Review

<u>Def</u> A set $S \subseteq \mathbb{R}$ is called sequentially compact if every sequence in S has a convergent subsequence converging to $L \in S$

Example: ① <u>Theorem</u> $[a, b]$ is sequentially compact

Proof: $\forall \{x_n\} \subset [a, b]$

$a \leq x_n \leq b$ $\underset{\text{B-W Thm}}{\Rightarrow}$ There is a convergent subsequence x_{n_i}

$a \leq x_{n_i} \leq b$ $\underset{\text{Limit Location Thm}}{\Rightarrow}$ $a \leq \lim_{i \to \infty} x_{n_i} \leq b$

② $(a, b]$ is not sequentially compact.

Proof: $x_n = a + \frac{b-a}{n} \to a$

$\Rightarrow$ Any subsequence of $\{x_n\}$ goes to $a \notin S = (a, b]$.

③ $[0, +\infty)$ is not sequentially compact.

④ $S = \{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots, \frac{1}{n}, \cdots\}$ is not sequentially compact

Counter Example: $x_n = \frac{1}{n} \to 0 \notin S$.

④ $S = \{0\} \cup \{\frac{1}{n} : n \in \mathbb{N}\} = \{0, 1, \frac{1}{2}, \frac{1}{3}, \cdots\}$ is sequentially compact.

Proof: $\forall \{x_n\} \subset S$, $x_n \in S \Rightarrow 0 \leq x_n \leq 1$

Bolzano-Weistrauss Thm

$\Rightarrow$ A converging subsequence $\{x_{n_i}\}$

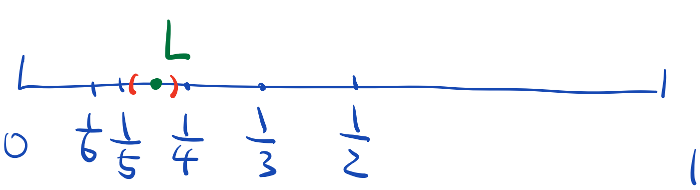
$\left.\begin{array}{l}\text{Limit Location Thm} \\ 0 \leq x_{n_i} \leq 1\end{array}\right\} \Rightarrow \quad 0 \leq \underbrace{\lim_{i \to \infty} x_{n_i}}_{= L} \leq 1$

Claim that $L \in S$.

Proof by contradiction: assume $L \notin S$.

Then $L \neq 0, 1, \frac{1}{2}, \frac{1}{3}, \cdots$ and $0 < L < 1$.

Let $\frac{1}{m}$ be the nearest number in S to L



Let $\varepsilon = |L - \frac{1}{m}| > 0$, then

$(L-\varepsilon, L-\varepsilon)$ contains no $\frac{1}{n}$, thus no x_{n_i}.

$\Rightarrow \forall i$, $|x_{n_i} - L| \geq \varepsilon$

$\Rightarrow \exists\, \varepsilon = |L - \frac{1}{m}| > 0$ s.t. $|x_{n_i} - L| \geq \varepsilon,\ \forall i$

Contradiction with $x_{n_i} \to L$.

Theorem $f(x)$ is locally bounded on $[a,b]$

$\Rightarrow f(x)$ is bounded on $[a,b]$

Proof: Assume $f(x)$ is not bounded

$\Rightarrow \forall M \geq 0,\ \exists\, x_M \in [a,b]$ s.t. $|f(x_M)| > M$

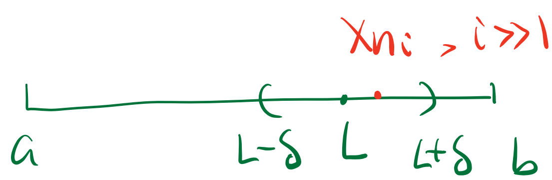
$\Rightarrow$ For $M = n,\ \exists\, x_n \in [a,b]$ s.t. $|f(x_n)| > n$

([a,b] is compact) $\Rightarrow \{x_n\}$ has a converging subsequence

$x_{n_i} \to L \in [a,b]$

$|f(x_{n_i})| > n_i \geq i \Rightarrow \lim\limits_{i \to \infty} |f(x_{n_i})| = +\infty$

Contradiction with Locally bounded at L.



Theorem (Boundeness Thm) $f(x)$ is continuous on a compact interval $I \Rightarrow f(x)$ is bounded on I.

Remark/Theorem: An interval is (sequentially) compact if and only if it's finite and closed.

