rho^2 e}\right)\left(1 \pm \frac{u}{a_0}\right)\rho e
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\widetilde{e}\left(\mathbf{w} \pm \frac{1}{a_0}\mathbf{f}(\mathbf{w})\right) > 0 &\iff \frac{p^2}{2(a_0 \pm u)^2\rho^2 e} < 1 \\
&\iff \frac{p^2}{2\rho^2 e} < (a_0 \pm u)^2.
\end{aligned}
$$

So (2.9) implies $\widetilde{e}(\mathbf{w} \pm \frac{1}{a_0}\mathbf{f}(\mathbf{w})) > 0$, thus $\mathbf{w} \pm \frac{1}{a_0}\mathbf{f}(\mathbf{w}) \in G$. ■

We discuss the high order schemes now. Here we only consider the first order Euler forward time discretization; strong stability preserving high order Runge-Kutta [13] and multi-step [12] time discretization will keep the validity of positivity-preserving property since G is convex. A general high order finite volume scheme, or the scheme satisfied by the cell averages of a DG method solving (1.1) with $\mathbf{s}(\mathbf{w}, x) = \mathbf{0}$, has the following form

$$\overline{\mathbf{w}}_j^{n+1} = \overline{\mathbf{w}}_j^n - \lambda \left[\mathbf{h}\left(\mathbf{w}_{j+\frac{1}{2}}^-, \mathbf{w}_{j+\frac{1}{2}}^+\right) - \mathbf{h}\left(\mathbf{w}_{j-\frac{1}{2}}^-, \mathbf{w}_{j-\frac{1}{2}}^+\right)\right], \tag{2.10}$$

where $\mathbf{h}$ is defined in (2.8). $\overline{\mathbf{w}}_j^n$ is the approximation to the cell average of the exact solution $\mathbf{v}(x,t)$ in the cell $I_j = [x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}]$ at time level n, and $\mathbf{w}_{j+\frac{1}{2}}^-$, $\mathbf{w}_{j+\frac{1}{2}}^+$ are the high order approximations of the point values $\mathbf{v}(x_{j+\frac{1}{2}}, t^n)$ within the cells I_j and I_{j+1} respectively. These values are either reconstructed from the cell averages $\overline{\mathbf{w}}_j^n$ in a finite volume method or read directly from the evolved polynomials in a DG method. We assume that there is a polynomial vector $\mathbf{q}_j(x) = (\rho_j(x), m_j(x), E_j(x))^T$ (either reconstructed in a finite volume method or evolved in a DG method) with degree k, where $k \geq 1$, defined on I_j such that $\overline{\mathbf{w}}_j^n$ is the cell average of $\mathbf{q}_j(x)$ on I_j, $\mathbf{w}_{j-\frac{1}{2}}^+ = \mathbf{q}_j(x_{j-\frac{1}{2}})$ and $\mathbf{w}_{j+\frac{1}{2}}^- = \mathbf{q}_j(x_{j+\frac{1}{2}})$.

We need the N-point Legendre Gauss-Lobatto quadrature rule on the interval $I_j = [x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}]$, which is exact for the integral of polynomials of degree up to $2N-3$. We would need to choose N such that $2N-3 \geq k$. Denote these quadrature points on I_j as

$$S_j = \{x_{j-\frac{1}{2}} = \widehat{x}_j^1, \widehat{x}_j^2, \cdots, \widehat{x}_j^{N-1}, \widehat{x}_j^N = x_{j+\frac{1}{2}}\}. \tag{2.11}$$

Let $\widehat{w}_\alpha$ be the Legendre Gauss-Lobatto quadrature weights for the interval $[-\frac{1}{2}, \frac{1}{2}]$ such that $\sum\limits_{\alpha=1}^{N} \widehat{w}_\alpha = 1$, with $2N-3 \geq k$. Following the same lines as in [16], we can prove

Theorem 2.3. For a finite volume scheme or the scheme satisfied by the cell averages of a DG method (2.10) with the Lax-Friedrichs flux (2.8) and (2.9) solving the Euler equations with an equation of state satisfying (2.5), if $\mathbf{q}_j(\widehat{x}_j^\alpha) \in G$ for all j and α, then $\overline{\mathbf{w}}_j^{n+1} \in G$ under the CFL condition $\lambda a_0 \leq \widehat{w}_1$. ■

A simple linear scaling limiter can enforce $\mathbf{q}_j(\widehat{x}_j^\alpha) \in G$ without destroying the accuracy if $\overline{\mathbf{w}}_j^n \in G$, see [16] for details. We refer to the RKDG scheme with this limiter as the positivity-preserving DG method.

Notice that even though we discuss only the Lax-Friedrichs flux, any other positivity preserving first order numerical fluxes, such as the Godunov flux, will also work for our high order positivity preserving schemes.

2.2 The Euler equations with a source term

The notations are the same as in the previous subsection. A general high order finite volume scheme, or the scheme satisfied by the cell averages of a DG method solving (1.1), has the following form

$$\overline{\mathbf{w}}_j^{n+1} = \overline{\mathbf{w}}_j^n - \lambda \left[\mathbf{h}\left(\mathbf{w}^-_{j+\frac{1}{2}}, \mathbf{w}^+_{j+\frac{1}{2}}\right) - \mathbf{h}\left(\mathbf{w}^-_{j-\frac{1}{2}}, \mathbf{w}^+_{j-\frac{1}{2}}\right)\right] + \lambda \int_{I_j} \mathbf{s}(\mathbf{q}_j(x), x)dx,$$

where the integral can be approximated by quadratures with sufficient accuracy. Let us assume that we use a Gauss quadrature with L points, which is exact for single variable polynomials of degree $2L-1 \geq k$. We assume $\{x_j^\beta : \beta = 1, \cdots, L\}$ denote the Gauss quadrature points on the interval I_j and w_β be the quadrature weights for the interval $[-\frac{1}{2}, \frac{1}{2}]$ such that $\sum\limits_{\beta=1}^{L} w_\beta = 1$. Replace the integral by the Gauss quadrature, the scheme now becomes

$$\overline{\mathbf{w}}_j^{n+1} = \overline{\mathbf{w}}_j^n - \lambda \left[\mathbf{h}\left(\mathbf{w}^-_{j+\frac{1}{2}}, \mathbf{w}^+_{j+\frac{1}{2}}\right) - \mathbf{h}\left(\mathbf{w}^-_{j-\frac{1}{2}}, \mathbf{w}^+_{j-\frac{1}{2}}\right)\right] + \Delta t \sum_{\beta=1}^{L} w_\beta \mathbf{s}(\mathbf{q}_j(x_j^\beta), x_j^\beta). \tag{2.12}$$

