- Given $A \in \mathbb{R}^{n\times n}$, A_{ij} denotes its (i,j)-th entry, $\tilde{A}_{ij}$ denotes the cofactor matrix of A_{ij}, obtained by deleting the row and column containing A_{ij}

Ex: $A = \begin{pmatrix} 1 & -1 & 2 & 3 \\ 3 & 4 & \boxed{1} & 2 \\ 2 & -5 & -3 & 8 \\ -6 & 2 & -4 & 1 \end{pmatrix}$ $\quad A_{23} = 1$

The cofactor matrix of A_{23} is $\underbrace{\begin{pmatrix} 1 & -1 & 3 \\ 2 & -5 & 8 \\ -6 & 2 & 1 \end{pmatrix}}_{\tilde{A}_{23}}$

- Definition of det by cofactor expansion

① $A \in \mathbb{R}^{1\times 1}$ $\quad A = [a]$, $\quad \det(A) = a$.

② $A \in \mathbb{R}^{2\times 2}$ $\quad A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$$\det(A) = |A| = (-1)^{1+1} \cdot a \cdot d + (-1)^{1+2} \cdot c \cdot b = ad - bc$$

③ $A \in \mathbb{R}^{3\times 3}$ $\quad A = \begin{bmatrix} a & b & c \\ d & e & f \\ i & g & h \end{bmatrix}$

$$\det(A) = |A| = (-1)^{1+2} \cdot b \cdot \begin{vmatrix} d & f \\ i & h \end{vmatrix} + (-1)^{2+2} \cdot e \cdot \begin{vmatrix} a & c \\ i & h \end{vmatrix} + (-1)^{3+2} \cdot g \cdot \begin{vmatrix} a & c \\ d & f \end{vmatrix}$$

$$= -b\cdot(dh-if)$$
$$+\,e\,(ah-ic)$$
$$-\,g\,(af-cd)$$

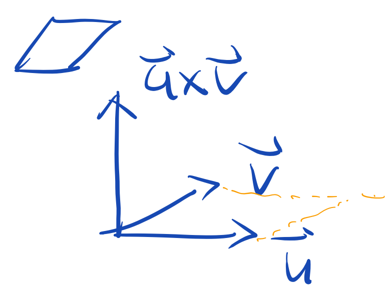
Example: The cross product of $\vec{u}=\begin{pmatrix}a\\b\\c\end{pmatrix}$, $\vec{v}=\begin{pmatrix}d\\e\\f\end{pmatrix}$

is $\vec{u}\times\vec{v}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ a & b & c\\ d & e & f\end{vmatrix}$

$$= (-1)^{1+1}\cdot\vec{i}\cdot\begin{vmatrix}b & c\\ e & f\end{vmatrix} \qquad \vec{i}=\begin{pmatrix}1\\0\\0\end{pmatrix}$$
$$+ (-1)^{1+2}\cdot\vec{j}\cdot\begin{vmatrix}a & c\\ d & f\end{vmatrix} \qquad \vec{j}=\begin{pmatrix}0\\1\\0\end{pmatrix}$$
$$+ (-1)^{1+3}\cdot\vec{k}\cdot\begin{vmatrix}a & b\\ d & e\end{vmatrix} \qquad \vec{k}=\begin{pmatrix}0\\0\\1\end{pmatrix}$$
$$= \vec{i}\cdot(bf-ce)-\vec{j}\,(af-cd)+\vec{k}\,(ae-bd)$$

① $\|\vec{u}\times\vec{v}\| = \|\vec{u}\|\cdot\|\vec{v}\|\cdot\sin\theta$

is the area of



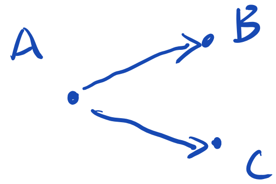
② $\vec{u}\times\vec{v}$ is $90°$ to both $\vec{u}$ and $\vec{v}$

Ex: A $(1, 1, 1)$

B $(2, 1, 1)$

C $(1, 2, 3)$

Find area of $\triangle ABC$.



