Review

an infinite series $\sum_{n=0}^{\infty} a_n = a_0 + a_1 + a_2 + \cdots + a_n + \cdots$

Example: ① $\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} + \cdots$ diverges

Harmonic Series

② $\sum_{n=0}^{\infty} r^n \begin{cases} = \frac{1}{1-r}, & \text{if } |r|<1 \\ \text{diverges}, & \text{if } |r| \geq 1 \end{cases}$

Geometric Series

$$\sum_{n=0}^{\infty} \frac{1}{2^n} = 1 + \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^n} + \cdots = 2$$

$$S_n = 1 + \frac{1}{2} + \cdots + \frac{1}{2^n} = \frac{1-(\frac{1}{2})^{n+1}}{1-\frac{1}{2}} \longrightarrow 2$$

③ $\sum_{n=0}^{\infty} \frac{1}{n!} = 1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!} + \cdots = e$

exponential series

Theorem (n-th term Test)

$$\sum a_n \text{ converges} \Rightarrow \lim_{n\to\infty} a_n = 0$$

Remark: $\lim_{n\to\infty} a_n$ is not $0 \Rightarrow \sum a_n$ diverges

Theorem (Tail Convergence)

$\sum_{n=N_0}^{\infty} a_n$ converges for some $N_0 \Rightarrow \sum_{n=0}^{\infty} a_n$ converges

$\Rightarrow \sum_{n=N}^{\infty} a_n$ converges, $\forall N \in \mathbb{N}$

Comparison Theorem series Assume $0 \le a_n \le b_n$, $\forall n$

① $\sum b_n$ converges $\Rightarrow$ $\sum a_n$ converges

and $\sum a_n \le \sum b_n$

② $\sum a_n$ diverges $\Rightarrow$ $\sum b_n$ diverges

Squeeze Thm is for sequence

Def

① $\sum a_n$ is absolutely convergent if $\sum |a_n|$ converges;

② $\sum a_n$ is conditionally convergent if $\sum a_n$ converges but $\sum |a_n|$ diverges.

Example: ① $\sum (-1)^n \frac{1}{2^n}$ and $\sum (-1)^n \frac{1}{n!}$ are absolutely convergent.

③ $\sum (-1)^n \frac{1}{n}$ is conditionally convergent

Theorem $\sum |a_n|$ converges $\Rightarrow$ $\sum a_n$ converges

Proof: Define

$$a_n^+ = \begin{cases} |a_n|, & \text{if } a_n > 0 \\ 0, & \text{otherwise} \end{cases}, \quad a_n^- = \begin{cases} |a_n|, & \text{if } a_n < 0 \\ 0, & \text{otherwise} \end{cases}$$

Then $a_n = a_n^+ - a_n^-$ and $|a_n| = a_n^+ + a_n^-$

$$\left.\begin{array}{l} 0 \le a_n^+ \le |a_n| \\ 0 \le a_n^- \le |a_n| \\ \text{Comparison Theorem} \end{array}\right\} \Rightarrow \sum a_n^+ \text{ and } \sum a_n^- \text{ converges}$$

$$\left.\begin{array}{l} a_n = a_n^+ - a_n^- \\ \text{Linearity Theorem} \end{array}\right\} \Rightarrow \sum a_n = \sum a_n^+ - \sum a_n^- \text{ converges}$$

<u>Theorem (Ratio Test)</u> Assume $a_n \neq 0$, $n \gg 1$

and $\lim\limits_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L$.

Then $L < 1 \Rightarrow \sum |a_n|$ converges

$L > 1 \Rightarrow \sum a_n$ diverges

$L = 1$: no conclusion. $\begin{cases} ① \ \sum \frac{1}{n} & \text{diverges} \\ ② \ \sum (-1)^{n-1} \frac{1}{n} & \text{converges} \end{cases}$

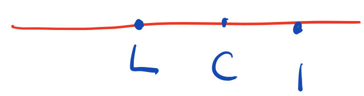
<u>Theorem (n-th root test)</u> Assume $\lim\limits_{n\to\infty} |a_n|^{\frac{1}{n}} = L$.

Then $L < 1 \Rightarrow \sum |a_n|$ converges

$L > 1 \Rightarrow \sum a_n$ diverges

$L = 1$: no conclusion.