The exactness of the quadrature rule for polynomials of degree k implies

$$\overline{\mathbf{w}}_j^n = \frac{1}{\Delta x}\int_{I_j} \mathbf{q}_j(x)dx = \sum_{\beta=1}^{L} w_\beta \mathbf{q}_j(x_j^\beta). \tag{2.13}$$

Plug (2.13) in, then (2.12) becomes

$$\begin{aligned}
\overline{\mathbf{w}}_j^{n+1} &= \frac{1}{2}\left(\overline{\mathbf{w}}_j^n - 2\lambda \left[\mathbf{h}\left(\mathbf{w}^-_{j+\frac{1}{2}}, \mathbf{w}^+_{j+\frac{1}{2}}\right) - \mathbf{h}\left(\mathbf{w}^-_{j-\frac{1}{2}}, \mathbf{w}^+_{j-\frac{1}{2}}\right)\right]\right) \\
&\quad + \frac{1}{2}\left(\overline{\mathbf{w}}_j^n + 2\Delta t \sum_{\beta=1}^{L} w_\beta \mathbf{s}(\mathbf{q}_j(x_j^\beta), x_j^\beta)\right) \\
&= \frac{1}{2}\mathbf{H} + \frac{1}{2}\sum_{\beta=1}^{L} w_\beta \left(\mathbf{q}_j(x_j^\beta) + 2\Delta t \mathbf{s}(\mathbf{q}_j(x_j^\beta), x_j^\beta)\right)
\end{aligned}$$

where

$$\mathbf{H} = \overline{\mathbf{w}}_j^n - 2\lambda \left[\mathbf{h}\left(\mathbf{w}^-_{j+\frac{1}{2}}, \mathbf{w}^+_{j+\frac{1}{2}}\right) - \mathbf{h}\left(\mathbf{w}^-_{j-\frac{1}{2}}, \mathbf{w}^+_{j-\frac{1}{2}}\right)\right].$$

Thus $\overline{\mathbf{w}}_j^{n+1}$ in (2.12) is a convex combination of $\mathbf{H}$ and $\mathbf{q}_j(x_j^\beta) + \Delta t \mathbf{s}(\mathbf{q}_j(x_j^\beta), x_j^\beta)$ ($\beta = 1, \cdots, L$). Assuming $\mathbf{q}_j(\widehat{x}_j^\alpha) \in G$ for all j and α, then $\mathbf{H} \in G$ under the CFL condition $\lambda a_0 \le \frac{1}{2}\widehat{w}_1$ by the Theorem 2.3. To ensure $\overline{\mathbf{w}}_j^{n+1} \in G$, we only need to consider the sufficient condition for $\mathbf{q}_j(x_j^\beta) + 2\Delta t \mathbf{s}(\mathbf{q}_j(x_j^\beta), x_j^\beta) \in G$.

Suppose the following is true, for any $\mathbf{w} \in G$, there exists a time step restriction

$$\Delta t \le A_{\mathbf{s}}(\mathbf{w}, x) \tag{2.14}$$

such that $\mathbf{w} + 2\Delta t \mathbf{s}(\mathbf{w}, x) \in G$. Here $A_{\mathbf{s}}(\mathbf{w}, x)$ is a function of $\mathbf{w}$ and x, which is positive for positive density and pressure. Its explicit form depends on the specific source term $\mathbf{s}$ in (1.1). Then we have the following result.

Theorem 2.4. Consider a finite volume scheme or the scheme satisfied by the cell averages of a DG method (2.12) with the Lax-Friedrichs flux (2.8) and (2.9) solving the Euler equations with a source term (1.1). Assume the equation of state satisfies (2.5). If $\mathbf{q}_j(\widehat{x}_j^\alpha), \mathbf{q}_j(x_j^\beta) \in G$ for all α, β and j, then $\overline{\mathbf{w}}_j^{n+1} \in G$ under the CFL conditions $\lambda a_0 \le \frac{1}{2}\widehat{w}_1$ and

$$\Delta t \le \min_{\beta, j} A_{\mathbf{s}}(\mathbf{q}_j(x_j^\beta), x_j^\beta). \tag{2.15}$$

■

Remark: We only showed how to construct the one-dimensional schemes. For two-dimensional positivity-preserving high order schemes solving the Euler equations with a source term, it is straightforward to follow the ideas of Theorem 2.4 and [16] on rectangular meshes or [17] on triangular meshes.

The limiter in [16] can enforce $\mathbf{q}_j(\widehat{x}_j^\alpha), \mathbf{q}_j(x_j^\beta) \in G$ without destroying high order accuracy if $\overline{\mathbf{w}}_j^n \in G$.

2.3 Limiter and implementation for the DG method

We only discuss the one-dimensional algorithm. Two-dimensional algorithms on rectangular and triangular meshes are straightforward by following [16, 17].

At time level n, assuming the DG polynomial in cell I_j is $\mathbf{q}_j(x) = (\rho_j(x), m_j(x), E_j(x))^T$ with degree k , and the cell average of $\mathbf{q}_j(x)$ is $\overline{\mathbf{w}}_j^n = \left(\overline{\rho}_j^n, \overline{m}_j^n, \overline{E}_j^n\right)^T \in G$, then the algorithm flowchart of our algorithm for the Euler forward is

- Set up a small number $\varepsilon = \min_j\{10^{-13}, \overline{\rho}_j^n, p(\overline{\mathbf{w}}_j^n)\}$.

