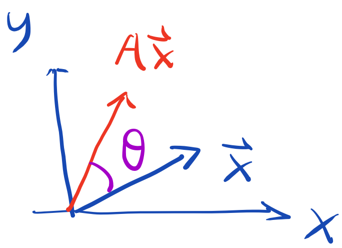
Chapter 6 Eigenvalues & Eigenvectors

- Any matrix $A \in \mathbb{R}^{n\times n}$ corresponds to a mapping L_A from $\mathbb{R}^n$ to $\mathbb{R}^n$:

$$L_A : \mathbb{R}^n \longrightarrow \mathbb{R}^n$$
$$\vec{x} \longmapsto A\vec{x}$$

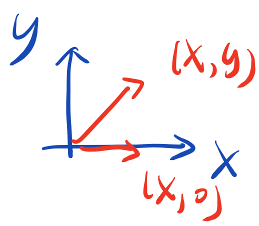
Example: ① $A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$



$$L_A : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$
$$\begin{pmatrix} x \\ y \end{pmatrix} \longmapsto A\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos\theta\, x - \sin\theta\, y \\ \sin\theta\, x + \cos\theta\, y \end{pmatrix}$$

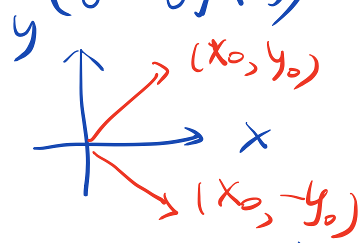
Rotation of angle θ counterclockwise

② $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$



$$L_A : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$
$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto A\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ 0 \end{pmatrix}$$

③ $A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$



$$L_A : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$
$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto A\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ -y \end{pmatrix}$$

- Question: For a square matrix $A \in \mathbb{R}^{n\times n}$, is there a nonzero vector $\vec{v}$ s.t. $A\vec{v}$ is parallel to $\vec{v}$?

Which direction the mapping L_A does NOT change?

- Def (Eigenvalues & Eigenvectors)

For $A \in \mathbb{R}^{n\times n}$, $\vec{v} \in \mathbb{R}^n$ or $\vec{v} \in \mathbb{C}^n$ is called an eigen-vector of A if $A\vec{v} = \lambda \vec{v}$ for some number λ and $\vec{v} \neq \vec{0}$.

The number λ is called eigen-value of A, associated with the eigen-vector $\vec{v}$.

Example: ① If $\begin{cases} A\vec{v} = \vec{0} \\ \vec{v} \neq \vec{0} \end{cases}$, $\vec{v}$ is an eigenvector for eigenvalue 0.

$A\vec{v} = 0 \cdot \vec{v}$

② $\underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}}_{A} \cdot \underbrace{\begin{pmatrix} x \\ 0 \end{pmatrix}}_{\vec{v}} = \begin{pmatrix} x \\ 0 \end{pmatrix} = \underbrace{1}_{\lambda} \cdot \underbrace{\begin{pmatrix} x \\ 0 \end{pmatrix}}_{\vec{v}}$ $\quad x \neq 0$

- How to find Eigenvalues:

$$A\vec{v} = \lambda \vec{v} \iff A\vec{v} - \lambda\vec{v} = \vec{0}$$
$$\iff A\vec{v} - \lambda I \vec{v} = \vec{0}$$
$$\iff (A - \lambda I)\vec{v} = \vec{0}$$

$\Longleftrightarrow (A-\lambda I)\vec{x}=\vec{0}$ has a nonzero sol.

$\Longleftrightarrow A-\lambda I$ is singular (not invertible)

$\Longleftrightarrow |A-\lambda I|=0$

$\det(A-\lambda I)$ is a polynomial of λ

Example: $A=\begin{pmatrix} 5 & 2 & 0 \\ 2 & 5 & 0 \\ -3 & 4 & 6 \end{pmatrix}$

$$\det(A-\lambda I) = \det\left[\begin{pmatrix} 5 & 2 & 0 \\ 2 & 5 & 0 \\ -3 & 4 & 6 \end{pmatrix} - \lambda\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right]$$

$$= \begin{vmatrix} 5-\lambda & 2 & 0 \\ 2 & 5-\lambda & 0 \\ -3 & 4 & 6-\lambda \end{vmatrix}$$

$$= (6-\lambda)\cdot(-1)^{3+3}\cdot\begin{vmatrix} 5-\lambda & 2 \\ 2 & 5-\lambda \end{vmatrix}$$

$$= (6-\lambda)\cdot\left[(5-\lambda)^2-4\right]$$

$$= (6-\lambda)(\lambda^2-10\lambda+25-4)$$

$$= -(\lambda-6)(\lambda^2-10\lambda+21)$$

$$= -(\lambda-6)(\lambda-7)(\lambda-3) = 0$$

<u>Def</u> $\det(A-\lambda I)$ is a polynomial of degree n, called characteristic polynomial of $A\in\mathbb{R}^{n\times n}$

Its roots are eigenvalues of A.

- How to find eigenvectors:

If λ is known, then solve $\vec{v}$ in $(A-\lambda I)\vec{v}=\vec{0}$.

