How to solve a linear system (Gaussian Elimination):

Step 0: Find the augmented matrix $[A \mid \vec{b}]$

Step I: Do row operations to get REF or RREF

Step II: Find solutions from REF or RREF

Case 1: $\left[\begin{array}{ccc|c} 1 & 0 & 0 & a \\ 0 & 1 & 0 & b \\ 0 & 0 & 1 & c \end{array}\right] \Rightarrow \begin{cases} x = a \\ y = b \\ z = c \end{cases}$

Case 2: $\left[\begin{array}{ccc|c} 1 & 0 & 0 & a \\ 0 & 1 & 0 & b \\ 0 & 0 & 0 & c \end{array}\right]$ and $c \neq 0 \Rightarrow$ no solution

Case 3: $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right]$

Mark pivots

Mark columns without pivots

Corresponding unknown is free

$z = t$, $y = 3$, $x = 2$

$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ t \end{pmatrix}$ where t is any real number.

Ex: $\begin{cases} x + y + z = 0 \\ z - y = 1 \end{cases}$

Sol: $\begin{cases} x + y + z = 0 \\ 0 \cdot x - y + z = 1 \end{cases}$

$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & -1 & 1 & 1 \end{array}\right]$

① Find REF

② Mark pivots

③ Mark columns (in coefficient matrix) without pivots

The corresponding variable is free:

Set $z = t$, plug in and solve it

$$\begin{cases} y = -1 + t \\ x = -y - z = 1 - 2t \end{cases}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1-2t \\ -1+t \\ t \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + \begin{pmatrix} -2t \\ t \\ t \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + t \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}, \quad \underbrace{\forall}_{\text{for any}}\, t \underbrace{\in}_{\text{in}} \underbrace{\mathbb{R}}_{\text{set of all real numbers}}$$

Remark: REF is NOT unique

RREF is unique

A few concepts for matrices.

Let $\begin{cases} c \in \mathbb{R} \text{ be a scalar} \\ A \in \mathbb{R}^{m\times n} \text{ be a matrix} \end{cases}$, then define

① Scalar multiplication to a matrix :

cA is define as multiplying c to each entry.

Example: $2\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 2 & 4 \\ 6 & 8 \end{pmatrix}$

② Matrix Addition : $A, B \in \mathbb{R}^{m\times n}$

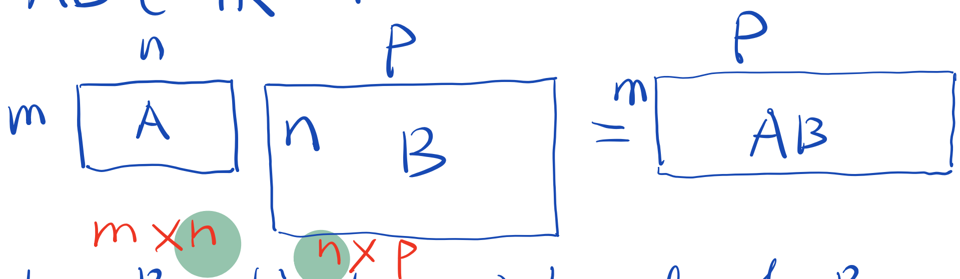
$A+B$ is defined as addition for each entry.

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} + \begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix} = \begin{pmatrix} 6 & 8 \\ 10 & 12 \end{pmatrix}$$

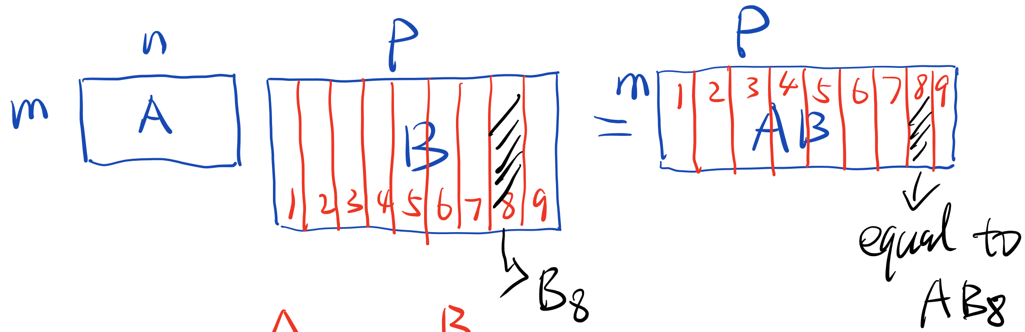
③ Matrix-Matrix Multiplication : $\begin{cases} A \in \mathbb{R}^{m\times n} \\ B \in \mathbb{R}^{n\times p} \end{cases}$