Proof: ① Show it's bounded above by contradiction

② Show it's bounded below by contradiction

Let I be this compact interval

① Assume it's not bounded above.

$\Rightarrow \forall M, \exists x_M \in I$ s.t. $f(x_M) > M$

$\Rightarrow$ For $M = n$, $\exists x_n \in I$, $f(x_n) > n$

I is compact $\Rightarrow \exists x_{n_i} \to L \in I$

$f(x)$ is cont. $\Rightarrow \lim f(x_{n_i}) \to f(L)$

$f(x_n) > n \Rightarrow f(x_{n_i}) > n_i \geq i$

$\Rightarrow \lim\limits_{i \to \infty} f(x_{n_i}) = +\infty$

Contradiction with $f(L)$ being a real number.

Theorem (Maximum Theorem) $f(x)$ is continuous on a compact interval I

$\Rightarrow \max\limits_{x \in I} f(x)$ and $\min\limits_{x \in I} f(x)$ exist,

which means that $\exists x^*, x_* \in I$ s.t.

$f(x^*) = \sup\limits_{x \in I} f(x)$, $f(x_*) = \inf\limits_{x \in I} f(x)$

Remark: $\sup_{x \in I} f(x) = \sup \{ f(x) : x \in I \}$

Example: 1) $\frac{1}{x}$ has no max/min on $(0,1)$ or $(1,+\infty)$

2) $\frac{1}{x}$ has max/min on $[1,2]$

Proof: We only prove max since it's similar for min.

$\left. \begin{array}{r} f(x) \text{ is continuous on a compact interval } I \\ \text{Boundedness Theorem} \end{array} \right\}$

$\Rightarrow \exists M$ s.t. $f(x) \leq M, \ \forall x \in I$

$\Rightarrow \{ f(x) : x \in I \}$ has one upper bound

(Completeness for Set) $\Rightarrow \sup \{ f(x) : x \in I \}$ exists

$\Rightarrow \sup_{x \in I} f(x) = K$ exists

Claim that $\forall n, \ \exists x_n \in I$ s.t. $f(x_n) > K - \frac{1}{n}$

(Proof of claim: assume not, then

$\exists n, \ \forall x \in I, \ f(x) \leq K - \frac{1}{n}$

$\Rightarrow K - \frac{1}{n}$ is one upper bound of $\{ f(x) : x \in I \}$

Contradiction with $\sup \{ f(x) : x \in I \} = K$.)

$\left.\begin{array}{l} x_n \in I \\ I \text{ is compact} \end{array}\right\} \Rightarrow$ There is a subsequence $x_{n_i} \to L \in I$

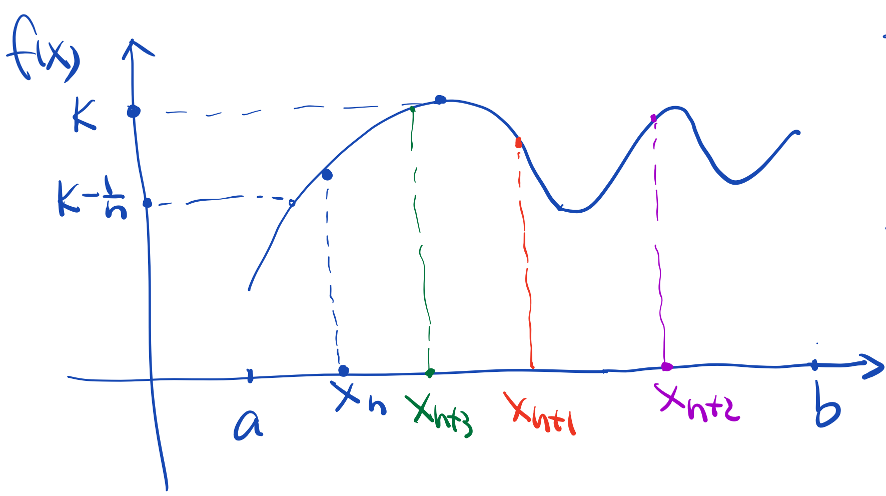
$$K - \frac{1}{n_i} < f(x_{n_i}) \le K$$

Squeeze Thm $\Rightarrow$ $f(x_{n_i}) \to K$

$f(x)$ is cont. $\Rightarrow$ $f(x_{n_i}) \to f(L)$

(Uniqueness of Limit) $\Rightarrow$ $f(L) = K = \sup_{x \in I} f(x)$

$\Rightarrow$ $\max_{x \in I} f(x) = f(L)$



$f(x_n) \to K$

$x_{n_i} \to L$

$f(x_{n_i}) \to K$

$f(x_{n_i}) \to f(L)$

<u>Theorem</u> $f(x)$ is continuous on a compact interval I, then $f(I)$ is a compact interval

where $f(I) = \{f(x) : x \in I\}$

Proof: By Max Thm, $\begin{cases} M = \max_{x \in I} f(x) \\ m = \min_{x \in I} f(x) \end{cases}$ exist.

