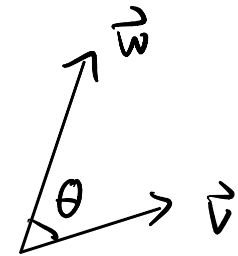
$\|c\vec{v}\| = |c| \cdot \|\vec{v}\|$

- $\dfrac{\vec{v}}{\|\vec{v}\|} = \dfrac{1}{\|\vec{v}\|} \cdot \vec{v}$ is a unit vector (in dir of $\vec{v}$) if $\vec{v} \neq \vec{0}$.




<u>Formula</u>: $\vec{v} \cdot \vec{w} = \|\vec{v}\| \|\vec{w}\| \cos\theta$

$0 \le \theta \le \pi$

<u>Ex</u>: $\vec{v} \cdot \vec{v} = \|\vec{v}\|^2$

$\theta = 0 \Rightarrow \cos\theta = 1$

- $\vec{v}, \vec{w} \neq \vec{0}$. $\vec{v} \cdot \vec{w} = 0 \Leftrightarrow \cos\theta = 0 \Leftrightarrow \theta = \frac{\pi}{2}$

$\vec{v}$ & $\vec{w}$ are perpendicular (orthogonal) $\Leftrightarrow \vec{v} \cdot \vec{w} = 0$

Notation: $\vec{v} \perp \vec{w}$

- Cauchy-Schwartz inequality.

$$\underline{|\vec{v}\cdot\vec{w}|} = \|\vec{v}\|\,\|\vec{w}\|\,|\cos\theta| \leq \underline{\|\vec{v}\|\,\|\vec{w}\|}$$

$$-1 \leq \cos\theta \leq 1 \qquad \leq 1$$

In dim 3

$$|v_1w_1 + v_2w_2 + v_3w_3| \leq \sqrt{v_1^2+v_2^2+v_3^2}\cdot\sqrt{w_1^2+w_2^2+w_3^2}$$

## Matrix

$A$ is matrix of size $m \times n$

$$A = \underbrace{\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}}_{n \text{ columns.}} \Bigg\} \ m \text{ rows}$$

entries.

$1 \le i \le m$

$i$th row

$$a_{ij} = A(i,j) = A_{ij}$$

$j$ th column

$1 \le j \le n$

$A$ has $mn$ entries

$m$ rows

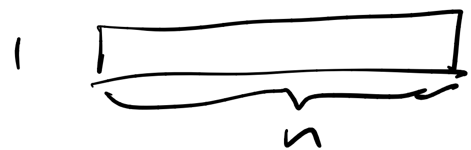
$n$ columns

- square matrix $m = n$

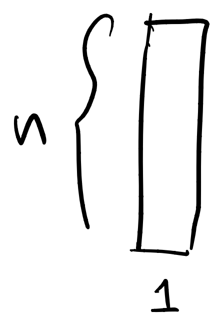
$$B = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \qquad 3\times 3 \text{ matrix.}$$

$$B_{23} = 6 \qquad B_{31} = 7.$$

- A row vector of dim $n$ is a matrix of size $1 \times n$




- A column vector of dim $n$ is — — — — $n \times 1$



# Matrix - Vector multiplication

$$\begin{pmatrix} \boxed{\begin{matrix}1\\-1\\0\end{matrix}} & \boxed{\begin{matrix}0\\1\\-1\end{matrix}} & \boxed{\begin{matrix}0\\0\\1\end{matrix}} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = ?$$

def 1

def 2

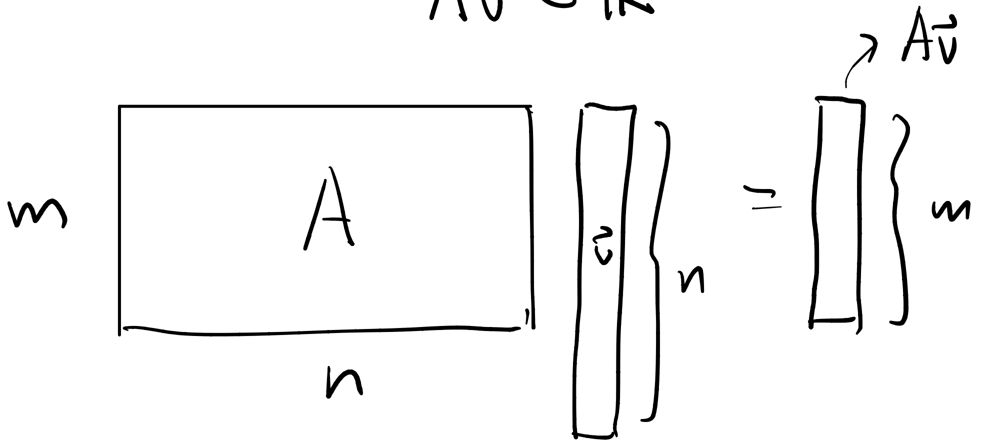
① linear combinations

$$x\begin{pmatrix}1\\-1\\0\end{pmatrix} + y\begin{pmatrix}0\\1\\-1\end{pmatrix} + z\begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}x\\-x+y\\-y+z\end{pmatrix}$$