AB is called product of A and B.

1) $AB \in \mathbb{R}^{m\times p}$



2) Let B_j $(j=1,\dots,p)$ be cols of B,
then j-th col of AB is equal to AB_j

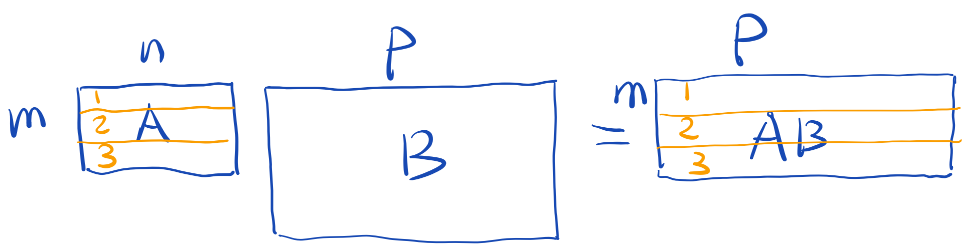


Example: $\underset{A}{\begin{pmatrix}1&2\\-1&0\end{pmatrix}}\underset{B}{\begin{pmatrix}\boxed{\begin{matrix}0\\1\end{matrix}}&\boxed{\begin{matrix}1\\3\end{matrix}}\end{pmatrix}} = \begin{pmatrix}\boxed{\begin{matrix}2\\0\end{matrix}}&\boxed{\begin{matrix}7\\-1\end{matrix}}\end{pmatrix}$

$$\left.\begin{aligned}\begin{pmatrix}\boxed{1\quad 2}\\\boxed{-1\quad 0}\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix}&=\begin{pmatrix}2\\0\end{pmatrix}\\\begin{pmatrix}\boxed{1\quad 2}\\\boxed{-1\quad 0}\end{pmatrix}\begin{pmatrix}1\\3\end{pmatrix}&=\begin{pmatrix}7\\-1\end{pmatrix}\end{aligned}\right\}\Rightarrow$$

$$\begin{pmatrix}1&2\\-1&0\end{pmatrix}\begin{pmatrix}0&1\\1&3\end{pmatrix}=\begin{pmatrix}\boxed{\begin{matrix}2\\0\end{matrix}}&\boxed{\begin{matrix}7\\-1\end{matrix}}\end{pmatrix}$$

2) Let A_i $(i=1,\dots,m)$ be rows of A, then i-th row of AB is equal to A_iB

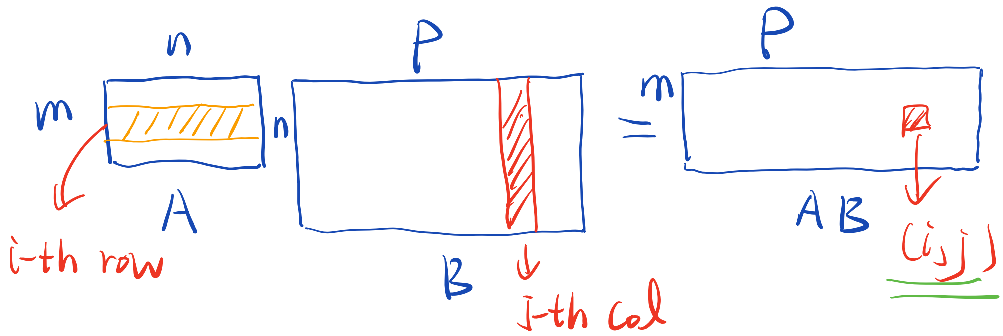


Ex: $\begin{pmatrix}\boxed{1\quad 2}\\\boxed{-1\quad 0}\end{pmatrix}\begin{pmatrix}0&1\\1&3\end{pmatrix} = \begin{pmatrix}\boxed{2\quad 7}\\\boxed{0\quad -1}\end{pmatrix}$

