Def $f(x)$ is defined on I

$\sup\limits_{x\in I} f(x)$ is $\sup\{f(x) : x\in I\}$

$\max\limits_{x\in I} f(x)$ is $\max\{f(x) : x\in I\}$

smallest upper bound

$\inf\limits_{x\in I} f(x)$ is $\inf\{f(x) : x\in I\}$

$\min\limits_{x\in I} f(x)$ is $\min\{f(x) : x\in I\}$

largest lower bound

Theorem (Completeness for functions)

$f(x)$ is defined on an interval I

① $f(x)$ is bounded above $\Rightarrow$ $\sup\limits_{x\in I} f(x)$ exists

② $f(x)$ is bounded below $\Rightarrow$ $\inf\limits_{x\in I} f(x)$ exists

Estimating functions:

① $|f(x)g(x)| \leq |f(x)|\cdot|g(x)|$

② $|f(x)+g(x)| \leq |f(x)| + |g(x)|$

③ $f(x)$ is bounded $\Leftrightarrow$ $A \leq f(x) \leq B$

$\Leftrightarrow |f(x)| \leq K$

Approximation: $f(x) \underset{\varepsilon}{\approx} g(x)$ means

$|f(x)-g(x)| < \varepsilon$

or $g(x) - \varepsilon < f(x) < g(x) + \varepsilon$

Example: Find a δ-neighborhood of 0, over which $\sin x \underset{\varepsilon}{\approx} x$, $\varepsilon = 0.001$

For $\varepsilon = 0.001$, want $\delta > 0$ s.t. $\forall x \in (-\delta, \delta)$, $|\sin x - x| < \varepsilon = 0.001$

Sol: ① In HW#8 P1, we will prove

$$|\sin x - x| < \frac{x^3}{3!} \quad \text{for } 0 < x < 1$$

② $\frac{x^3}{3!} < 0.001 \iff x^3 < 0.006 \Leftarrow x < 0.18$

So we can take $\delta = 0.18$.

1) $\forall x \in (-\delta, \delta)$, $|\sin x - x| < 0.001$

2) $\sin x \underset{0.001}{\approx} x$, if $x \underset{0.18}{\approx} 0$

<u>Def</u> "for $x \approx x_0$" (for x near x_0)

means $x \in (x_0 - \delta, x_0 + \delta)$

"for $x \underset{\delta}{\approx} x_0$ with some $\delta > 0$"

Example: ① "$x^4 < x^2$ for $x \approx 0$" is true

② "$x^3 < x$ for $x \approx 0$" is false

$x^3 < x$ only for $x \in (0, 1)$

<u>Def</u> ① "for $x \gg 1$" for large x

means

"for x in $(a, +\infty)$ for some a"

② "for $x << -1$" for negatively large x

means

"$x \in (-\infty, a)$"

③ "for $|x| >> 1$" means

"$|x| \in (a, +\infty)$" $\Leftrightarrow$ $x \in (-\infty, -a) \cup (a, +\infty)$

<u>Def</u> ① "at $+\infty$" means "for $x >> 1$"

② "at $-\infty$" means "for $x << -1$"

Example: $f(x) = x^n + a_1 x^{n-1} + a_2 x^{n-2} + \cdots + a_n$

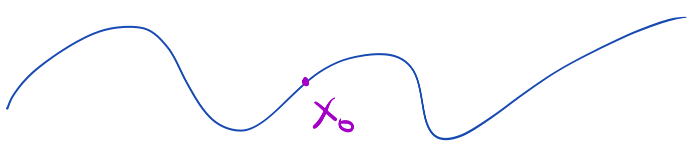
1) $f(x) > 0$ at $+\infty$ (for $x >> 1$)

2) If n is odd, $f(x) < 0$ at $-\infty$

If n is even, $f(x) > 0$ at $-\infty$

3) $\frac{1}{f(x)}$ is bounded at $\pm\infty$ (for $|x| >> 1$)

<u>Def</u> (local behavior)



① $f(x)$ is <u>locally increasing at x_0</u> if $f(x)$ is increasing for $x \approx x_0$

② $f(x)$ is <u>locally bounded at x_0</u> if $f(x)$ is bounded for $x \approx x_0$

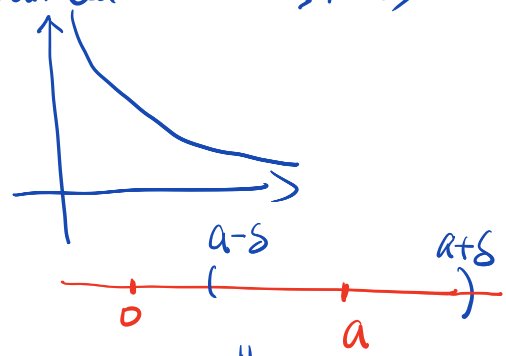
③ $f(x)$ is <u>locally positive at x_0</u> if $f(x)$ is positive for $x \approx x_0$