- In each cell, modify the density first: evaluate $\rho_{\min} = \min_{\alpha,\beta}\{\rho_j(\widehat{x}_j^\alpha), \rho_j(x_j^\beta)\}$ and get $\widehat{\rho}_j(x)$ by
$$\widehat{\rho}_j(x) = \theta_1(\rho_j(x) - \overline{\rho}_j^n) + \overline{\rho}_j^n, \quad \theta_1 = \min\left\{\frac{\overline{\rho}_j^n - \varepsilon}{\overline{\rho}_j^n - \rho_{\min}}, 1\right\}.$$
Set $\widehat{\mathbf{q}}_j(x) = (\widehat{\rho}_j(x), m_j(x), E_j(x))^T$.

- Define
$$Q_j = \{\widehat{\mathbf{q}}_j(\widehat{x}_j^\alpha), \widehat{\mathbf{q}}_j(x_j^\beta) : \alpha = 1, \cdots, N, \quad \beta = 1, \cdots, L\}.$$
Then modify the pressure: for each $\mathbf{q} \in Q_j$, if $p(\mathbf{q}) < \varepsilon$, then solve the following equation for $s_\mathbf{q}$,
$$p\left[(1 - s_\mathbf{q})\overline{\mathbf{w}}_j^n + s_\mathbf{q}\mathbf{q}\right] = \varepsilon.$$
If $p(\mathbf{q}) \geq \varepsilon$, then set $s_\mathbf{q} = 1$. Calculate
$$\widetilde{\mathbf{q}}_j(x) = \theta_2\left(\widehat{\mathbf{q}}_j(x) - \overline{\mathbf{w}}_j^n\right) + \overline{\mathbf{w}}_j^n, \quad \theta_2 = \min_{\mathbf{q}\in Q_j} s_\mathbf{q}.$$

- Use $\widetilde{\mathbf{q}}_j(x)$ instead of $\mathbf{q}_j(x)$ in the DG scheme with Euler forward in time under the CFL condition $\lambda a_0 \leq \frac{1}{2}\widehat{w}_1$ and (2.15). ■

For SSP high order time discretizations, we need to use the limiter in each stage for a Runge-Kutta method or in each step for a multistep method. See [16] for more details about the limiter. We point out that (2.15) could be very restrictive if we enforce it every time step. To be efficient, we could implement (2.15) only when a preliminary calculation to the next time step produces negative density or pressure.

The implementation for a finite volume method is similar, but it will be a little bit more complicated for the WENO procedure since there are only reconstructed nodal values but no polynomials in each cell after the WENO reconstruction. One way to implement the limiter is to construct polynomials using the nodal values and cell averages, see [15] for details. We are also exploring other, simpler ways to implement this positivity preserving limiter for WENO finite volume schemes. These implementation details and numerical tests will be reported elsewhere.

3 Examples and numerical tests

3.1 The Euler equations with axial symmetry

In this subsection, we only consider the equation of state for the thermally ideal gases. The conservative form of the Euler equations governing two-dimensional circular symmetric or three-dimensional spherical symmetric flow of a compressible inviscid fluid with the equation of state (1.4) can be written as:

$$(B(r)\mathbf{w})_t + (B(r)\mathbf{f}(\mathbf{w}))_r = (0, B'(r)p, 0)^T \tag{3.16}$$

where $\mathbf{w}$ and $\mathbf{f}(\mathbf{w})$ are defined in (1.2) with $B(r) = r$ for two-dimensional circular symmetric flow and $B(r) = r^2$ for three-dimensional spherical symmetric flow.

Denote $\mathbf{W} = (\widehat{\rho}, \widehat{m}, \widehat{E})^T = B(r)\mathbf{w}$, and $\widehat{p} = (\gamma-1)(\widehat{E} - \frac{1}{2}\frac{\widehat{m}^2}{\widehat{\rho}})$. Let d denote the dimension, i.e., $d = 2$ or $d = 3$. Then $B(r)\mathbf{f}(\mathbf{w}) = \mathbf{f}(\mathbf{W})$ and the source can be written as $B'(r)p = (d-1)\frac{\widehat{p}}{r}$. (3.16) becomes $\mathbf{W}_t + \mathbf{f}(\mathbf{W})_r = \left(0, (d-1)\frac{\widehat{p}}{r}, 0\right)^T$.

For simplicity, we abuse the notation by setting $r = x$ and replacing $\mathbf{W}$ by $\mathbf{w}$, then we can consider

$$\mathbf{w}_t + \mathbf{f}(\mathbf{w})_x = \left(0, (d-1)\frac{p}{x}, 0\right)^T, \quad x \geq 0. \tag{3.17}$$

Lemma 3.1. The condition (2.14) for (3.17) is

$$\Delta t < -\frac{\gamma x u}{2(d-1)c^2} + \frac{x}{2c}\sqrt{\frac{\gamma^2 u^2}{(d-1)^2 c^2} + \frac{2\gamma}{(d-1)^2(\gamma-1)}}. \tag{3.18}$$

Proof: Assume $\mathbf{w} \in G$. To ensure $\mathbf{w} + 2\Delta t\mathbf{s}(\mathbf{w}, x) = (\rho, m + 2(d-1)\Delta t\frac{p}{x}, E) \in G$, it suffices to ensure the "pressure" $p(\mathbf{w} + 2\Delta t\mathbf{s}(\mathbf{w}, x)) > 0$. Notice that $p = (\gamma-1)(E - \frac{1}{2}\frac{m^2}{\rho})$ and $c^2 = \gamma\frac{p}{\rho}$, we get

$$\begin{aligned} p(\mathbf{w} + 2\Delta t\mathbf{s}(\mathbf{w}, x)) &= (\gamma-1)(E - \frac{1}{2}\frac{\left(m + 2(d-1)\Delta t\frac{p}{x}\right)^2}{\rho}) \\ &= \left[\frac{1}{\gamma-1} - 2(d-1)\frac{u}{x}\Delta t - 2(d-1)^2\frac{c^2}{\gamma x^2}\Delta t^2\right] p. \end{aligned}$$

So we only need $\frac{1}{\gamma-1} - 2(d-1)\frac{u}{x}\Delta t - 2(d-1)^2\frac{c^2}{\gamma x^2}\Delta t^2 > 0$. Solving the quadratic equation gives us the result. ■

Remark: We can easily verify that the right hand side of the time step restriction (3.18) is positive provided that $c > 0$ and $x > 0$. By (2.15), this condition has to hold for all the Gauss quadrature points of each interval $x = x_j^\beta$. Since the smallest of such points is $x_1^1 = O(\Delta x)$ away from $x = 0$, the time step restriction (3.18) is similar to the usual CFL condition $\Delta t < C\Delta x$.