$$\underbrace{(1\quad 2)\begin{pmatrix}\boxed{\begin{matrix}0\\1\end{matrix}}&\boxed{\begin{matrix}1\\3\end{matrix}}\end{pmatrix}}=(2\quad 7)$$

$$\begin{pmatrix} -1 & 0 \end{pmatrix}\begin{pmatrix} \boxed{\begin{matrix}0\\1\end{matrix}} & \boxed{\begin{matrix}1\\3\end{matrix}} \end{pmatrix} = \begin{pmatrix} 0 & -1 \end{pmatrix}$$

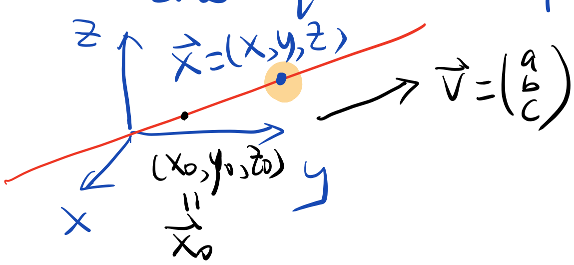
3) The (i,j)-entry of AB is equal to the dot product of A_i and B_j



Ex: $\begin{pmatrix} 1 & 2 \\ \boxed{-1 \quad 0} \end{pmatrix}\begin{pmatrix} 0 & \boxed{1} \\ 1 & \boxed{3} \end{pmatrix} = \begin{pmatrix} * & * \\ * & \boxed{*} \end{pmatrix}$

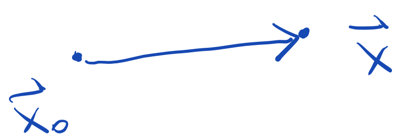
In general, $AB \neq BA$

The Line Equation in point-direction form



A line passes the point $\vec{x}_0=(x_0,y_0,z_0)$ and is parallel to $\vec{v}=\begin{pmatrix}a\\b\\c\end{pmatrix}$

Let $\vec{x}=(x,y,z)$ be any point on this line.



The vector from $\vec{x}_0$ to $\vec{x}$ is obtained by

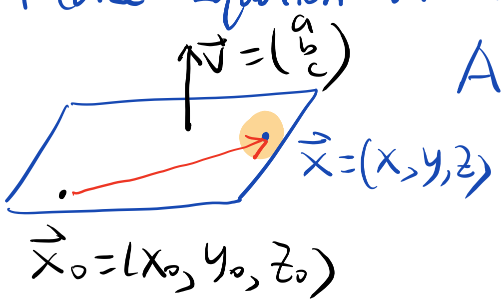
$$\vec{x}-\vec{x}_0 = \begin{pmatrix} x-x_0 \\ y-y_0 \\ z-z_0 \end{pmatrix},$$ which is parallel to $\vec{v}$.

$\Leftrightarrow \quad \vec{x} - \vec{x}_0 = t\vec{v}$ for some $t \in \mathbb{R}$.

$$\Leftrightarrow \quad \begin{pmatrix} x - x_0 \\ y - y_0 \\ z - z_0 \end{pmatrix} = t \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

$$\Rightarrow \quad \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} + t \begin{pmatrix} a \\ b \\ c \end{pmatrix}, \quad t \in \mathbb{R}.$$

Plane Equation in 3D



A plane $\begin{cases} \text{passes a point } \vec{x}_0 \\ \perp \vec{v} \end{cases}$

$\Leftrightarrow$ The vector $(\vec{x} - \vec{x}_0) \perp \vec{v}$

$$\Leftrightarrow \quad \begin{pmatrix} x - x_0 \\ y - y_0 \\ z - z_0 \end{pmatrix} \cdot \begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0$$

$$\Leftrightarrow \quad a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

$$\Leftrightarrow \quad ax + by + cz = ax_0 + by_0 + cz_0$$

We usually write it as $ax + by + cz = d$.

