Excercise (True or False)

① $\sum a_n$ converges $\Rightarrow \sum (-1)^n a_n$ converges

② If a power series converges absolutely at $x = c > 0$, then it converges for any $x \in [-c, c]$.

③ $n a_n \to 1 \Rightarrow \sum a_n$ diverges

④ $\sum \sin(\frac{1}{n})$ diverges

$a_n = \frac{1}{n^p}$

$n a_n \to 0 \Leftrightarrow p > 1$

⑤ $\left.\begin{array}{l} n a_n \to 0 \\ a_n \geq 0 \\ a_n \text{ decreasing} \end{array}\right\} \Rightarrow \sum a_n$ converges

Solution: ① F. $a_n = (-1)^n \frac{1}{n}$

② T.

③ T.

Asymptotic Comparison Thm

$\left|\frac{a_n}{b_n}\right| \to 1$, then $\sum |a_n|$ converges $\Leftrightarrow \sum |b_n|$ converges

$n a_n \to 1 \Rightarrow \dfrac{a_n}{1/n} \to 1$

④ T.

Intuition: $\dfrac{\sin(\frac{1}{n})}{(\frac{1}{n})} \to 1$ by L'Hospital's Rule

Proof: HW#5 P4, $\sin x > \frac{1}{2} x \Rightarrow \sin(\frac{1}{n}) > \frac{1}{2}\frac{1}{n}$

Comparison Thm for Series

Useful inequality $\frac{1}{2}x < \sin x < x$ if $x>0$ is close to 0

⑤ F. $a_n = \frac{1}{n\ln(n)}$, $a_n \searrow$, $na_n \to 0$

Integral Test? $\Rightarrow$ a_n diverges

$\int \frac{1}{x\ln x}dx = \ln(\ln x)$ | HW#6 P1

Homework 7

Due on Oct 20th on gradescope.

1. (30 pts) Page 124, Exercise 8.4/1. Only need to prove $\sum_{n=0}^{\infty} s_n x^n = \frac{f(x)}{1-x}$.

2. (20 pts) Page 124, Problem 8-1.

3. (30 pts) Page 124, Problem 8-2.

4. (20 pts) Page 112, Problem 7-6.

$\boxed{P1}$ Prove $\sum_{n=0}^{\infty} a_n x^n = f(x)$ converges for $|x|<1$

$\Rightarrow \sum_{n=0}^{\infty} S_n x^n = \frac{f(x)}{1-x}$ for $|x|<1$

Hint: $\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots$

$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$

Theorem (Multiplication of power series)

If $\sum a_n x^n = f(x)$ and $\sum b_n x^n = g(x)$ for $|x|<K$,

then $\sum C_n x^n = f(x)g(x)$, $|x|<K$.

where $C_n = a_0 b_n + a_1 b_{n-1} + \cdots + a_n b_0 = \sum_{i+j=n} a_i b_j$

$\boxed{P2}$ Find radius of convergence for $\sum (\sin n) x^n$

Hint: $\sum (\sin n)$ diverges (why?)

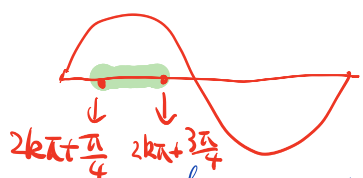
$$|\sin n| \leq 1 \Rightarrow |(\sin n) x^n| \leq |x^n|$$

$\boxed{P3}$ Show that ① $\sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n+1}}$ converges

and ② its multiplication with itself diverges.

$\boxed{P4}$ ① $[2k\pi + \frac{\pi}{4}, 2k\pi + \frac{3\pi}{4}]$ contain an integer

② Use ① to show $\sum \frac{(\sin n)}{n}$ is not absolutely convergent.



Chapter 9 Functions of one variable

Definition $f(x)$ is a mapping from its domain (some subset of $\mathbb{R}$) to $\mathbb{R}$.

