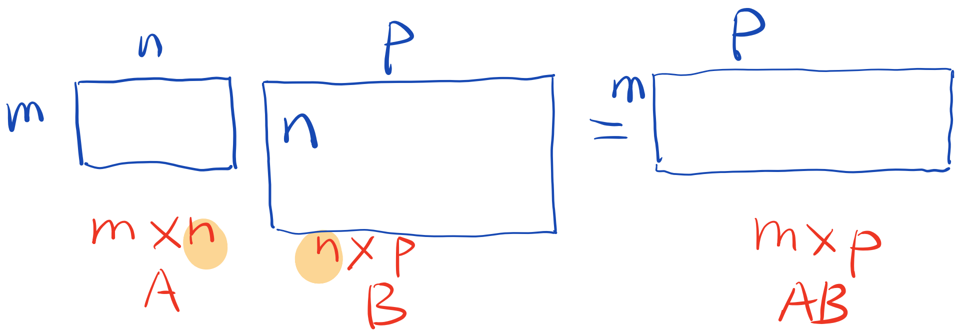
- Matrix – Matrix Multiplication: $\left.\begin{array}{l} A \in \mathbb{R}^{m\times n} \\ B \in \mathbb{R}^{n\times p} \end{array}\right\} \Rightarrow AB \in \mathbb{R}^{m\times p}$

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ -1 & 1 & 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \\ 0 & 1 & -1 \\ 2 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 7 & 5 & 2 \\ 2 & 1 & 1 \end{pmatrix}$$



- Rules of Matrix–Matrix Multiplication:

① For $A, B \in \mathbb{R}^{n\times n}$, usually $AB \neq BA$

$$\begin{pmatrix} 1 & 2 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 3 \end{pmatrix} = \begin{pmatrix} 2 & 7 \\ 0 & -1 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} & 0 \\ & \end{pmatrix}$$

② For three matrices A, B, C, if ABC is well-define (sizes match), then

$$(AB)C = A(BC)$$

Example: $\begin{pmatrix} 1 & 2 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$

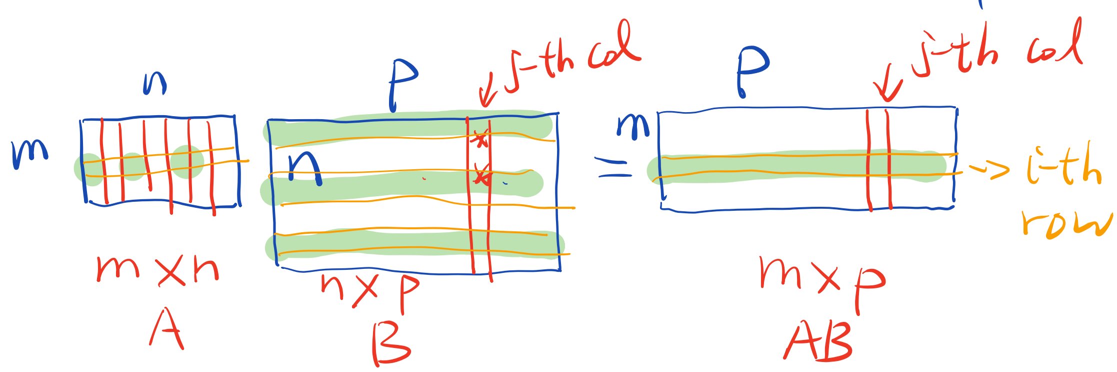
③ $A(B+C) = AB + AC$

$(A+B)C = AC + BC$

- Definition: Identity matrix is a square matrix with $\begin{cases} a_{ii} = 1, & \forall i \\ a_{ij} = 0, & \text{if } i \neq j \end{cases}$

$$\underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}_{I_{2\times2} \text{ or } I_2}, \quad \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{I_{3\times3},\ I_3}, \quad \underbrace{\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}}_{I_{4\times4},\ I_4}$$