② dot product.

$$\begin{pmatrix} \boxed{1 \quad 0 \quad 0} \\ \boxed{-1 \quad 1 \quad 0} \\ \boxed{0 \quad -1 \quad 1} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} (1,0,0)\cdot\begin{pmatrix}x\\y\\z\end{pmatrix} \\ (-1,1,0)\cdot\begin{pmatrix}x\\y\\z\end{pmatrix} \\ (0,-1,1)\begin{pmatrix}x\\y\\z\end{pmatrix} \end{pmatrix} = \begin{pmatrix}x\\-x+y\\-y+z\end{pmatrix}$$

In general, $A$ of size $m \times n$, $\vec{v} \in \mathbb{R}^n$, $A\vec{v} \in \mathbb{R}^m$




Ex: $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix} = v_1 \begin{pmatrix} 1 \\ 5 \end{pmatrix} + v_2 \begin{pmatrix} 2 \\ 6 \end{pmatrix} + v_3 \begin{pmatrix} 3 \\ 7 \end{pmatrix} + v_4 \begin{pmatrix} 4 \\ 8 \end{pmatrix}$

$$= \begin{pmatrix} v_1 + 2v_2 + 3v_3 + 4v_4 \\ 5v_1 + 6v_2 + 7v_3 + 8v_4 \end{pmatrix}$$

Ex: $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = 1\cdot\begin{pmatrix} a \\ c \end{pmatrix} + 0\cdot\begin{pmatrix} b \\ d \end{pmatrix} = \begin{pmatrix} a \\ c \end{pmatrix}$

Ex: $\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}\begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

The identity matrix of size $n$ ($n\times n$ square matrix)

| $n =$ | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| $I_n$ | $(1)$ | $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ | $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ | $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ |

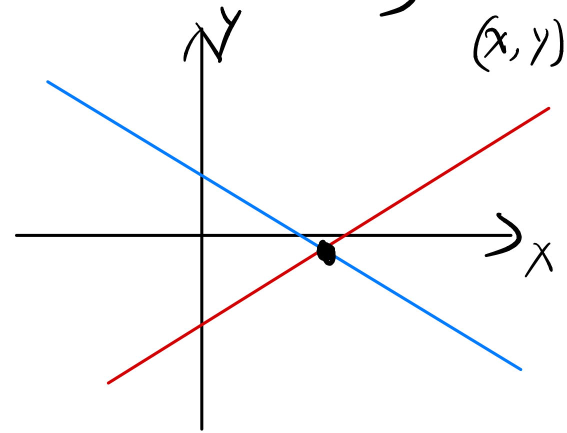
Ex: $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

prop: $I_n \vec{x} = \vec{x}$, $\vec{x} \in \mathbb{R}^n$

---

linear system (= system of linear equations)

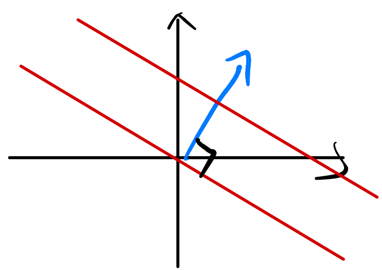
$$\begin{cases} 2x + 3y = 1 \\ x - 2y = 1 \end{cases}$$




$$y = \frac{1-2x}{3} = -\frac{2}{3}x + \frac{1}{3}$$

$2x + 3y = 0$

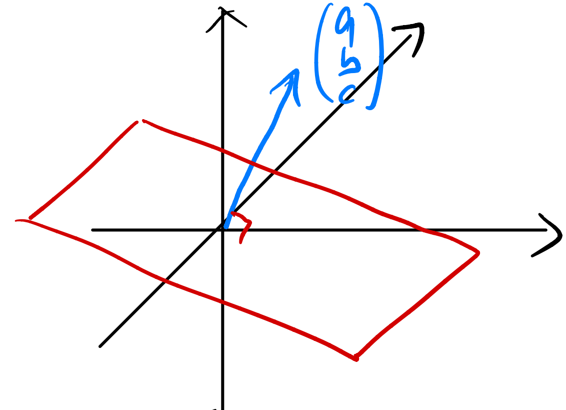
$\begin{pmatrix} 2 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix}$



in $\mathbb{R}^3$

$ax + by + cz = 0$

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$




geometrically: solving linear system

$\Longleftrightarrow$ finding the intersection of lines/planes/...

next:
$$\begin{cases} x + 2y + 3z = 1 \\ 3x + y + z = 0 \\ x + y + z = 0 \end{cases} \xrightarrow[\text{form}]{\text{matrix}} A \cdot \vec{x} = \vec{b}$$

coef matrix
$$A = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \qquad \vec{b} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \qquad \vec{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$