$$f(x): D_f \rightarrow \mathbb{R}$$
$$x \mapsto f(x)$$

D_f is domain of $f(x)$

Example: $\ln(x): \mathbb{R}_+ \rightarrow \mathbb{R}$

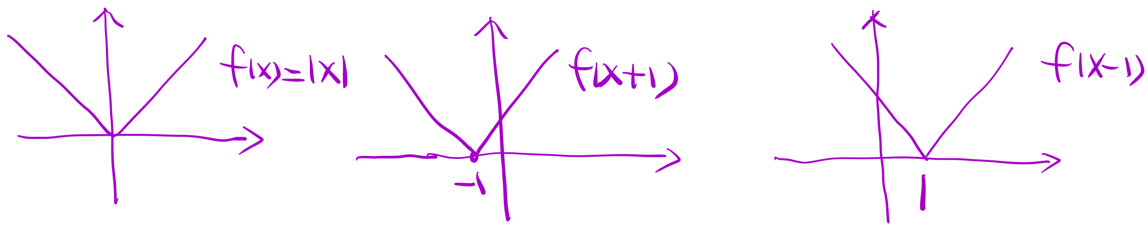
$$x \rightarrow \ln(x)$$

Composition $f(x), g(x)$

$$f \circ g(x) = f(g(x))$$

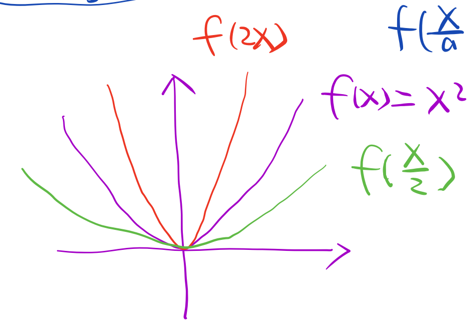
Translation $f(x)$ $a > 0$

$f(x+a)$, $f(x-a)$



<u>Change of scale</u> $a>1$

$f(\frac{x}{a})$, $f(ax)$

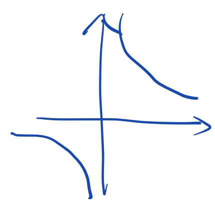


<u>Def</u>

$f(x)$ is $\begin{cases} \text{increasing if } f(a)\le f(b), \ \forall a,b\in D_f, \ a<b \\ \text{strictly increasing if } f(a)<f(b), \ \forall a,b\in D_f, \ a<b \\ \text{monotone if either } \nearrow \text{ or } \searrow . \end{cases}$

Example: ① $f(x)\equiv 1$ is increasing.

② $f(x)=\frac{1}{x}$ is not monotone



<u>Def</u> $f(x)$ is even if $f(-x)=f(x)$, $\forall x\in D_f$.

$f(x)$ is odd if $f(-x)=-f(x)$, $\forall x\in D_f$.

Ex: ① $\left.\begin{matrix} \sin x \\ x^3 \\ x \end{matrix}\right\}$ is odd ② $\left.\begin{matrix} \cos x \\ x^2 \\ x^4 \\ |x| \\ f(|x|) \end{matrix}\right\}$ is even

Theorem Suppose the domain of $f(x)$ is symmetric around 0, then $f(x)$ has a unique representation:

$$f(x) = E(x) + O(x)$$

where $E(x)$ is even and $O(x)$ is odd.

Proof: $E(x) = [f(x) + f(-x)]/2$ is Even

$O(x) = [f(x) - f(-x)]/2$ is Odd

$$O(-x) = [f(-x) - f(x)]/2 = -[f(x) - f(-x)]/2$$

Def $f(x)$ is periodic if $f(x+c) = f(x)$, $\forall x \in D_f$.

c is called a period.

The smallest such c is the minimal period.

Ex: Prove that if $f(x)$ is even and monotone, it is constant.

① Assume $\nearrow$, $a > 0$, $f(a) = f(-a)$,

for any $x \in (-a, a)$, $f(-a) \le f(x) \le f(a)$

$\Rightarrow f(x)$ is constant on $[-a, a]$

② Assume $\searrow$,

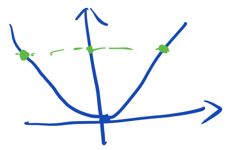
Ex: Show $\sin(x)$ is not a polynomial.

Proof: $\sin(x)$ has infinitely many roots.

Def Inverse function

$$y = f^{-1}(x) \Leftrightarrow x = f(y)$$

Ex: ① $y = x^2$ does not have an inverse function



② $y = x^2$ has an inverse on $x \in [0, +\infty)$

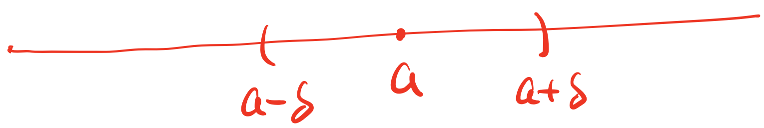
$\Updownarrow$ $\quad\quad\quad\quad$ $y = \sqrt{x}$

$\sqrt{y} = x$

③ The inverse of $\begin{cases} y = x^2 \\ x \in (-\infty, 0] \end{cases}$ is $y = -\sqrt{x}$.

Chapter 10

Def $a \in \mathbb{R}$, $\delta > 0$, the δ-neighborhood of a is $(a-\delta, a+\delta)$



Same meaning
$\begin{cases} ① & x \in (a-\delta, a+\delta) \\ ② & a-\delta < x < a+\delta \\ ③ & |x-a| < \delta \\ ④ & x \underset{\delta}{\approx} a \end{cases}$

Def ① b is an upper bound of $f(x)$ on an interval I if $f(x) \le b$, $\forall x \in I$

② $f(x)$ is bounded above on $I \iff f(x)$ has one upper bound on I.

<u>Def</u> $f(x)$ is defined on I

$\sup_{x \in I} f(x)$ is $\sup\{f(x) : x \in I\}$

$\max_{x \in I} f(x)$ is $\max\{f(x) : x \in I\}$

smallest upper bound

$\inf_{x \in I} f(x)$ is $\inf\{f(x) : x \in I\}$

$\min_{x \in I} f(x)$ is $\min\{f(x) : x \in I\}$

smallest upper bound

Ex: ① $I = (0,1)$ $\quad f(x) = x$

$$\sup_{x \in I} f(x) = 1$$

$$\max_{x \in I} f(x) \text{ DNE}$$

② $I = [0,1)$, $f(x) = x$

$$\inf_{x \in I} f(x) = \min_{x \in I} f(x) = 0$$

$$\sup_{x \in I} f(x) = 1, \quad \max_{x \in I} f(x) \text{ DNE}.$$