Example 3.1. The Sedov point-blast wave is a typical low density and low pressure problem involving shocks and the solution is circular or spherical symmetric. The exact solution formula can be found in [11, 7]. We test the third order RKDG method with TVB limiter and the positivity-preserving limiter [16] for three blast waves. The high order RKDG scheme with TVB limiter without the positivity preserving limiter will blow up due to the presence of negative density or pressure for these tests.

The first one is two-dimensional circular symmetric flow, i.e., we solve the equations (3.17) with $d = 2$. For the initial condition, the density is 1, velocity is zero, total energy is 10^{-12} everywhere except that the energy in the first cell is the constant $\frac{E_0}{\Delta x}$ with $E_0 = 0.979264$ (emulating a δ-function at the center). $\gamma = 1.4$. The computational results at $t = 1$ are shown in Figure 3.1. We can see the solution is captured very well.

To study how the numerical solutions are affected by the geometric projection (from the Cartesian to the radical coordinates) around the shock front, we compare the computational result of the positivity-preserving RKDG method solving (3.17) in this paper with the result of the positivity-preserving RKDG method solving the two-dimensional Euler equations in Cartesian coordinates in [16] on the same mesh size. See Figure 3.2. We can observe that the shock is better resolved for the result in Cartesian coordinates, however, the computational cost is much larger since it is a two-dimensional computation.

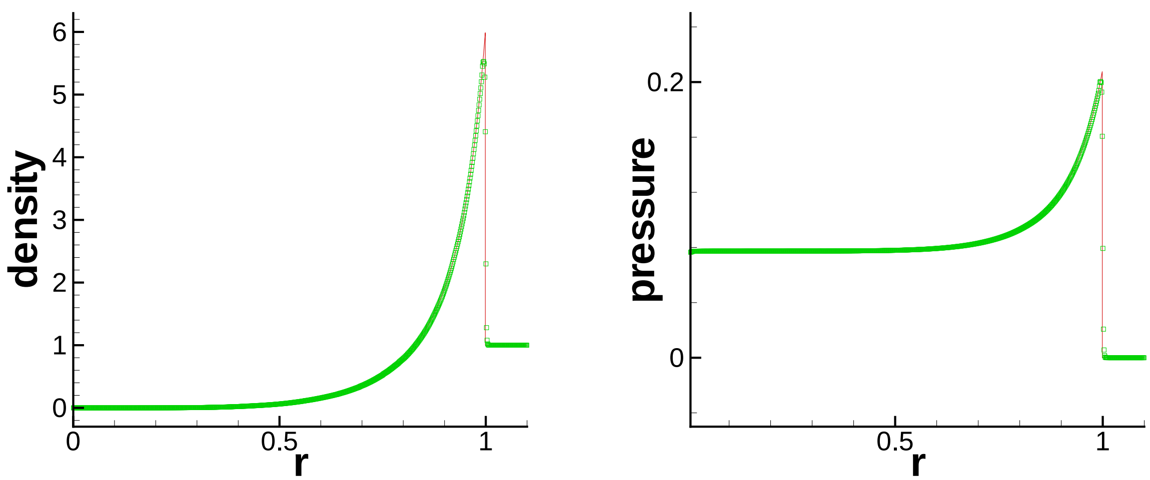


Figure 3.1: Example 3.1: Circular symmetric flow, 2D Sedov blast. The third order positivity-preserving RKDG scheme with TVB limiter. The solid line is the exact solution. Symbols are numerical solutions. $\Delta x = \frac{1.1}{800}$.

Example 3.2. The second test case for the Sedov point-blast wave example is the three-dimensional spherical symmetric flow, i.e., we solve the equations (3.17) with $d = 3$. The initial conditions are the same as in the previous example. The computational results at $t = 1$ are shown in Figure 3.3. We again observe a well captured solution.

Example 3.3. The last test case for the Sedov point-blast wave example is the three-dimensional cylindrical symmetric flow. The governing equations are

$$(r\mathbf{w})_t + (r\mathbf{f}(\mathbf{w}))_r + (r\mathbf{g}(\mathbf{w}))_z = \mathbf{s}(\mathbf{w})$$