Linear System

$$\begin{cases} 2x + 4y - 2z = 2 \\ 4x + 9y - 3z = 8 \\ -2x - 3y + 7z = 10 \end{cases}$$

Matrix Form

$$\begin{pmatrix} 2 & 4 & -2 \\ 4 & 9 & -3 \\ -2 & -3 & 7 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 8 \\ 10 \end{pmatrix}$$

Augmented Matrix

$$\left(\begin{array}{ccc|c} 2 & 4 & -2 & 2 \\ 4 & 9 & -3 & 8 \\ -2 & -3 & 7 & 10 \end{array}\right)$$

Gaussian Elimination: use row ops to get either REF or RREF

Row Echelon Form (REF)
- ① zero rows are at bottom
- ② position of leading coefs

first nonzero entry in each row

1) $\left(\begin{array}{ccc|c} 1 & 0 & 0 & 4 \\ 0 & 2 & 0 & 5 \\ 0 & 0 & 3 & 6 \end{array}\right)$ ✓

2) $\left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right)$ ✓

4) $\left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{array}\right)$ ✗

3) $\left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right)$ ✗

5) $\left(\begin{array}{ccc|c} 2 & 0 & 0 & 1 \\ 2 & 2 & 2 & 2 \\ 0 & 0 & 2 & 3 \end{array}\right)$ ✗

Reduced Row Echelon Form RREF
- ① already a REF
- ② all pivots are ones
- ③ any col containing leading ones has only one nonzero entry.

1) $\left(\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right)$ ✓

3) $\left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right)$ ✗

2) $\left(\begin{array}{ccc|c} 1 & 0 & 0 & a \\ 0 & 1 & 0 & b \\ 0 & 0 & 1 & c \end{array}\right)$ ✓

4) $\left(\begin{array}{ccc|c} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \end{array}\right)$ ✗

Ex: Matrix Form

$$\begin{pmatrix} 2 & 4 & -2 \\ 4 & 9 & -3 \\ -2 & -3 & 7 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 8 \\ 10 \end{pmatrix}$$

Augmented Matrix

$$\left(\begin{array}{ccc|c} 2 & 4 & -2 & 2 \\ 4 & 9 & -3 & 8 \\ -2 & -3 & 7 & 10 \end{array}\right)$$

How to solve it by Gaussian Elimination:

$$\left(\begin{array}{ccc|c} 2 & 4 & -2 & 2 \\ 4 & 9 & -3 & 8 \\ -2 & -3 & 7 & 10 \end{array}\right)$$

$$\xRightarrow{\frac{1}{2}r1 \to r1} \left(\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 4 & 9 & -3 & 8 \\ -2 & -3 & 7 & 10 \end{array}\right)$$

$$\xRightarrow{-4r1+r2 \to r2} \left(\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 0 & 1 & 1 & 4 \\ -2 & -3 & 7 & 10 \end{array}\right)$$

$$\xrightarrow{2r1+r3 \to r3} \left(\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 0 & 1 & 1 & 4 \\ 0 & 1 & 5 & 12 \end{array}\right)$$

$$\xrightarrow{r3-r2 \to r3} \left(\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 4 & 8 \end{array}\right)$$

$$\xrightarrow{r1-2r2 \to r1} \left(\begin{array}{ccc|c} 1 & 0 & -3 & -7 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 4 & 8 \end{array}\right)$$

$$\xrightarrow{\frac{1}{4}r3 \to r3} \left(\begin{array}{ccc|c} 1 & 0 & -3 & -7 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 1 & 2 \end{array}\right)$$

either

① Solve it by substitution backwards:

$z=2 \Rightarrow y=2 \Rightarrow x=-1$

or RREF

② $$\xrightarrow[3r3+r1 \to r1]{r2-r3 \to r2} \left(\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 2 \end{array}\right)$$

Different Scenarios 4 RREF or REF

① $\left(\begin{array}{ccc|c} 1 & 0 & 0 & a \\ 0 & 1 & 0 & b \\ 0 & 0 & 1 & c \end{array}\right) \Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$

② $\left(\begin{array}{ccc|c} 1 & 0 & 0 & a \\ 0 & 1 & 0 & b \\ 0 & 0 & 0 & 1 \end{array}\right) \Rightarrow$ contradiction $\Rightarrow$ no sol at all.

