- Singular Value Decomposition (SVD)

$\boxed{\text{Theorem}}$ For any $A \in \mathbb{R}^{n\times n}$, there are $U, \Sigma, V \in \mathbb{R}^{n\times n}$

s.t. $A = U\Sigma V^T$

$$\boxed{A} = \boxed{U}\ \boxed{\begin{matrix}\sigma_1 & & 0\\ & \ddots & \\ 0 & & \sigma_n\end{matrix}}\ \boxed{V^T} \qquad \underset{U^T}{\boxed{\begin{matrix}\text{—}\\ \text{—}\\ \text{—}\end{matrix}}}\ \underset{U}{\boxed{|\,|\,|}} = I$$

$U^TU = I \qquad \sigma_i \geq 0 \qquad V^TV = I$

① $\Sigma = \begin{pmatrix}\sigma_1 & & 0\\ & \ddots & \\ 0 & & \sigma_n\end{pmatrix}$, $\sigma_i \geq 0$ are singular values

$$\sigma(A) = \sqrt{\lambda(A^TA)} = \sqrt{\lambda(AA^T)}$$

$\hookrightarrow$ eigenvalue

② Cols of U are orthonormal : left singular vectors

$$U^TU = UU^T = I$$

③ Cols of V are orthonormal : right singular vectors

$$V^TV = VV^T = I$$

④ $A = U\Sigma V^T$

$\Leftrightarrow A = U\Sigma V^{-1}$

$\Leftrightarrow AV = U\Sigma$

$\Leftrightarrow A[\vec{v}_1\ \vec{v}_2\ \vec{v}_3] = [\vec{u}_1\ \vec{u}_2\ \vec{u}_3]\begin{bmatrix}\sigma_1 & 0 & 0\\ 0 & \sigma_2 & 0\\ 0 & 0 & \sigma_3\end{bmatrix}$

$\Leftrightarrow [A\vec{v}_1\ A\vec{v}_2\ A\vec{v}_3] = [\sigma_1\vec{u}_1\ \sigma_2\vec{u}_2\ \sigma_3\vec{u}_3]$

$\Leftrightarrow A\vec{v}_i = \sigma_i\vec{u}_i$ The order must match

⑤ $A = U\Sigma V^T \Rightarrow AA^T = U\Sigma V^T V \Sigma U^T = U\Sigma^2 U^T$

Review: A is real positive definite if ① $A = A^T$ ② $\vec{x}^T A \vec{x} > 0$, $\forall \vec{x} \in \mathbb{R}^n$, $\vec{x} \neq \vec{0}$

• Theorem A real symmetric $A \in \mathbb{R}^{n\times n}$ is positive (semi-)definite if and only if all eigenvalues of A are positive. (non-negative)

$\Rightarrow \begin{cases} \vec{u}_i \text{ are orthonormal eigenvectors of } AA^T. \\ \sigma_i^2 \text{ are eigenvalues of } AA^T. \end{cases}$

$\vec{x}^T A^T A \vec{x} = (A\vec{x})^T A\vec{x} = \|A\vec{x}\|^2$

$A = U\Sigma V^T \Rightarrow A^TA = V\Sigma U^T U \Sigma V^T = V\Sigma^2 V^T$

$\Rightarrow \begin{cases} \vec{v}_i \text{ are orthonormal eigenvectors of } A^TA. \\ \sigma_i^2 \text{ are eigenvalues of } A^TA. \end{cases}$

And we have pick right order and signs of $\vec{u}_i, \vec{v}_i$ s.t.

$$A\vec{v}_i = \sigma_i \vec{u}_i$$

Example: $A = \begin{bmatrix} 2 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & -2 & 0 \end{bmatrix}$. Find its SVD.

① $A^TA = \begin{bmatrix} 2 & 2 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & -2 & 0 \end{bmatrix} = \begin{bmatrix} 8 & 2 & 0 \\ 2 & 5 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$\lambda(A^TA) = 9, 4, 0$

$\Rightarrow \sigma(A) = 3, 2, 0$

② Corresponding eigenvectors of A^TA are

$\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -\frac{1}{2} \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

Orthonormal eigenvectors $\left(\frac{\vec{v}_i}{\|\vec{v}_i\|}\right)$:

$$\frac{1}{\sqrt{5}}\begin{pmatrix}2\\1\\0\end{pmatrix},\quad \frac{2}{\sqrt{5}}\begin{pmatrix}-\frac{1}{2}\\1\\0\end{pmatrix},\quad \begin{pmatrix}0\\0\\1\end{pmatrix}$$

$$V = \begin{pmatrix}2/\sqrt{5} & -1/\sqrt{5} & 0\\ 1/\sqrt{5} & 2/\sqrt{5} & 0\\ 0 & 0 & 1\end{pmatrix}$$