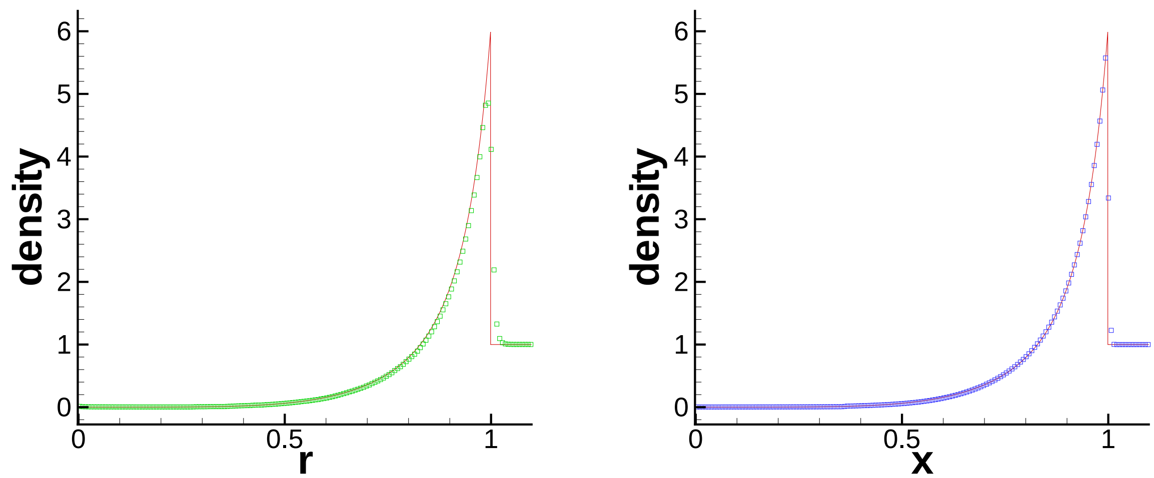


Figure 3.2: Example 3.1: 2D Sedov blast. The left is the result of the third order positivity-preserving RKDG method solving (3.17) with the mesh size $\Delta r = \frac{1.1}{160}$; the right is the projection along the line $y = 0$ of the result computed by the third order positivity-preserving RKDG method solving the two-dimensional Euler equations in Cartesian coordinates [16] on a 160×160 mesh over the square $[0, 1.1] \times [0, 1.1]$.

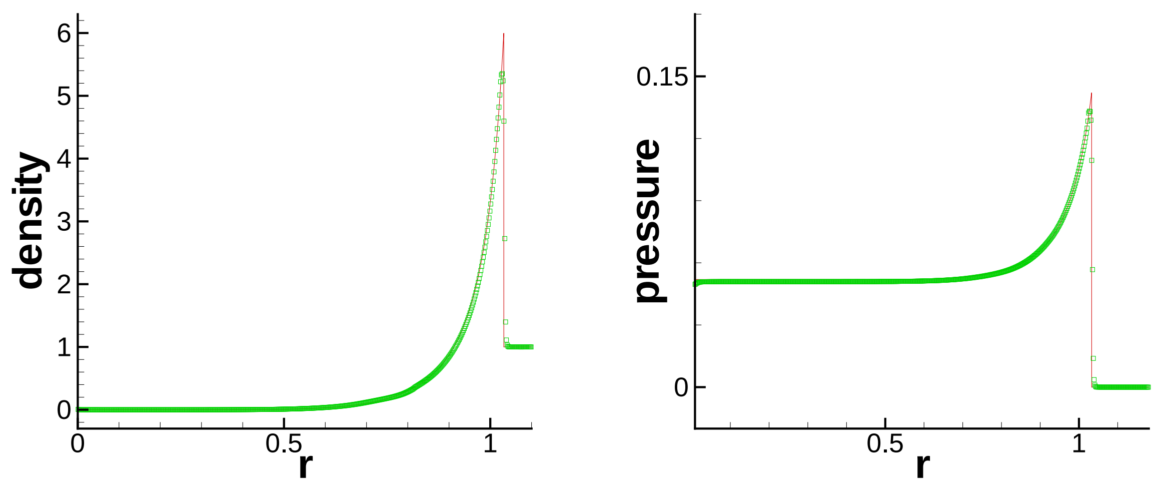


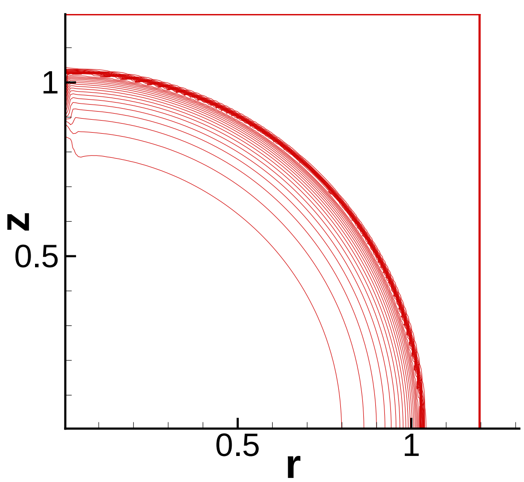
Figure 3.3: Example 3.2: Spherical symmetric flow, 3D Sedov blast. The third order positivity-preserving RKDG scheme with TVB limiter. The solid line is the exact solution. Symbols are numerical solutions. $\Delta x = \frac{1.1}{600}$.

$$\mathbf{w} = \begin{pmatrix} \rho \\ m \\ n \\ E \end{pmatrix}, \quad \mathbf{f}(\mathbf{w}) = \begin{pmatrix} m \\ \rho u^2 + p \\ \rho uv \\ (E+p)u \end{pmatrix}, \quad \mathbf{g}(\mathbf{w}) = \begin{pmatrix} n \\ \rho uv \\ \rho v^2 + p \\ (E+p)v \end{pmatrix}, \quad \mathbf{s}(\mathbf{w}) = \begin{pmatrix} 0 \\ p \\ 0 \\ 0 \end{pmatrix}$$

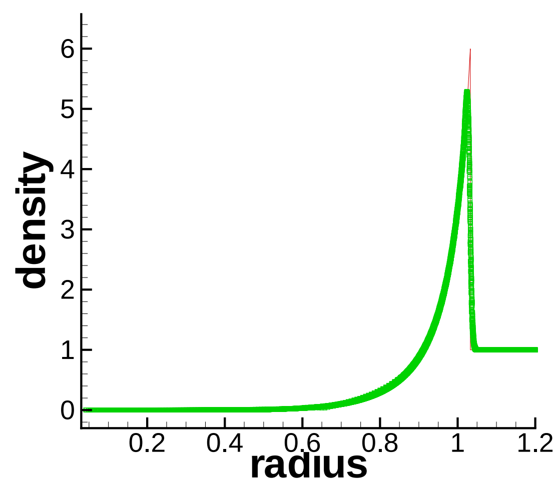
with

$$m = \rho u, \quad n = \rho v, \quad E = \frac{1}{2}\rho u^2 + \frac{1}{2}\rho v^2 + \rho e, \quad p = (\gamma - 1)\rho e.$$

The initial energy E_0 is still taken as 0.979264 and the computational domain is $[0, 1.2] \times [0, 1.2]$, see [16] for the boundary conditions. The computational results at $t = 1$ are shown in Figure 3.4.



