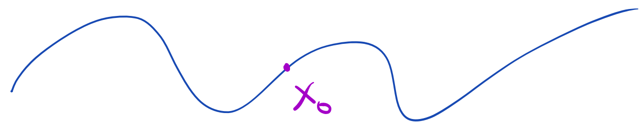
Question: Any example/operation that is correct for finite sum but not necessarily true for infinite series?

We cannot multiply or rearrange terms in conditional convergent series.

Review

Def (local behavior)



① $f(x)$ is <u>locally increasing at x_0</u> if $f(x)$ is increasing for $x \approx x_0$

② $f(x)$ is <u>locally bounded at x_0</u> if $f(x)$ is bounded for $x \approx x_0$

③ $f(x)$ is <u>locally positive at x_0</u> if $f(x)$ is positive for $x \approx x_0$

Def $f(x)$ is locally bounded on an interval I if $\forall x_0 \in I$, $f(x)$ is bounded for $x \approx x_0$

Example: ① $f(x) = \frac{1}{x}$ is not bounded on $(0, +\infty)$

② $f(x) = \frac{1}{x}$ is locally bounded on $(0, +\infty)$

Def $f(x)$ is locally increasing on an interval I if $\forall x_0 \in I$, $f(x)$ is increasing for $x \approx x_0$

Example: ① $f(x) = \frac{1}{x}$ is not decreasing

② $f(x) = \frac{1}{x}$ is locally decreasing on $(0, +\infty)$

$f(x) = \frac{1}{x}$ is locally decreasing on $(-\infty, 0)$

Def $f(x)$ is continuous at x_0 if $f(x)$ is defined for $x \approx x_0$

and $\forall \varepsilon > 0$, $f(x) \underset{\varepsilon}{\approx} f(x_0)$, for $x \approx x_0$

$\forall \varepsilon > 0, \exists \delta > 0$, s.t. $\forall x \in (x_0 - \delta, x_0 + \delta)$, $|f(x) - f(x_0)| < \varepsilon$.

Def for $x \approx a^+$ means "$\exists \delta$, $x \in [a, a+\delta)$"

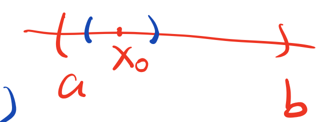
for $x \approx a^-$ means "$\exists \delta$, $x \in (a-\delta, a]$".

Def Assume $f(x)$ is defined for relevant x-values.

① $f(x)$ is right continuous at x_0 : $\forall \varepsilon > 0$, $f(x) \underset{\varepsilon}{\approx} f(x_0)$, for $x \approx x_0^+$

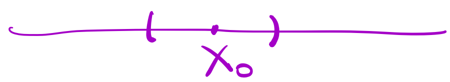
② $f(x)$ is left continuous at x_0 : $\forall \varepsilon > 0$, $f(x) \underset{\varepsilon}{\approx} f(x_0)$, for $x \approx x_0^-$

③ $f(x)$ is continuous on $[a, b]$



if $f(x)$ is $\begin{cases} \text{continuous on } (a, b) \\ \text{right continuous at } a \\ \text{left continuous at } b \end{cases}$

Discontinuity



Def We say x_0 is a point of discontinuity of $f(x)$

if $\begin{cases} ① \ f(x) \text{ is not continuous at } x_0 \\ ② \ x_0 \text{ is isolated} : f(x) \text{ is continuous at all points near } x_0 \end{cases}$

① $f(x)$ is cont. at x_0: $\forall \varepsilon>0, \exists \delta>0, \forall x \in (x_0-\delta, x_0+\delta)$
$|f(x)-f(x_0)|<\varepsilon$

$f(x)$ is discont. at x_0: $\exists \varepsilon>0, \forall \delta>0, \exists x \in (x_0-\delta, x_0+\delta)$
s.t. $|f(x)-f(x_0)| \geq \varepsilon$

② $\exists \delta>0$, $f(x)$ is continuous at any point in $(x_0-\delta, x_0)$
and $(x_0, x_0+\delta)$

Two kinds of points of discontinuity

I. Removable discontinuity means that
$f(x)$ will be continuous if we change $f(x_0)$

Example: $f(x)=\frac{1}{x}\sin x$ is not defined at $x_0=0$

Redefine $f(0)=0$, then it is discontinuous at $x_0=0$.

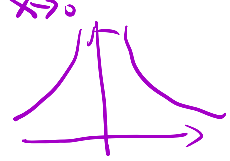
Redefine $f(0)=1$, then $f(x)$ is continuous at $x_0=0$.