$\vec{u} = \overrightarrow{AB} = B - A = \begin{pmatrix}1\\0\\0\end{pmatrix}$

$\vec{v} = \overrightarrow{AC} = C - A = \begin{pmatrix}0\\1\\2\end{pmatrix}$

Area of $\triangle = \frac{1}{2}\|\vec{u} \times \vec{v}\|$

Example: $\vec{u} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$, $\vec{v} = \begin{pmatrix} d \\ e \\ f \end{pmatrix}$, $\vec{w} = \begin{pmatrix} i \\ g \\ h \end{pmatrix}$

The tripple product is $\vec{u} \times \vec{v} \cdot \vec{w}$

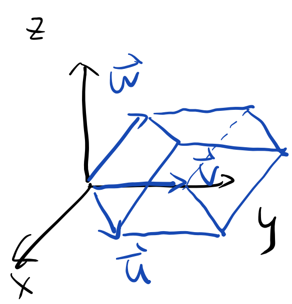
same

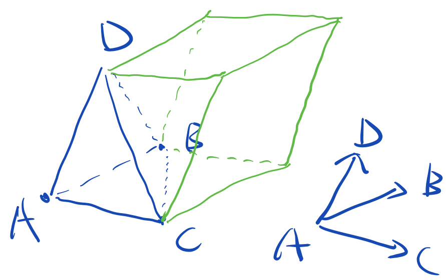
① First compute $\vec{u} \times \vec{v}$

Then compute $(\vec{u} \times \vec{v}) \cdot \vec{w}$

② $\vec{u} \times \vec{v} \cdot \vec{w} = \begin{vmatrix} a & b & c \\ d & e & f \\ i & g & h \end{vmatrix} = \begin{vmatrix} a & d & i \\ b & e & g \\ c & f & h \end{vmatrix}$

③ $|\vec{u} \times \vec{v} \cdot \vec{w}|$ is volume of parallelipiped





- $A \in \mathbb{R}^{4\times 4}$, $\det(A)$ is defined by cofactor expansion.

$$\det\begin{pmatrix} 1 & -1 & 2 & 3 \\ 3 & 4 & 1 & 2 \\ 2 & -5 & -3 & 8 \\ -6 & 2 & -4 & 1 \end{pmatrix} \overset{\text{(2nd row)}}{=} (-1)^{2+1} \cdot \boxed{3} \cdot \begin{vmatrix} -1 & 2 & 3 \\ -5 & -3 & 8 \\ 2 & -4 & 1 \end{vmatrix}$$

$$+ (-1)^{2+2} \cdot \boxed{4} \cdot \begin{vmatrix} 1 & 2 & 3 \\ 2 & -3 & 8 \\ -6 & -4 & 1 \end{vmatrix}$$

$$+(-1)^{2+3}\cdot\boxed{1}\cdot\begin{vmatrix}1&-1&3\\2&-5&8\\-6&2&1\end{vmatrix}$$

$$+(-1)^{2+4}\cdot\boxed{2}\cdot\begin{vmatrix}1&-1&2\\2&-5&-3\\-6&2&-4\end{vmatrix}$$

- Row/Col Ops

① Type I will change sign

$$\begin{vmatrix}a&b&c\\d&e&f\\i&g&h\end{vmatrix}=-\begin{vmatrix}a&c&b\\d&f&e\\i&h&g\end{vmatrix}$$

② Type II : factor a number out of any row/col

$$\begin{vmatrix}2&4&6\\1&2&3\\4&5&6\end{vmatrix}=2\begin{vmatrix}1&2&3\\1&2&3\\4&5&6\end{vmatrix}\qquad\begin{bmatrix}2&4&6\\2&2&2\\4&4&4\end{bmatrix}=2\begin{bmatrix}1&2&3\\1&1&1\\2&2&2\end{bmatrix}$$

$$\begin{vmatrix}2&4&6\\2&2&2\\4&4&4\end{vmatrix}=8\begin{vmatrix}1&2&3\\1&1&1\\2&2&2\end{vmatrix}$$

③ Type III : does not change det

Example:

$$(C1\cdot(-\tfrac{1}{2})+C4\to C4)$$

$$\begin{vmatrix}2&0&0&1\\0&1&3&-3\\-2&-3&-5&2\\4&-4&4&-6\end{vmatrix}=\begin{vmatrix}2&0&0&0\\0&1&3&-3\\-2&-3&-5&3\\4&-4&4&-8\end{vmatrix}$$

$$= 4 \begin{vmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 3 & -3 \\ -2 & -3 & -5 & 3 \\ 1 & -1 & 1 & -2 \end{vmatrix} = 8 \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 3 & -3 \\ -2 & -3 & -5 & 3 \\ 1 & -1 & 1 & -2 \end{vmatrix}$$