(a) 20 equally spaced contour lines from 0 to 5.5



(b) Projection to radical coordinates. The solid line is the exact solution. Symbols are numerical solutions.

Figure 3.4: Example 3.3: Cylindrical symmetric flow, 3D Sedov blast. The third order positivity-preserving RKDG scheme with TVB limiter on a 160×160 mesh.

3.2 The Euler equations with gravity

The two-dimensional Euler equations with standard gravity in the y-direction for the thermally ideal gases take the following form:

$$\begin{pmatrix} \rho \\ m \\ n \\ E \end{pmatrix}_t + \begin{pmatrix} m \\ \rho u^2 + p \\ \rho u v \\ (E+p)u \end{pmatrix}_x + \begin{pmatrix} n \\ \rho u v \\ \rho v^2 + p \\ (E+p)v \end{pmatrix}_y = \begin{pmatrix} 0 \\ 0 \\ -\rho g \\ -\rho v g \end{pmatrix}$$

with

$$m = \rho u, \quad n = \rho v, \quad E = \frac{1}{2}\rho u^2 + \frac{1}{2}\rho v^2 + \rho e, \quad p = (\gamma - 1)\rho e.$$

By the same calculation as in Lemma 3.1, we get (2.14) for the gravity source as

$$\Delta t \leq \frac{1}{\sqrt{2\gamma(\gamma-1)}}\frac{c}{g}, \quad c = \sqrt{\gamma \frac{p}{\rho}}. \tag{3.19}$$

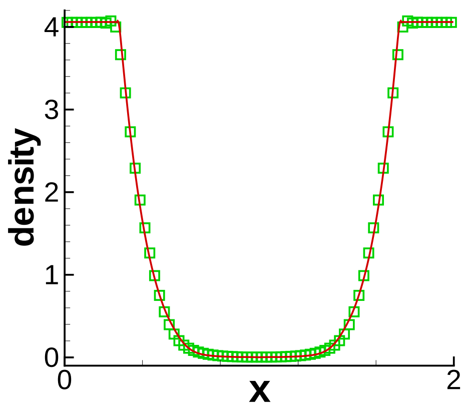
Clearly, the right hand side of the time step restriction (3.19) is positive provided $c > 0$.

Example 3.4. We test a two-dimensional Riemann problem. The domain is $[0, 2] \times [0, 2]$. Initially $(\rho, u, p) = (7, -1, 0.2)$ if $x \leq 1$; $(\rho, u, p) = (7, 1, 0.2)$ if $x \geq 1$, and $v = 0$ everywhere. The boundary conditions for the top and the bottom are reflection, and the boundary conditions for the left and the right are outflow. Without the gravity, i.e., $g = 0$, the exact solution contains two rarefaction waves with vacuum emerging, see [8, 16]. Here we set $g = 1$ to test the robustness of the positivity-preserving DG method since there are both low pressure and low density in the solution. TVB limiter is not needed because there are no shocks. The numerical results at $t = 0.6$ are shown in Figure 3.5, demonstrating very clean and well resolved solutions. The RKDG scheme with only the TVB limiter will blow up for any g.

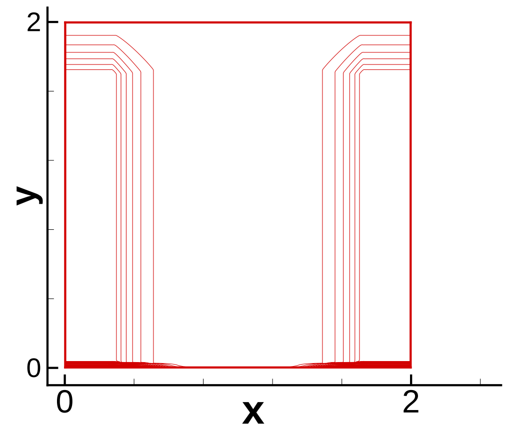
3.3 The Euler equations with non-equilibrium chemistry

We consider the three species model with a more general equation of state in [14]. The model involves three species, O_2, O and N_2 ($\rho_1 = \rho_O, \rho_2 = \rho_{O_2}$ and $\rho_3 = \rho_{N_2}$) with the reaction:

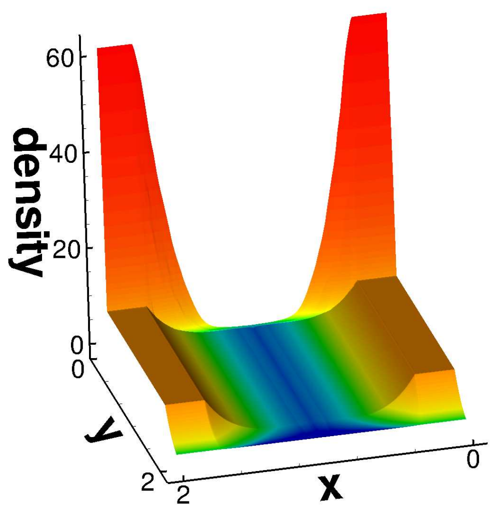
$$O_2 + N_2 \rightleftharpoons O + O + N_2.$$



(a) Cut along the line $y = 1.7875$.



(b) 60 equally spaced contour lines from 0 to 60.



(c) Surface

Figure 3.5: Example 3.4: The third order positivity-preserving RKDG scheme. The solid line in (a), the contour lines in (b) and the surface in (c) are the numerical solution of the 400×400 mesh. The symbols in (a) are the one of the 80×80 mesh.