Def $f(x)$ is locally bounded on an interval I if

$\forall x_0 \in I$, $f(x)$ is bounded for $x \approx x_0$

Example: ① $f(x) = \frac{1}{x}$ is not bounded on $(0, +\infty)$

② $f(x) = \frac{1}{x}$ is locally bounded on $(0, +\infty)$



Proof of ②:

Want to show

" $\forall a \in (0, +\infty)$, $\exists \delta > 0$, $\exists K$

$|f(x)| \leq K$, $\forall x \in (a-\delta, a+\delta)$ "

for any $a > 0$, let $\delta = \frac{a}{2}$, then

$\forall x \in (\frac{a}{2}, \frac{3a}{2})$, $f(x) = \frac{1}{x} \in (\frac{2}{3a}, \frac{2}{a})$

$|f(x)| < \frac{2}{a}$

$\forall a \in (0, +\infty)$, pick $\delta = \frac{a}{2}$, $K = \frac{2}{a}$, $\forall x \in (a-\delta, a+\delta)$, $|f(x)| \leq K$. #

Def $f(x)$ is locally increasing on an interval I if

$\forall x_0 \in I$, $f(x)$ is increasing for $x \approx x_0$

Example: ① $f(x) = \frac{1}{x}$ is not decreasing

② $f(x) = \frac{1}{x}$ is locally decreasing on $(0, +\infty)$

$f(x) = \frac{1}{x}$ is locally decreasing on $(-\infty, 0)$

Ex (T or F) :

① $f(x) = \sqrt{x}$ is locally bounded on $(0, +\infty)$

② $f(x)$ is bounded on its domain $\Rightarrow$ $f(x)$ is locally bounded on its domain

③ $f(x)$ is locally bounded on $I \Rightarrow f(x)$ is bounded on I.

Sol: ① T

② T

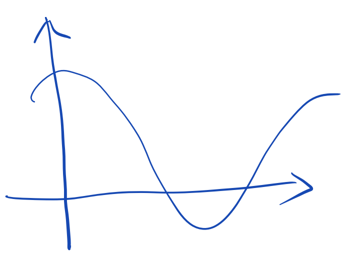
③ F. $f(x) = \frac{1}{x}$ on $(0, +\infty)$

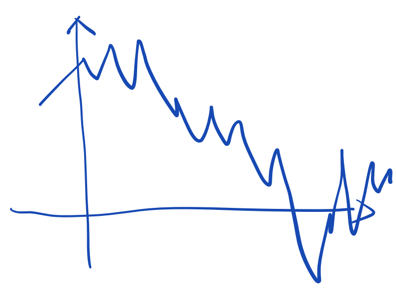
Theorem

$f(x)$ is locally bounded on $[a, b] \Rightarrow f(x)$ is bounded on $[a, b]$

Chapter II Continuity and Limits

What is a continuous function?





Def $f(x)$ is continuous at x_0 if $f(x)$ is defined for $x \approx x_0$ and $\forall \varepsilon > 0$, $f(x) \underset{\varepsilon}{\approx} f(x_0)$, for $x \approx x_0$

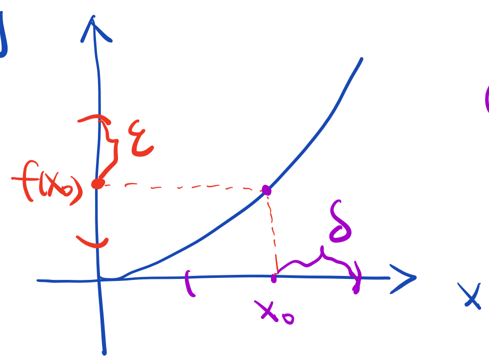
$\forall \varepsilon>0,\ \exists \delta>0,\ \text{s.t. } \forall x\in(x_0-\delta, x_0+\delta),\ |f(x)-f(x_0)|<\varepsilon.$

Example: Show $f(x)=x^2$ is continuous on $I=(-a,a)$, $a>0$

Sol: For any fixed $x_0\in I$,

for any $\varepsilon>0$, $\forall x\in(x_0-\delta,x_0+\delta)$ (δ must be small enough s.t. $(x_0-\delta, x_0+\delta)\subset I$)

$$
\begin{aligned}
&|f(x)-f(x_0)| \\
&= |x^2-x_0^2| \\
&= |x-x_0|\cdot|x+x_0| \\
&\leq |x-x_0|\cdot(|x|+|x_0|) \\
&\leq \delta\cdot(a+a) \\
&= 2a\delta = \varepsilon \quad \text{if } \delta=\frac{\varepsilon}{2a}.
\end{aligned}
$$



Continuity means $\forall\varepsilon, \exists\delta$

$$x \underset{\delta}{\approx} x_0 \Rightarrow f(x) \underset{\varepsilon}{\approx} f(x_0)$$

Def for $x\approx a^+$ means "$\exists\delta,\ x\in[a, a+\delta)$"

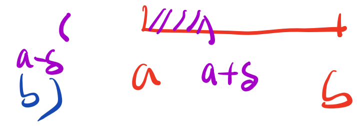
for $x\approx a^-$ means "$\exists\delta,\ x\in(a-\delta, a]$".