Claim $f(I) = [m, M]$.

Assume $f(x^*) = M$
$f(x_*) = m$, $x^*, x_* \in I$.

WLOG, assume $x_* < x^*$.

Intermediate Value Theorem

$\Rightarrow \forall L \in [m, M], \exists x \in [x_*, x^*]$ s.t. $f(x) = L$.

Uniform Continuity on some interval

① $f(x)$ is continuous on (a, b):

δ depends on x_0, ε.

$\forall x_0 \in (a, b), \forall \varepsilon > 0, \exists \delta > 0$ s.t.

$\forall x \in (x_0 - \delta, x_0 + \delta) \cap (a, b), |f(x) - f(x_0)| < \varepsilon$.

② Def Uniform Continuity

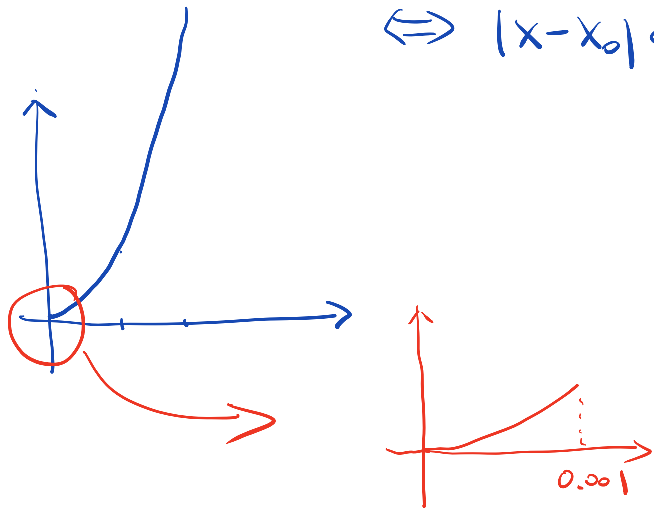
$f(x)$ is uniformly continuous on (a, b) if

$\forall x_0 \in (a, b), \forall \varepsilon > 0$, there is $\delta > 0$ which only depends on ε s.t.

$\forall x \in (x_0-\delta, x_0+\delta)\cap(a,b)$, $|f(x)-f(x_0)|<\varepsilon$.

③ Example: $f(x)=x^2$ is not uniformly continuous on $[0,+\infty)$

$$\forall \varepsilon>0,\ |x^2-x_0^2|<\varepsilon \Leftrightarrow |x-x_0|\cdot|x+x_0|<\varepsilon$$
$$\Leftrightarrow |x-x_0|<\frac{\varepsilon}{|x+x_0|}=\delta$$



③ Example: $f(x)=x^2$ is uniformly continuous on $[1,2]$

$$\forall \varepsilon>0,\ |x^2-x_0^2|<\varepsilon \Leftrightarrow |x-x_0|\cdot|x+x_0|<\varepsilon$$
$$\Leftrightarrow |x-x_0|<\frac{\varepsilon}{|x+x_0|}=\delta$$
$$\Uparrow$$

$\left.\begin{array}{l} x_0\in[1,2] \\ x\in[1,2] \end{array}\right\} \Rightarrow \frac{1}{4}\leq\frac{1}{x+x_0}\leq\frac{1}{2}$ $\quad |x-x_0|<\frac{\varepsilon}{2}$ or $\frac{\varepsilon}{4}$?

(*) $\left.\begin{array}{l} |x-x_0|<\frac{\varepsilon}{4} \\ \frac{1}{4}\leq\frac{1}{x+x_0} \end{array}\right\} \Rightarrow |x-x_0|<\frac{\varepsilon}{4}\leq\frac{\varepsilon}{|x+x_0|}$

$\Rightarrow |x^2-x_0^2|<\varepsilon$

$\forall x_0 \in [1,2]$, $\forall \varepsilon > 0$, $\exists \delta = \frac{\varepsilon}{4}$, s.t.

$\forall x \in (x_0 - \delta, x_0 + \delta) \cap [1,2]$, $|x^2 - x_0^2| < \varepsilon$ by (*).