The governing equations are

$$\begin{pmatrix} \rho_1 \\ \rho_2 \\ \rho_3 \\ \rho u \\ E \end{pmatrix}_t + \begin{pmatrix} \rho_1 u \\ \rho_2 u \\ \rho_3 u \\ \rho u^2 + p \\ (E+p)u \end{pmatrix}_x = \begin{pmatrix} 2M_1\omega \\ -M_2\omega \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

and

$$\rho = \sum_{s=1}^{3} \rho_s, \quad p = RT\sum_{s=1}^{3} \frac{\rho_s}{M_s}, \quad E = \sum_{s=1}^{3} \rho_s e_s(T) + \rho_1 h_1^0 + \frac{1}{2}\rho u^2$$

where the enthalpy h_1^0 is a constant, R is the universal gas constant, M_s is the molar mass of species s, and the internal energy $e_s(T) = 3R/2M_s$ and $5R/2M_s$ for monoatomic and diatomic species respectively. The rate of the chemical reaction is given by

$$\omega = \left(k_f(T)\frac{\rho_2}{M_2} - k_b(T)\left(\frac{\rho_1}{M_1}\right)^2 \right) \sum_{s=1}^{3} \frac{\rho_s}{M_s}, \quad k_f = CT^{-2}e^{-E/T},$$

$$k_b = k_f / \exp\left(b_1 + b_2 \log z + b_3 z + b_4 z^2 + b_5 z^3\right), \quad z = 10000/T$$

where b_i, C and E are constants which can be found in [14, 3].

The eigenvalues of the Jacobian $\mathbf{f}'(\mathbf{w})$ are $(u, u, u, u+c, u-c)$ where $c = \sqrt{\gamma\frac{p}{\rho}}$ with $\gamma = 1 + \frac{p}{T\sum_{s=1}^{3}\rho_s e_s'(T)}$. So if we take $a_0 = ||(|u|+c)||_\infty$ in the Lax-Friedrichs flux (2.8), then all the results in Section 2 will hold. The condition (2.14) for this source is

$$\Delta t < \begin{cases} \frac{\rho_2}{2M_2\omega}, & \text{if} \quad \omega > 0 \\ -\frac{\rho_1}{4M_1\omega}, & \text{if} \quad \omega < 0 \end{cases}. \tag{3.20}$$

Clearly, the right hand side of the time step restriction (3.20) is positive.

Example 3.5. This example is a shock tube problem for the reactive flows with high pressure on the left and low pressure on the right initially in the chemical equilibrium ($\omega = 0$). The initial conditions are:

$$(p_L, T_L) = (1000N/m^2, 8000K), \quad (p_R, T_R) = (1N/m^2, 8000K)$$

with zero velocity everywhere and the densities satisfying

$$\frac{\rho_O}{2M_O} + \frac{\rho_{O_2}}{M_{O_2}} = \frac{21}{79}\frac{\rho_{N_2}}{M_{N_2}},$$

where $M_O = 0.016, M_{O_2} = 0.032$ and $M_{N_2} = 0.028$. The initial densities of O, O_2 and N_2 are $5.251896311257204 \times 10^{-5}$, $3.748071704863518 \times 10^{-5}$ and $2.962489471973072 \times 10^{-4}$ on the left, and $8.341661837019181 \times 10^{-8}$, $9.455418692098664 \times 10^{-11}$ and $2.748909430004963 \times 10^{-7}$ on the right.

Low densities and low pressure will emerge in the solution, which may cause blow-ups for the high order schemes. Our numerical solutions of the positivity-preserving RKDG scheme with the TVB limiter at $t = 0.0001$ are shown in Figure 3.6. We obtain clean and grid converged solutions for this test case.

3.4 High Mach number astrophysical jets with radiative cooling

To simulate the well-collimated supersonic outflow from a central compact object in astrophysical context, namely, the gas flows and shock wave patterns which are revealed by the Hubble Space Telescope images, one can implement theoretical models in a gas dynamics simulator. We consider the two-dimensional model with radiative cooling in [4], which is governed by

$$\begin{pmatrix} \rho \\ m \\ n \\ E \end{pmatrix}_t + \begin{pmatrix} m \\ \rho u^2 + p \\ \rho u v \\ (E+p)u \end{pmatrix}_x + \begin{pmatrix} n \\ \rho u v \\ \rho v^2 + p \\ (E+p)v \end{pmatrix}_y = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \Lambda(T) \end{pmatrix}$$

with

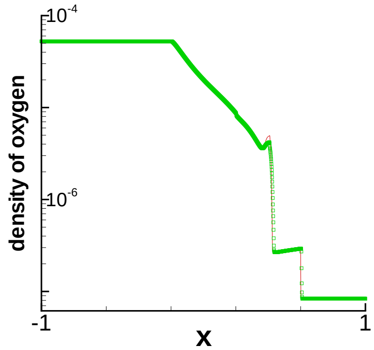
$$m = \rho u, \quad n = \rho v, \quad E = \frac{1}{2}\rho u^2 + \frac{1}{2}\rho v^2 + \rho e, \quad p = (\gamma - 1)\rho e = \rho R T.$$

The cooling law is approximated by

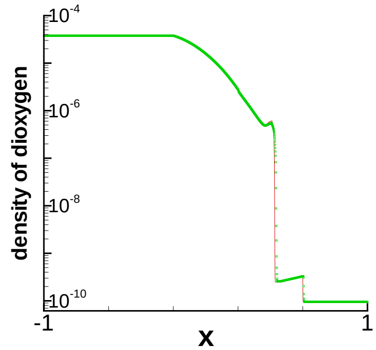
$$\Lambda(T) = \begin{cases} -\tilde{\Lambda}(p^2 - p_a^2) & T > T_a \\ 0 & \text{otherwise} \end{cases},$$

where P_a and T_a are the ambient pressure and temperature and $\tilde{\Lambda}$ is a constant. The condition (2.14) for this cooling term is

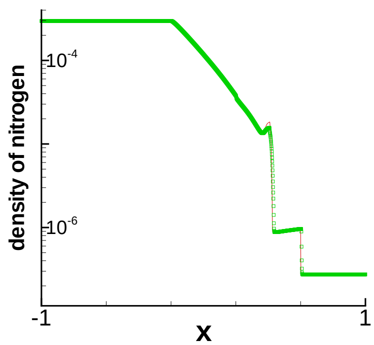
$$\Delta t \leq \frac{p}{2(\gamma - 1)\tilde{\Lambda}(p^2 - p_a^2)} \quad \text{if} \quad p > p_a.$$