II. Essential discontinuity

$f(x)$ is discontinuous at x_0 for any value of $f(x_0)$

Example: ① $f(x)=\sin\frac{1}{x}$ at $x_0=0$ $\lim_{x\to 0}\sin(\frac{1}{x})$ DNE

② $f(x)=\frac{1}{|x|}$ at $x_0=0$



③ ______

______

Def (Limit of a function) $x \underset{\neq}{\approx} x_0 : \exists \delta>0, \forall x \in (x_0-\delta, x_0) \cup (x_0, x_0+\delta)$

Let $f(x)$ be defined for $x \approx x_0$ but not necessarily for $x = x_0$ (we denote it by $x \underset{\neq}{\approx} x_0$),

we say $f(x) \to L$ as $x \to x_0$ if

$$\forall \varepsilon > 0, \quad f(x) \underset{\varepsilon}{\approx} L \text{ for } x \underset{\neq}{\approx} x_0$$

$$\forall \varepsilon > 0, \exists \delta > 0, \text{ s.t. } \forall x \in (x_0-\delta, x_0) \cup (x_0, x_0+\delta),$$

$$|f(x) - L| < \varepsilon$$

We also write $\lim_{x \to x_0} f(x) = L$

Ex: (T or F) $\lim_{x \to x_0} f(x)$ has nothing to do with $f(x_0)$

T

Def (One sided limits)

$\lim_{x \to x_0^+} f(x) = L : \forall \varepsilon > 0, \exists \delta > 0, \forall x \in (x_0, x_0+\delta)$
$|f(x) - L| < \varepsilon$

$\lim_{x \to 0^+} \log x$

$\lim_{x \to x_0^-} f(x) = L : \forall \varepsilon > 0, \exists \delta > 0, \forall x \in (x_0-\delta, x_0)$
$|f(x) - L| < \varepsilon$

Def (Limits at ∞) $\forall \varepsilon > 0, \exists M$ s.t. $\forall x \geq M, |f(x) - L| < \varepsilon$

$\lim_{x \to +\infty} f(x) = L : \forall \varepsilon > 0, f(x) \underset{\varepsilon}{\approx} L$ for $x \gg 1$.

Def (Infinite Limits)

$\lim_{x \to x_0} f(x) = +\infty$: $\forall\, b > 0$, $f(x) > b$ for $x \underset{\neq}{\approx} x_0$.

$\lim_{x \to 0} \frac{1}{|x|} = +\infty$

<u>Theorem</u> If $f(x) \to L$, $g(x) \to M$ as $x \to x_0$, then

① $af(x) + b\,g(x) \to aL + bM$, as $x \to x_0$

② $f(x)\,g(x) \to LM$ as $x \to x_0$

③ $g(x)/f(x) \to M/L$ as $x \to x_0$ if $\begin{cases} f(x) \neq 0 \\ L \neq 0 \end{cases}$

<u>Theorem</u> As $x \to x_0$

① $\left.\begin{array}{l} f(x) \to +\infty \\ \left.\begin{array}{l} g(x) \to +\infty \\ \text{or } g(x) \text{ is bounded below} \end{array}\right\} \end{array}\right\} \Rightarrow f(x) + g(x) \to +\infty$

② $\left.\begin{array}{l} f(x) \to +\infty \\ \left.\begin{array}{l} g(x) \to L > 0 \\ \text{or } g(x) > k > 0 \text{ for some } k \end{array}\right\} \end{array}\right\} \Rightarrow f(x)\,g(x) \to +\infty$

③ $f(x) \to +\infty \Rightarrow \frac{1}{f(x)} \to 0$

④ $f(x) > 0$, $f(x) \to 0 \Rightarrow \frac{1}{f(x)} \to +\infty$

<u>Squeeze Theorem</u>

① $f(x) \le g(x) \le h(x)$ for $x \underset{\neq}{\approx} x_0$

$\left.\begin{array}{l} f(x) \to L \\ h(x) \to L \end{array}\right.$ as $x \to x_0 \Rightarrow g(x) \to L$

② $f(x) \geq g(x)$ for $x \underset{\neq}{\approx} x_0$

$$\lim_{x \to x_0} g(x) = +\infty \Rightarrow \lim_{x \to x_0} f(x) = +\infty$$

<u>Limit Location Theorem</u>

Assume limits exist.

① $f(x) \leq M$ for $x \underset{\neq}{\approx} x_0 \Rightarrow \lim_{x \to x_0} f(x) \leq M$

② $f(x) \leq g(x)$ for $x \underset{\neq}{\approx} x_0 \Rightarrow \lim_{x \to x_0} f(x) \leq \lim_{x \to x_0} g(x)$

<u>Function location Theorem</u>

$$\lim_{x \to x_0} f(x) < M \Rightarrow f(x) < M \text{ for } x \underset{\neq}{\approx} x_0.$$

Ex: Describe $\lim_{x \to 0} f(x)$ DNE

① "$\lim_{x \to 0} f(x)$ exists" means:

1) $\exists L \in \mathbb{R}$ s.t. $\lim_{x \to 0} f(x) = L$

2) $\exists L \in \mathbb{R}$ s.t. $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t.