Proof of Ratio Test for the case $L < 1$:



$\lim\limits_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L < 1 \Rightarrow$ There is a number $c \in (L, 1)$

s.t. $\lim\limits_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L < c$

Sequence Location Thm $\Rightarrow \quad \left|\frac{a_{n+1}}{a_n}\right| < c < 1, \quad n >> 1$

$\Rightarrow \exists N \in \mathbb{N}$ s.t. $\left|\frac{a_{n+1}}{a_n}\right| < c, \ \forall n \geq N.$

$$\Rightarrow \begin{cases} \left|\frac{a_{N+1}}{a_N}\right| < c \\ \left|\frac{a_{N+2}}{a_{N+1}}\right| < c \\ \left|\frac{a_{N+3}}{a_{N+2}}\right| < c \end{cases}$$

$$\Rightarrow \left|\frac{a_{N+k}}{a_N}\right| < c^k$$

$$\Rightarrow 0 \leq \underbrace{|a_{N+k}|}_{A_k} < \underbrace{c^k |a_N|}_{B_k}$$

$$0 \leq A_k \leq B_k$$

$$\sum_{k=0}^{\infty} B_k = \sum_{k=0}^{\infty} c^k |a_N| = |a_N| \left(\sum_{k=0}^{\infty} c^k\right)$$
$$= |a_N| \cdot \frac{1}{1-c}$$

Comparison Thm $\Rightarrow \sum_{k=0}^{\infty} A_k = \sum_{k=0}^{\infty} |a_{N+k}|$ converges

Tail Convergence Thm $\Rightarrow \sum_{n=0}^{\infty} |a_n|$ converges

---

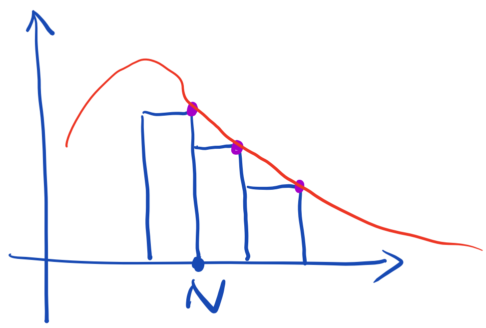
Theorem (Integral Test)

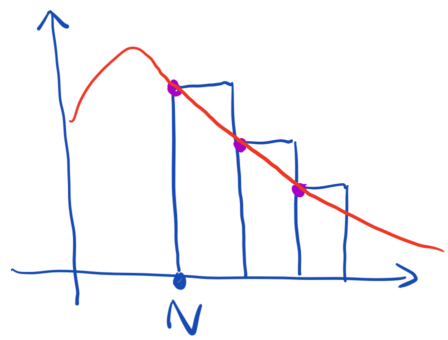
Assume $f(x) \geq 0$ and $f(x)$ is decreasing for any $x \geq N$, for some $N \in \mathbb{N}$.

Then ① $\sum f(n)$ converges if $\int_N^{\infty} f(x)\,dx$ is finite

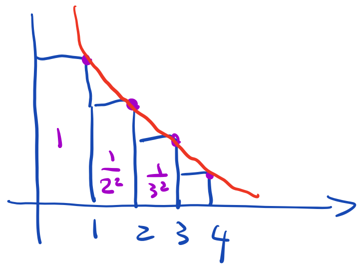
② $\sum f(n)$ diverges if $\int_N^{\infty} f(x)\,dx$ is infinite

Proof:

① 

② 

Example : $\sum_{n=1}^{N} \frac{1}{n^2}$ converges



$\int_0^n \frac{1}{x^2}dx = -\frac{1}{x}\Big|_0^n \to +\infty$ is useless

$\int_1^n \frac{1}{x^2}dx = \left(-\frac{1}{x}\right)_1^n = 1-\frac{1}{n} \to 1$

Integral Test $\Rightarrow$ Convergence of Series

Theorem Asymptotic Comparison Test

If $\lim \left|\frac{a_n}{b_n}\right| = 1$, then

$\sum |a_n|$ converges $\Leftrightarrow$ $\sum |b_n|$ converges

$\sum |a_n|$ diverges $\Leftrightarrow$ $\sum |b_n|$ diverges

Proof is an excercise.

Example: ① $\sum_{n=2}^{\infty} \frac{1}{n^3-2n+1}$ converges by asymptotic comparison with $\sum \frac{1}{n^3}$ $\frac{\frac{1}{n^3-2n+1}}{\frac{1}{n^3}} = \frac{n^3}{n^3-2n+1} \to 1$

We know $\sum \frac{1}{n^3}$ converges by integral test.

② $\sum \sqrt{\frac{4n}{n^2+1}}$ diverges because

$$\lim \frac{\sqrt{\frac{4n}{n^2+1}}}{\frac{2}{\sqrt{n}}} = 1$$

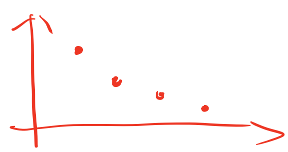
We know $\sum \frac{2}{\sqrt{n}}$ diverges by integral test.

<u>Theorem</u> (Cauchy Test for alternating series)

If $\{a_n\}$ is positive and strictly decreasing and $\lim_{n\to\infty} a_n = 0$, then $\sum (-1)^n a_n$ converges.

Ex: $\sum (-1)^n \frac{1}{\sqrt{n}}$ converges.

HW #5 Hints



1. (20 pts) Let $a_n \geq 0$ be decreasing and assume $\sum_{n=1}^{\infty} a_n$ converges. Prove $na_n \to 0$. (See Problem 7-1 on Page 111).