$A\vec{v}_i = \sigma_i \vec{u}_i$

$\Leftrightarrow \vec{u}_i = \frac{A\vec{v}_i}{\sigma_i}$, $\sigma_i \neq 0$

③ $$AA^T = \begin{pmatrix}4 & 4 & 0\\ 4 & 5 & -2\\ 0 & -2 & 4\end{pmatrix}$$

$\lambda(AA^T) = 9, 4, 0$

eigenvectors: $\begin{pmatrix}-4\\-5\\2\end{pmatrix}$, $\begin{pmatrix}1\\0\\2\end{pmatrix}$, $\begin{pmatrix}-2\\2\\1\end{pmatrix}$

orthonormal eigen-vectors $\pm\frac{1}{3\sqrt{5}}\begin{pmatrix}4\\5\\-2\end{pmatrix}$, $\pm\frac{1}{\sqrt{5}}\begin{pmatrix}-1\\0\\-2\end{pmatrix}$, $\pm\frac{1}{3}\begin{pmatrix}-2\\2\\1\end{pmatrix}$

The sign is to be determined by $A\vec{v}_i = \sigma_i \vec{u}_i$

$$A\vec{v}_2 = \begin{bmatrix}2 & 0 & 0\\ 2 & 1 & 0\\ 0 & -2 & 0\end{bmatrix}\frac{2}{\sqrt{5}}\begin{bmatrix}-\frac{1}{2}\\1\\0\end{bmatrix} = \underset{\substack{\parallel\\2}}{\sigma_2}\cdot\frac{1}{\sqrt{5}}\begin{bmatrix}-1\\0\\-2\end{bmatrix}$$

$$A\vec{v}_3 = \begin{bmatrix}2 & 0 & 0\\ 2 & 1 & 0\\ 0 & -2 & 0\end{bmatrix}\begin{bmatrix}0\\0\\1\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix} = \underset{\substack{\parallel\\0}}{\sigma_3}\cdot(\pm)\frac{1}{3}\begin{pmatrix}-2\\2\\1\end{pmatrix}$$

$$\Rightarrow \quad U = \begin{pmatrix} \frac{4}{3\sqrt{5}} & \frac{-1}{\sqrt{5}} & -\frac{2}{3} \\ \frac{5}{3\sqrt{5}} & 0 & \frac{2}{3} \\ \frac{-2}{3\sqrt{5}} & \frac{-2}{\sqrt{5}} & \frac{1}{3} \end{pmatrix}$$

$$\Rightarrow \quad A = U \begin{pmatrix} 3 & & \\ & 2 & \\ & & 0 \end{pmatrix} V^T \iff \begin{cases} A\vec{v}_i = \sigma_i \vec{u}_i \\ U^T U = I \\ V^T V = I \end{cases}$$

Summary of How to Compute SVD:

Step I: Find eigenvalues of A^TA, denoted by $\lambda(A^TA)$

Singular values of A are $\sigma_i(A) = \sqrt{\lambda_i(A^TA)}$

Step II: Find orthonormal eigenvectors of A^TA,

and they are right singular vectors of A: $\vec{v}_i$

Step III: ① If $\sigma_i > 0$, then $\vec{u}_i = \dfrac{A\vec{v}_i}{\sigma_i}$

(Or

If all the eigenspaces of AA^T are one-dimensional,

we pick $+$ or $-$ s.t. $A\vec{v}_i = \sigma_i \vec{u}_i$)

② If $\sigma_i = 0$, find orthonormal eigenvectors of AA^T

s.t. $A\vec{v}_i = \sigma_i \vec{u}_i$

Linear Transformation (Chapter 8)

<u>Definition</u> Let V and W be two abstract vector spaces, then a mapping

$$T: V \longrightarrow W$$
$$\vec{v} \longmapsto T(\vec{v})$$

is called a Linear Transformation if

① $T(\vec{u}+\vec{v}) = T(\vec{u}) + T(\vec{v})$, $\forall \vec{u}, \vec{v} \in V$

② $T(a\vec{v}) = aT(\vec{v})$, $\forall \vec{v} \in V$, $\forall a \in \mathbb{R}$

<u>Remark</u>: $\left.\begin{matrix} ① \\ ② \end{matrix}\right\} \Longleftrightarrow T(a\vec{u}+b\vec{v}) = aT(\vec{u}) + bT(\vec{v})$

So a linear transformation preserves a linear combination

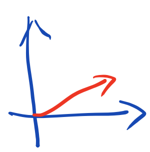
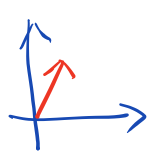
Remark: $T(a\vec{v}) = aT(\vec{v}) \Rightarrow T(\vec{0}) = \vec{0}$

Examples of linear transformation:

1) Given any matrix $A \in \mathbb{R}^{m\times n}$,

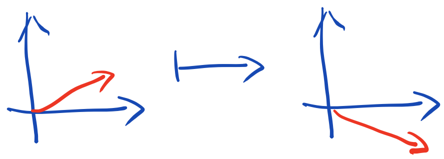
$$L_A: \mathbb{R}^n \longrightarrow \mathbb{R}^m \quad \text{is a linear transformation}$$
$$\vec{v} \longmapsto A\vec{v}$$