$\forall x \in (-\delta, 0) \cup (0, \delta)$, $|f(x) - L| < \varepsilon$

② Negation of "$\lim_{x \to 0} f(x)$ exists" means:

$\forall L \in \mathbb{R}$, $\exists \varepsilon > 0$ s.t. $\forall \delta > 0$

$\exists x \in (-\delta, 0) \cup (0, \delta)$, $|f(x) - L| \geq \varepsilon$.

Homework 8

Due on Oct 27th before 10am on gradescope.

1. (30 pts) Consider the Maclaurin seris for $\sin x$:

$$\sin x = \sum_{n=1}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$$

   (a) (10 pts) Prove that the series $\sum_{n=1}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$ converges for any $x \in [0,1]$. Thus $\sum_{n=1}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$ also converges for any $x \in [-1,0]$ since the only difference is a sign.

   (b) (15 pts) Prove that $|\sin x - x| \leq \frac{|x|^3}{3!}$ for any $x \in [-1,1]$. Hint: follow the proof of Alternating Series Test Theorem.

   (c) (5 pts) Use the estimate above to show $|x| < 0.1 \Rightarrow |\sin x - x| < 0.001$.

2. (10 pts) Prove that $\sum_{n=1}^{N} a_n \cos(nx)$ is bounded on $(-\infty, +\infty)$.

3. (10 pts) Show that $\int_0^1 \frac{x^4}{1+x^6} dx \leq \frac{1}{5}$ by estimating the integrand.

4. (10 pts) For what values of $k > 0$ are the function $f(x)$ bounded for $x \approx 0+$?

   (a) $f(x) = \int_x^1 (1/t^k) dt$.

   (b) $f(x) = \int_x^1 (e^t/t^k) dt$.

$$\frac{e^t}{t^k} = \frac{1 + t + \frac{t^2}{2!} + \cdots + \frac{t^k}{k!} + \cdots}{t^k}$$

5. (10 pts) Show that a function which is locally increasing on an interval I is increasing on I. Hint: try an indirect argument (or proof by contradiction) and use bisection to construct nested intervals.

6. (10 pts) If $f(x)$ is continuous at x_0, show $f(x)$ is locally bounded at x_0.

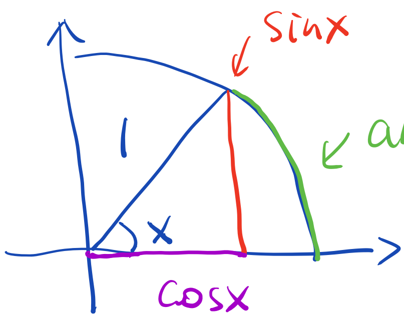
7. (20 pts) P167, Exercise 11.3/1. (a) $\lim_{x \to 0+} \frac{\sin x}{x} = 1$

$$\cos x \leq \frac{\sin x}{x} \leq \frac{1}{\cos x}$$

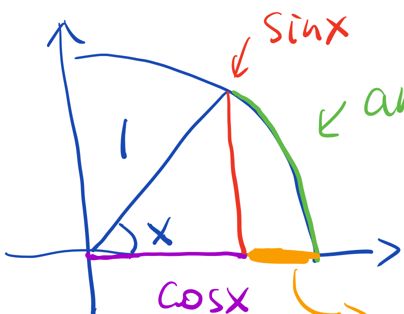
If we can show $\lim_{x \to 0^+} \cos x = 1$, then

we can use squeeze Thm.

How to show $\cos x \to 1$ as $x \to 0^+$?



Want $|\cos x - 1| < \varepsilon$



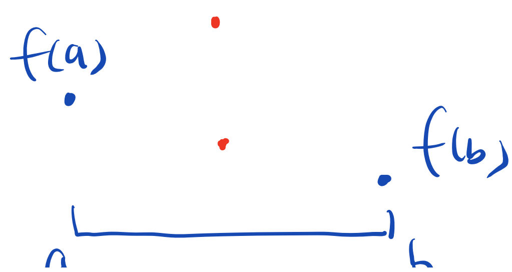
$|\cos x - 1| < ?$

$\boxed{P5}$: Indirect Proof Hint

Assume $f(x)$ is not increasing.

Then $\exists\, a, b \in I$ s.t.

$a < b \quad f(a) > f(b)$



Construct nested intervals s.t.

left end function value is higher than right end.

Let $c = \frac{a+b}{2}$, then three scenarios:

① $f(c) \geq f(a)$

② $f(b) < f(c) < f(a)$

③ $f(c) \leq f(b)$