$\forall \varepsilon > 0$, $\quad n a_{2n} \leq a_{n+1} + \cdots + a_{2n} < \varepsilon$, $\quad n >> 1$

$\Uparrow$ $\{a_n\}$ is $\downarrow$

$\Uparrow$ S_n is Cauchy

$|S_m - S_n| < \varepsilon$, $m, n >> 1$

$\Rightarrow |S_{n+1} - S_{2n}| < \varepsilon$

$2n a_{2n} < 2\varepsilon$

2. (20 pts) Page 111, Problem 7-2. Prove that if $|a_{n+1}/a_n| \leq |b_{n+1}/b_n|$ for $n \gg 1$, and $\sum b_n$ is absolutely convergent, then $\sum a_n$ is absolutely convergent.

$\exists N$ s.t. $\forall n \geqslant N$, $\left|\frac{a_{n+1}}{a_n}\right| \leqslant \left|\frac{b_{n+1}}{b_n}\right|$

$$\Rightarrow \begin{cases} \left|\frac{a_{N+1}}{a_N}\right| \leqslant \left|\frac{b_{N+1}}{b_N}\right| \\ \left|\frac{a_{N+2}}{a_{N+1}}\right| \leqslant \left|\frac{b_{N+2}}{b_N}\right| \\ \vdots \end{cases}$$

$$\Rightarrow \left|\frac{a_{N+k}}{a_N}\right| \leqslant \left|\frac{b_{N+k}}{b_N}\right|$$

$$\Rightarrow |a_{N+k}| \leqslant \left|\frac{a_N}{b_N}\right| \cdot |b_{N+k}|$$

3. (20 pts) Page 112, Problem 7-4. Prove that

$$\lim |a_{n+1}/a_n| = L \Rightarrow \lim |a_n|^{\frac{1}{n}} = L.$$

① $L=0$. Then $\forall \varepsilon>0$, $\exists N$, $\forall n \geqslant N$, $\left|\frac{a_{n+1}}{a_n}\right| < \varepsilon$

$$\Rightarrow \begin{cases} \left|\frac{a_{N+1}}{a_N}\right| < \varepsilon \\ \left|\frac{a_{N+2}}{a_{N+1}}\right| < \varepsilon \\ \left|\frac{a_{N+3}}{a_{N+2}}\right| < \varepsilon \\ \vdots \end{cases}$$

$$\Rightarrow \left|\frac{a_{N+k}}{a_N}\right| < \varepsilon^k$$

$$\Rightarrow |a_{N+k}| < \varepsilon^k |a_N|$$

$$\Rightarrow |a_{N+k}|^{\frac{1}{N+k}} < \varepsilon^{\frac{k}{N+k}} \, |a_N|^{\frac{1}{N+k}}$$

$$= \varepsilon^{\frac{N+k}{N+k}} \cdot \varepsilon^{\frac{-N}{N+k}} |a_N|^{\frac{1}{N+k}}$$

$$= \varepsilon \left|\frac{a_N}{\varepsilon^N}\right|^{\frac{1}{N+k}} < \varepsilon \cdot 2$$

For fixed ε, $\left|\frac{a_N}{\varepsilon^N}\right|^{\frac{1}{N+k}} \to 1$, as $k \to \infty$.

② $L > 0$, for any small $\varepsilon > 0$, $\exists N$, $\forall n \geq N$

$$0 < L - \varepsilon < \left|\frac{a_{n+1}}{a_n}\right| < L + \varepsilon$$

$$\Rightarrow (L-\varepsilon)^k |a_N| < |a_{N+k}| < (L+\varepsilon)^k |a_N|$$

$$\Rightarrow (L-\varepsilon)\left[\frac{|a_N|}{(L-\varepsilon)^N}\right]^{\frac{1}{N+k}} < |a_{N+k}|^{\frac{1}{N+k}} < (L+\varepsilon)\left[\frac{|a_N|}{(L+\varepsilon)^N}\right]^{\frac{1}{N+k}}$$

If we can show $\left[\frac{a_N}{(L+\varepsilon)^N}\right]^{\frac{1}{N+k}} < \frac{L+2\varepsilon}{L+\varepsilon}$,

then we have $|a_{N+k}|^{\frac{1}{N+k}} < L + 2\varepsilon$.

The key is that ε is fixed, N is fixed.

For fixed ε and N, $\lim\limits_{k \to \infty} \left[\frac{|a_N|}{(L+\varepsilon)^N}\right]^{\frac{1}{N+k}}$

$$= \lim_{k \to \infty} \left[\frac{|a_N|^{\frac{1}{N}}}{L+\varepsilon}\right]^{\frac{1}{k}}$$

$$= 1$$

$$\Rightarrow \left[\frac{|a_N|}{(L+\varepsilon)^N}\right]^{\frac{1}{N+k}} < 1 + \frac{\varepsilon}{L+\varepsilon}, \quad k \gg 1.$$