2) $T: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

 $\longmapsto$ 

Rotation by 30° is a linear transformation

3) $T: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$



Reflection w.r.t. x-axis is a linear transformation

4) $T: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

$\begin{pmatrix} x \\ y \end{pmatrix} \longmapsto \begin{pmatrix} x+2 \\ y \end{pmatrix}$ is NOT a linear transformation

because $T(\vec{0}) \neq \vec{0}$

5) Let $P_n(\mathbb{R})$ denote all single variable polynomials of degree at most n with real number coefficients

then consider differentiation and integration:

① $\frac{d}{dx}: P_3(\mathbb{R}) \longrightarrow P_2(\mathbb{R})$

$ax^3+bx^2+cx+d \longmapsto 3ax^2+2bx+c$

is a linear transformation

② $\int_0^x p(t)\,dt : P_2(\mathbb{R}) \longrightarrow P_3(\mathbb{R})$

$p(x)=ax^2+bx+c \longrightarrow \frac{a}{3}x^3+\frac{b}{2}x^2+cx$

is a linear transformation

5) Let V be the set

$$V = \{ a\sin(mx) + b\cos(nx),\ \forall m, n \in \underbrace{\mathbb{Z}}_{\text{all integers}},\ \forall a, b \in \mathbb{R} \}$$

Then V is an abstract vector space

$$\frac{d}{dx} : V \longrightarrow V$$
$$f(x) \longmapsto f'(x)$$

is a linear transformation

V is an infinite-dimensional vector space.

Consider two finite dimensional vector spaces

V with basis $\{\vec{v}_1, \cdots, \vec{v}_n\}$

and W with basis $\{\vec{w}_1, \cdots, \vec{w}_m\}$

Let T be a linear transformation

$$T : V \longrightarrow W$$
$$\vec{v} \longmapsto T(\vec{v})$$

Any $\vec{v} \in V$ can be written as

$$\vec{v} = a_1 \vec{v}_1 + a_2 \vec{v}_2 + \cdots + a_n \vec{v}_n$$

$\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$ is called coordinates of $\vec{v}$ under basis $\{\vec{v}_i\}$

$T(\vec{v}) \in W \Rightarrow T(\vec{v})$ can be written as

$$T(\vec{v}) = b_1 \vec{w}_1 + b_2 \vec{w}_2 + \cdots + b_m \vec{w}_m$$

$\begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$ is coordinates of $T(\vec{v})$ under basis $\{\vec{w}_i\}$

$$\begin{aligned} T(\vec{v}) &= T(a_1 \vec{v}_1 + \cdots + a_n \vec{v}_n) \\ &= a_1 T(\vec{v}_1) + \cdots + a_n T(\vec{v}_n) \\ &= [T(\vec{v}_1) \ \cdots \ T(\vec{v}_n)] \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} \end{aligned}$$

Let $T(\vec{v}_1) = A_{11} \vec{w}_1 + \cdots + A_{m1} \vec{w}_m = [\vec{w}_1 \cdots \vec{w}_m] \begin{bmatrix} A_{11} \\ \vdots \\ A_{m1} \end{bmatrix}$

$$T(\vec{v}_2) = A_{12}\vec{w}_1 + \cdots + A_{m2}\vec{w}_m = [\vec{w}_1 \cdots \vec{w}_m]\begin{bmatrix} A_{12} \\ \vdots \\ A_{m2} \end{bmatrix}$$

$$\vdots$$

$$T(\vec{v}_n) = A_{1n}\vec{w}_1 + \cdots + A_{mn}\vec{w}_m = [\vec{w}_1 \cdots \vec{w}_m]\begin{bmatrix} A_{1n} \\ \vdots \\ A_{mn} \end{bmatrix}$$

$$= [\vec{w}_1 \cdots \vec{w}_m]\underbrace{\begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ \vdots & \vdots & & \vdots \\ A_{m1} & A_{m2} & \cdots & A_{mn} \end{bmatrix}\begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}}_{\begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}}$$

$$\begin{tikzcd}
V \arrow[r, "T"] & W \\
\vec{v} \arrow[r, mapsto] \arrow[d, Leftrightarrow] & \vec{w} \arrow[d, Leftrightarrow] \\
\mathbb{R}^n & \mathbb{R}^m \\
\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \arrow[r, mapsto, "A \in \mathbb{R}^{m \times n}"] & \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix} = A\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}
\end{tikzcd}$$

Theorem T is a linear transformation

$T: V \longrightarrow W$ for

from an n-dimensional vector space V
to an m-dimensional vector space W.
Given a basis $\{\vec{v}_i\}$ for V
and a basis $\{\vec{w}_i\}$ for W,
there is a unique $A \in \mathbb{R}^{m \times n}$
representing T.