$0 \cdot x + 0 \cdot y + 0 \cdot z = 1$

③ $\left(\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right)$ The cols without pivots correspond to free parameters.

$\Rightarrow$ z is free

Set $z = t$, then solve for the others.

Second row $\Rightarrow$ $y - z = 1 \Rightarrow y = 1 + z = 1 + t$

First row $\Rightarrow$ $x + 2y + 3z = 0$

$$\begin{aligned} \Rightarrow x &= -2y - 3z \\ &= -2(1+t) - 3t \\ &= -5t - 2 \end{aligned}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -5t-2 \\ 1+t \\ t \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} -5t \\ t \\ t \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} -5 \\ 1 \\ 1 \end{pmatrix},$$

$\forall t \in \mathbb{R}$.

④ $\left(\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$ $x + y + z = 2$

$[\,1 \;\; 1 \;\; 1 \mid 2\,]$

Set $\begin{cases} y = s \\ z = t \end{cases}$

$x = 2 - y - z = 2 - s - t.$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2-s-t \\ s \\ t \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} -s \\ s \\ 0 \end{pmatrix} + \begin{pmatrix} -t \\ 0 \\ t \end{pmatrix}$$

$$= \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} + s\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \quad \forall s,t \in \mathbb{R}.$$

Ex: The equation $x+y+z+t=0$ represents a 3D plane in x-y-z-t space. Why?

$$\left[\begin{array}{cccc|c} 1 & \boxed{1} & \boxed{1} & \boxed{1} & 0 \end{array}\right]$$

Ex:
$$\begin{cases} x+y+z+t = 2 \\ 2x-y+t = 3 \end{cases}$$

Matrix Form

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 2 & -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 2 \\ 2 & -1 & 0 & 1 & 3 \end{array}\right]$$

$$\xrightarrow{-2r_1+r_2\to r_2} \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 2 \\ 0 & -3 & -2 & -1 & -1 \end{array}\right]$$

Set $\begin{cases} z = u \\ t = v \end{cases}$

$$-3y-2z-t=-1 \Rightarrow y = -\frac{2}{3}z - \frac{1}{3}t + \frac{1}{3}$$
$$= -\frac{2}{3}u - \frac{1}{3}v + \frac{1}{3}$$

$$x+y+z+t=2$$

$$\Rightarrow x = 2-y-z-t$$
$$= 2+\frac{2}{3}u+\frac{1}{3}v-\frac{1}{3}-u-v = -\frac{1}{3}u-\frac{2}{3}v+\frac{5}{3}$$

$$\Rightarrow \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = \begin{pmatrix} -\frac{1}{3}u - \frac{2}{3}v + 5/3 \\ -\frac{2}{3}u - \frac{1}{3}v + \frac{1}{3} \\ u \\ v \end{pmatrix}$$

$$= \begin{pmatrix} 5/3 \\ 1/3 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} -1/3\, u \\ -2/3\, u \\ u \\ 0 \end{pmatrix} + \begin{pmatrix} -2/3\, v \\ -1/3\, v \\ 0 \\ v \end{pmatrix}$$

$$= \begin{pmatrix} 5/3 \\ 1/3 \\ 0 \\ 0 \end{pmatrix} + u \begin{pmatrix} -1/3 \\ -2/3 \\ 1 \\ 0 \end{pmatrix} + v \begin{pmatrix} -2/3 \\ -1/3 \\ 0 \\ 1 \end{pmatrix},$$

$$\forall u, v \in \mathbb{R}.$$