(cofactor expansion along first row)

$$= 8 \cdot (-1)^{1+1} \cdot 1 \cdot \begin{vmatrix} 1 & 3 & -3 \\ -3 & -5 & 3 \\ -1 & 1 & -2 \end{vmatrix}$$

$$= 8 \begin{vmatrix} 1 & 3 & -3 \\ -3 & -5 & 3 \\ -1 & 1 & -2 \end{vmatrix}$$

$$= 8 \begin{vmatrix} 1 & 3 & -3 \\ 0 & 4 & -6 \\ 0 & 4 & -5 \end{vmatrix}$$

$$= 8 \cdot (-1)^{1+1} \cdot 1 \cdot \begin{vmatrix} 4 & -6 \\ 4 & -5 \end{vmatrix}$$

$$= 8 \cdot (-20 + 24) = 32.$$

- Quick facts about det

① $\det(I) = 1$

② zero row/col $\Rightarrow$ $\det = 0$

③ same row/col $\Rightarrow$ $\det = 0$

④ $\det(A^T) = \det(A)$

⑤ $\det(AB) = \det(A)\det(B)$

⑥ $\det(A^{-1}) = \dfrac{1}{\det(A)}$

⑦ A is invertible $\Leftrightarrow \det(A) \neq 0$

Proof: We can turn A into RREF(A) by row ops.

Row/Col ops do not change det to zero.

So $\det(A) = 0 \Leftrightarrow \det(\text{RREF}) = 0$.

$\Updownarrow$

RREF has zero row

$\Updownarrow$

$\text{rank}(A) < n$.

$\Updownarrow$

A is not invertible.

- Inverse Matrix and det.

Let $C_{ij} = (-1)^{i+j} \cdot |\tilde{A}_{ij}|$

If $A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$ is invertible, then

$$A^{-1} = \frac{1}{|A|} \begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix}$$

$$\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} \begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix} = \begin{bmatrix} \boxed{1} & \boxed{4} & \square \\ \square & \boxed{2} & \square \\ \square & \square & \boxed{3} \end{bmatrix}$$

$$\begin{aligned} 1 &= A_{11} \cdot C_{11} + A_{12} \cdot C_{12} + A_{13} \cdot C_{13} \\ &= A_{11} \cdot (-1)^{1+1} \cdot |\tilde{A}_{11}| \\ &\quad + A_{12} \cdot (-1)^{1+2} \cdot |\tilde{A}_{12}| \\ &\quad + A_{13} \cdot (-1)^{1+3} |\tilde{A}_{13}| = \det(A) \end{aligned}$$

$$④ = A_{11} \cdot C_{21} + A_{12} \cdot C_{22} + A_{13} \cdot C_{23}$$

$$= A_{11} \cdot (-1)^{2+1} \cdot |\tilde{A}_{21}| + A_{12} \cdot (-1)^{2+2} \cdot |\tilde{A}_{22}| + A_{13} \cdot (-1)^{2+3} \cdot |\tilde{A}_{23}|$$

$$\boxed{A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}}$$

$$= \det \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{11} & A_{12} & A_{13} \\ A_{31} & A_{32} & A_{33} \end{pmatrix} = 0$$

- Linear System Sol and det (Cramer's Rule)

$$A\vec{x} = \vec{b}$$

$$\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

If $\det(A) \neq 0$, then

$$\vec{x} = A^{-1}\vec{b}$$

$$= \frac{1}{|A|} \begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$$\Rightarrow \quad x_1 = \frac{1}{|A|}\left(C_{11}\cdot b_1 + C_{21}\cdot b_2 + C_{31}\cdot b_3 \right)$$

$$= \frac{1}{|A|}\left[b_1 \cdot (-1)^{1+1} \cdot |\widetilde{A_{11}}| + b_2 \cdot (-1)^{2+1} \cdot |\widetilde{A_{21}}| + b_3 \cdot (-1)^{3+1} \cdot |\widetilde{A_{31}}| \right]$$

$$\boxed{A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}}$$

$$x_1 = \frac{1}{|A|} \cdot \begin{vmatrix} b_1 & A_{12} & A_{13} \\ b_2 & A_{22} & A_{23} \\ b_3 & A_{32} & A_{33} \end{vmatrix}$$