Example: $A = \begin{pmatrix} 5 & 2 & 0 \\ 2 & 5 & 0 \\ -3 & 4 & 6 \end{pmatrix}$

$$|A-\lambda I| = -(\lambda-6)(\lambda-7)(\lambda-3) = 0$$

$$\Rightarrow \lambda_1 = 6, \quad \lambda_2 = 7, \quad \lambda_3 = 3$$

① Plug in $\lambda = 6$ in $(A-\lambda I)\vec{v} = \vec{0}$

$$\begin{pmatrix} 5-\lambda & 2 & 0 \\ 2 & 5-\lambda & 0 \\ -3 & 4 & 6-\lambda \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -1 & 2 & 0 \\ 2 & -1 & 0 \\ -3 & 4 & 0 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\left(\begin{array}{ccc|c} -1 & 2 & 0 & 0 \\ 2 & -1 & 0 & 0 \\ -3 & 4 & 0 & 0 \end{array}\right)$$

$(A-\lambda I)\vec{v} = \vec{0}$

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$\Rightarrow$ Sol is $\vec{v} = t\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad \forall t \in \mathbb{R}$

So $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ is an eigenvector

$\text{Span}\left\{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right\}$ is called Eigen Space for eigenvalue $\lambda_1 = 6$.

$t\begin{pmatrix}0\\0\\1\end{pmatrix}$ is an eigenvector if $t \neq 0$.

Def The subspace consisting of all eigenvectors for a particular eigenvalue λ and $\vec{0}$ is called eigenspace of λ.

Example: The eigenspace for $\lambda = 6$ is $\operatorname{Span}\left\{\begin{bmatrix}0\\0\\1\end{bmatrix}\right\}$.

T or F: $\operatorname{Null}(A - \lambda I)$ is the eigenspace of A for eigenvalue λ.

$\lambda_1 = 6$, $\lambda_2 = 7$, $\lambda_3 = 3$

② Plug in $\lambda = 7$ in $(A - \lambda I)\vec{v} = \vec{0}$

$$\begin{pmatrix} 5-\lambda & 2 & 0 \\ 2 & 5-\lambda & 0 \\ -3 & 4 & 6-\lambda \end{pmatrix}\begin{pmatrix}v_1\\v_2\\v_3\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

$$\begin{pmatrix} -2 & 2 & 0 \\ 2 & -2 & 0 \\ -3 & 4 & -1 \end{pmatrix}\begin{pmatrix}v_1\\v_2\\v_3\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\Rightarrow \vec{v} = t\begin{pmatrix}1\\1\\1\end{pmatrix}$, $\forall t \in \mathbb{R}$

$\Rightarrow$ The eigenspace for $\lambda_2 = 7$ is $\operatorname{Span}\left\{\begin{bmatrix}1\\1\\1\end{bmatrix}\right\}$

Verification: $\overset{A\cdot\vec{v}}{\begin{pmatrix}5&2&0\\2&5&0\\-3&4&6\end{pmatrix}\begin{pmatrix}1\\1\\1\end{pmatrix}} \overset{=}{=} \overset{\lambda\cdot\vec{v}}{\begin{pmatrix}7\\7\\7\end{pmatrix}} = 7\cdot\begin{pmatrix}1\\1\\1\end{pmatrix}$

③ Plug in $\lambda = 3$ in $(A-\lambda I)\vec{v} = \vec{0}$

$$\begin{pmatrix}5-\lambda&2&0\\2&5-\lambda&0\\-3&4&6-\lambda\end{pmatrix}\begin{pmatrix}v_1\\v_2\\v_3\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

$$\begin{pmatrix}2&2&0\\2&2&0\\-3&4&3\end{pmatrix}\begin{pmatrix}v_1\\v_2\\v_3\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

$$\left[\begin{array}{ccc|c}1&0&-3/7&0\\0&1&3/7&0\\0&0&0&0\end{array}\right]$$

$$\Rightarrow \vec{v} = t\begin{pmatrix}3/7\\-3/7\\1\end{pmatrix}, \quad \forall\, t\in\mathbb{R}$$

$t=7 \Rightarrow \begin{pmatrix}3\\-3\\7\end{pmatrix}$ is one eigen-vector for $\lambda_3 = 3$

The eigen-space is $\operatorname{Span}\left\{\begin{bmatrix}3\\-3\\7\end{bmatrix}\right\}$.

Verification $\begin{pmatrix}5&2&0\\2&5&0\\-3&4&6\end{pmatrix}\begin{pmatrix}3\\-3\\7\end{pmatrix} = \begin{pmatrix}9\\-9\\21\end{pmatrix} = 3\cdot\begin{pmatrix}3\\-3\\7\end{pmatrix}$

Example: $A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$

$$|A - \lambda I| = \begin{vmatrix} \cos\theta - \lambda & -\sin\theta \\ \sin\theta & \cos\theta - \lambda \end{vmatrix}$$

$$= (\cos\theta - \lambda)^2 + \sin^2\theta$$

$$= \lambda^2 - 2\cos\theta\,\lambda + \cos^2\theta + \sin^2\theta$$

$$= \lambda^2 - 2\cos\theta\,\lambda + 1 = 0$$

$$\Delta = 4\cos^2\theta - 4\cdot 1\cdot 1 = 4\cos^2\theta - 4$$

$$= 4(\cos^2\theta - 1)$$

$$= -4\sin^2\theta$$