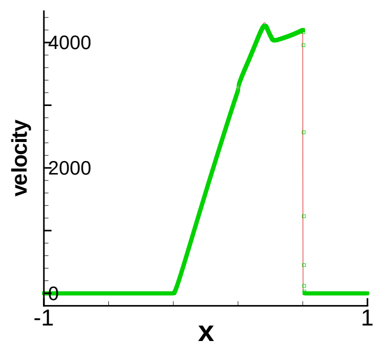
(a) ρ_O



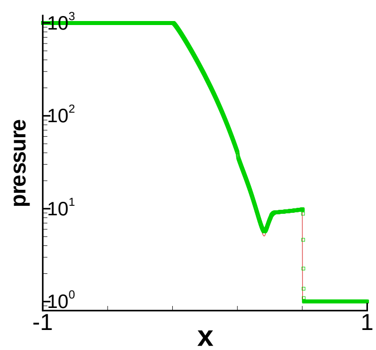
(b) ρ_{O_2}



(c) ρ_{N_2}



(d) Velocity



(e) Pressure

Figure 3.6: Example 3.5: The third order positivity-preserving RKDG scheme with TVB limiter. The solid line is the numerical solution of $\Delta x = \frac{2}{4000}$. The symbol is the numerical solution of $\Delta x = \frac{2}{2000}$.

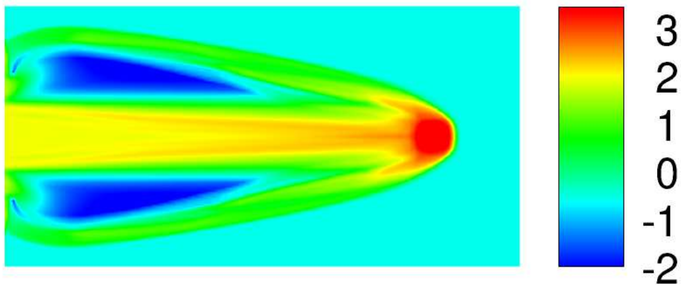
The velocity of the gas flow in the simulation is extremely high, and the Mach number could be hundreds or thousands. A big challenge for computation is, even for a state-of-the-art high order scheme solving the system, negative pressure could appear since the internal energy is very small compared to the huge kinetic energy. Moreover, the cooling source term increases the difficulty to preserve the positivity of the pressure. Therefore, we have a strong motivation to use the positivity-preserving schemes for this kind of problems.

Example 3.6. We compute a Mach 80 (i.e. the Mach number of the jet inflow is 80 with respect to the soundspeed in the jet gas) problem with the radiative cooling. γ is set as 5/3 and $\tilde{\Lambda} = 8.776$. The computation domain is $[0, 0.8] \times [0, 0.4]$, which is full of the ambient gas with $(\rho, u, v, p) = (5, 0, 0, 0.4127)$ initially. For the left boundary, $(\rho, u, v, p) = (5, 30, 0, 0.4127)$ if $y \in [-0.05, 0.05]$ and $(\rho, u, v, p) = (5, 0, 0, 0.4127)$ otherwise. The terminal time is 0.028. The computation is performed on a 256×128 mesh. See Figure 3.7.

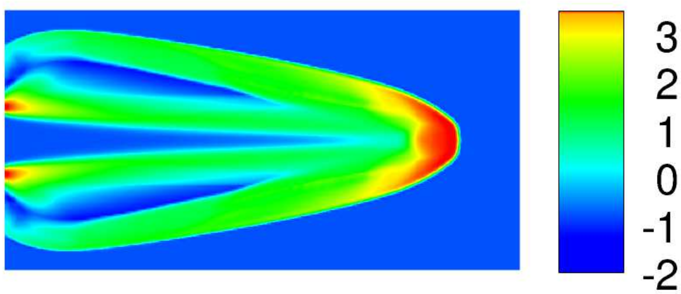
4 Concluding Remarks

In [16, 17], a general framework was established to construct high order schemes which can preserve the positivity of density and pressure for the compressible Euler equations in the gas dynamics. In this paper, we have shown that this framework also applies to a more general equation of state. Moreover, we extend it to the Euler equations with various source terms. We derived a sufficient condition for a high order finite volume scheme or the scheme satisfied by the DG method to satisfy the positivity-preserving condition. The same limiter in [16] can enforce it without destroying the high order accuracy.

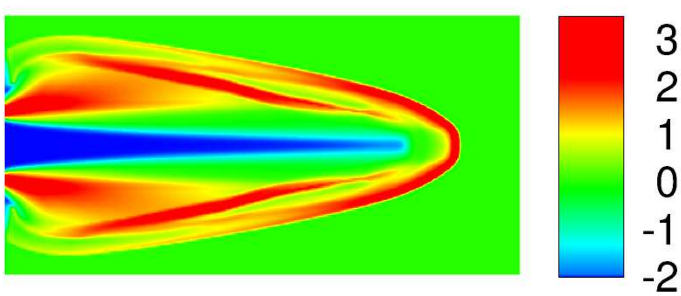
With the addition of the positivity-preserving limiter in this paper, which involves small additional computational cost, to the DG scheme or the finite volume scheme (e.g. ENO and WENO), the numerical solutions will satisfy the positivity property in the sense that the density and pressure of the cell average are positive under suitable CFL condition. We have tested the third order Runge-Kutta DG method with the positivity-preserving limiter for the Euler equations with four types of source terms: axial symmetry, gravity, chemical



(a) Density



(b) Pressure



(c) Temperature

Figure 3.7: Simulation of Mach 80 jet with radiative cooling. The third order positivity-preserving RKDG scheme with the TVB limiter. Scales are logarithmic.

reactions and radiative cooling.

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