- The j-th col in AB is a linear combination of all cols of A with j-th col of B as coef



$$\begin{pmatrix} 1 & 2 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 3 \end{pmatrix} = \begin{pmatrix} 2 & 7 \\ 0 & -1 \end{pmatrix}$$

- The i-th row in AB is a linear combination of all rows of B with i-th row of A as coef

- $\forall A \in \mathbb{R}^{n\times n}$, for the identity matrix I of the same size, we have

$$\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}\begin{pmatrix}a&b&c\\d&e&f\\g&h&i\end{pmatrix} = \begin{pmatrix}a&b&c\\d&e&f\end{pmatrix}$$

$$AI = A$$
$$IA = A$$

$$\begin{pmatrix}a&b&c\\d&e&f\\g&h&i\end{pmatrix}\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix} = \begin{pmatrix}a&b&c\\d&e&f\\g&h&i\end{pmatrix}$$

- Definition: $A \in \mathbb{R}^{n\times n}$ is invertible if there is a matrix $B \in \mathbb{R}^{n\times n}$ s.t.
$$AB = I \quad \text{and} \quad BA = I.$$

If A is invertible, B is called the inverse matrix of A, usually denoted as A^{-1}.
$$AA^{-1} = I \quad \text{and} \quad A^{-1}A = I$$

Example: $\begin{bmatrix}a&b\\c&d\end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix}d&-b\\-c&a\end{bmatrix}$ if $ad-bc \neq 0$

$$\begin{bmatrix}a&b\\c&d\end{bmatrix}\cdot\frac{1}{ad-bc}\begin{bmatrix}d&-b\\-c&a\end{bmatrix} = \begin{bmatrix}1&0\\0&1\end{bmatrix}$$

Remark: ① it can be proven that $AB=I \Rightarrow BA=I$
and $BA=I \Rightarrow AB=I$.
So only need to check $AB=I$
(or $BA=I$)
② A^{-1} is unique.

- Linear System

$$\begin{cases}2x+4y-2z=2\\4x+9y-3z=8\\-2x-3y+7z=10\end{cases}$$

Matrix Form $A\vec{x}=\vec{b}$

$$\begin{pmatrix} 2 & 4 & -2 \\ 4 & 9 & -3 \\ -2 & -3 & 7 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 8 \\ 10 \end{pmatrix}$$

Augmented Matrix $[A \mid \vec{b}]$

$$\left(\begin{array}{ccc|c} 2 & 4 & -2 & 2 \\ 4 & 9 & -3 & 8 \\ -2 & -3 & 7 & 10 \end{array}\right)$$

RREF is

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 2 \end{array}\right)$$

$\Rightarrow$ There is a unique sol $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix}$.

Facts for an $\boxed{n \times n}$ linear system $A\vec{x}=\vec{b}$

① Number of leading ones in RREF of $[A \mid \vec{b}]$ is n $\Leftrightarrow$ There is a unique sol.

② If A is invertible, then

$$A\vec{x} = \vec{b}$$

$$\Leftrightarrow \underline{A^{-1}A}\vec{x} = A^{-1}\vec{b}$$

$$\Leftrightarrow \underline{I}\vec{x} = A^{-1}\vec{b}$$

$\Leftrightarrow$ $\vec{x} = A^{-1}\vec{b}$ is a sol, and the only sol.

③ If number of leading ones in RREF of $[A|\vec{b}]$ is n, then A is invertible.

In other words, the following are equivalent:

① $A\vec{x}=\vec{b}$ has a unique sol

② REF or RREF of $[A|\vec{b}]$ has n pivots.

③ A is invertible

Example: If $A\vec{x}=\vec{b}$ has many sols (or no sol), then A is not invertible.