Def Assume $f(x)$ is defined for relevant x-values.

① $f(x)$ is right continuous at x_0 : $\forall\varepsilon>0$, $f(x)\underset{\varepsilon}{\approx} f(x_0)$, for $x\approx x_0^+$

② $f(x)$ is left continuous at x_0: $\forall \varepsilon > 0$, $f(x) \underset{\varepsilon}{\approx} f(x_0)$, for $x \approx x_0^-$

③ $f(x)$ is continuous on $[a, b]$

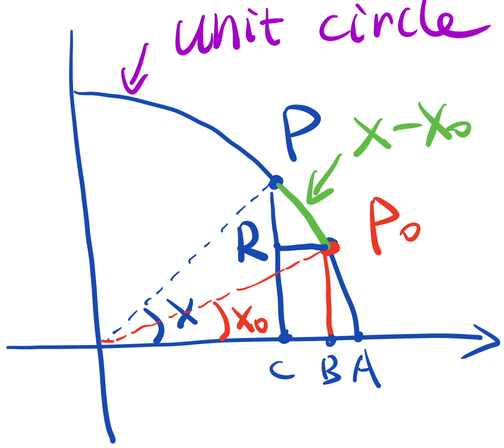


if $f(x)$ is $\begin{cases} \text{continuous on } (a, b) \\ \text{right continuous at } a \\ \text{left continuous at } b \end{cases}$

<u>Def</u> We say $f(x)$ is contiuous if its domain I is an interval and it is continuous on I.

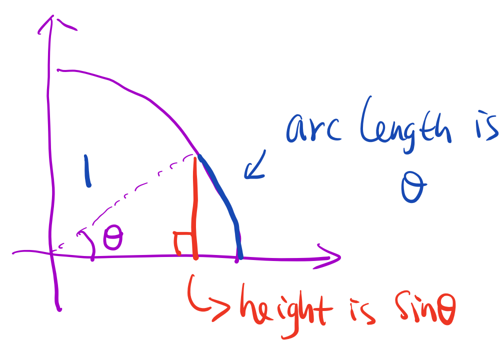
Example: $f(x) = \sin x$ is continuous

Sol:



Want to show

$|\sin x - \sin x_0| < \varepsilon$



① x and x_0 are angles

② The arc length of AP_0 is x_0

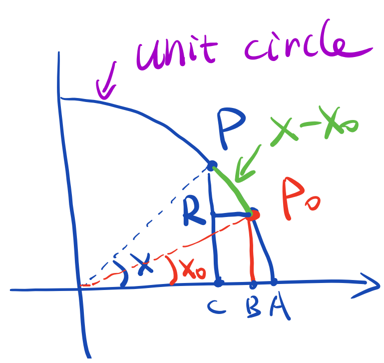
The arc length of AP is x

③ $PR = PC - RC$

$= PC - P_0B$

$= \sin x - \sin x_0$

④ Arc length $PP_0 > PR$



$\Rightarrow \quad |\sin x - \sin x_0| < |x - x_0|$

$\Rightarrow \quad \forall \varepsilon > 0, \quad \sin x \underset{\varepsilon}{\approx} \sin x_0 \text{ for } x \underset{\varepsilon}{\approx} x_0$

we pick $\delta = \varepsilon$.

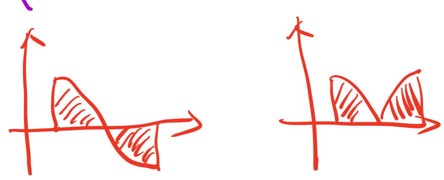
Example: Show $f(x) = \int_0^\pi \frac{\sin(xt)}{t}\,dt$ is continuous

Sol: for any fixed x_0,

$$|f(x) - f(x_0)| = \left| \int_0^\pi \frac{\sin(xt)}{t}\,dt - \int_0^\pi \frac{\sin(x_0 t)}{t}\,dt \right|$$

$$= \left| \int_0^\pi \frac{\sin xt - \sin x_0 t}{t}\,dt \right|$$

$$\leq \int_0^\pi \frac{|\sin xt - \sin x_0 t|}{t}\,dt \qquad \left(\left|\int_0^\pi f(x)\,dx\right| \leq \int_0^\pi |f(x)|\,dx \right)$$



$$\leq \int_0^\pi \frac{|xt - x_0 t|}{t}\,dt$$

$$= \pi\,|x - x_0|$$

$\Rightarrow \quad \forall \varepsilon > 0, \quad |f(x) - f(x_0)| \leq \pi\varepsilon, \text{ for } x \underset{\varepsilon}{\approx} x_0.$

By K-ε principle, $f(x)$ is continuous at x_0.

$\Rightarrow \quad f(x)$ is continuous.