- Gaussian Elimination for computing A^{-1}:

$$A = \begin{bmatrix} 2 & 4 & -2 \\ 4 & 9 & -3 \\ -2 & -3 & 7 \end{bmatrix}$$

Step 0: set up $[A | I]$

$$\left[\begin{array}{ccc|ccc} 2 & 4 & -2 & 1 & 0 & 0 \\ 4 & 9 & -3 & 0 & 1 & 0 \\ -2 & -3 & 7 & 0 & 0 & 1 \end{array}\right]$$

Step I: use row ops to transform it to RREF

Step II: ① if we get

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & * & * & * \\ 0 & 1 & 0 & * & * & * \\ 0 & 0 & 1 & * & * & * \end{array}\right]$$

inverse of A

② if we can't get n pivots, then A is not invertible.

Example:

$$\left[\begin{array}{ccc|ccc} 2 & 4 & -2 & 1 & 0 & 0 \\ 4 & 9 & -3 & 0 & 1 & 0 \\ -2 & -3 & 7 & 0 & 0 & 1 \end{array}\right]$$

$(\frac{1}{2}r1 \to r1)$
$$\left[\begin{array}{ccc|ccc} 1 & 2 & -1 & 1/2 & 0 & 0 \\ 4 & 9 & -3 & 0 & 1 & 0 \\ -2 & -3 & 7 & 0 & 0 & 1 \end{array}\right]$$

$(-4r1 + r2 \to r2)$
$$\left[\begin{array}{ccc|ccc} 1 & 2 & -1 & 1/2 & 0 & 0 \\ 0 & 1 & 1 & -2 & 1 & 0 \\ -2 & -3 & 7 & 0 & 0 & 1 \end{array}\right]$$

$(2r1 + r3 \to r3)$
$$\left[\begin{array}{ccc|ccc} 1 & 2 & -1 & 1/2 & 0 & 0 \\ 0 & 1 & 1 & -2 & 1 & 0 \\ 0 & 1 & 5 & 1 & 0 & 1 \end{array}\right]$$

$$(-2r_2+r_1\to r_1)\quad \left[\begin{array}{ccc|ccc} 1 & 0 & -3 & 9/2 & -2 & 0 \\ 0 & 1 & 1 & -2 & 1 & 0 \\ 0 & 1 & 5 & 1 & 0 & 1 \end{array}\right]$$

$$(-r_2+r_3\to r_3)\quad \left[\begin{array}{ccc|ccc} 1 & 0 & -3 & 9/2 & -2 & 0 \\ 0 & 1 & 1 & -2 & 1 & 0 \\ 0 & 0 & 4 & 3 & -1 & 1 \end{array}\right]$$

$$(\tfrac{1}{4}r_3\to r_3)\quad \left[\begin{array}{ccc|ccc} 1 & 0 & -3 & 9/2 & -2 & 0 \\ 0 & 1 & 1 & -2 & 1 & 0 \\ 0 & 0 & 1 & 3/4 & -1/4 & 1/4 \end{array}\right]$$

$$(r_2-r_3\to r_3)\quad \left[\begin{array}{ccc|ccc} 1 & 0 & -3 & 9/2 & -2 & 0 \\ 0 & 1 & 0 & -11/4 & 5/4 & -1/4 \\ 0 & 0 & 1 & 3/4 & -1/4 & 1/4 \end{array}\right]$$

$$(3r_3+r_1\to r_1)\quad \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 27/4 & -11/4 & 3/4 \\ 0 & 1 & 0 & -11/4 & 5/4 & -1/4 \\ 0 & 0 & 1 & 3/4 & -1/4 & 1/4 \end{array}\right]$$

$$\Rightarrow A^{-1}=\frac{1}{4}\begin{bmatrix} 27 & -11 & 3 \\ -11 & 5 & -1 \\ 3 & -1 & 1 \end{bmatrix}$$

$$\begin{pmatrix} 2 & 4 & -2 \\ 4 & 9 & -3 \\ -2 & -3 & 7 \end{pmatrix} \frac{1}{4} \begin{pmatrix} 27 & -11 & 3 \\ -11 & 5 & -1 \\ 3 & -1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$