Review

- Matrix-Matrix Multiplication: for $A \in \mathbb{R}^{m \times n}$, $B \in \mathbb{R}^{n \times p}$, $AB \in \mathbb{R}^{m \times p}$ can be computed or interpreted/viewed as

① $(AB)_{ij}$ is the dot product of i-th row of A and j-th col of B.

$$\underset{2\times4}{\begin{pmatrix} 1 & 2 & 3 & 4 \\ -1 & 1 & 0 & 2 \end{pmatrix}} \underset{4\times3}{\begin{pmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \\ 0 & 1 & -1 \\ 2 & 0 & 1 \end{pmatrix}} = \underset{2\times3}{\begin{pmatrix} 7 & 5 & 2 \\ 2 & 1 & 1 \end{pmatrix}}$$

② Each col in AB is obtained by multiplying A to each col of B

③ Each row in AB is obtained by multiplying each row of A to B

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ -1 & 1 & 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \\ 0 & 1 & -1 \\ 2 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 7 & 5 & 2 \\ 2 & 1 & 1 \end{pmatrix}$$

④ Each col in AB is a linear combination of cols in A.

⑤ Each row in AB is a linear combination of rows in B.

Ex: $\begin{pmatrix} 2 \\ 1 \end{pmatrix} = 1 \cdot \begin{pmatrix} 1 \\ -1 \end{pmatrix} + 0 \cdot \begin{pmatrix} 2 \\ 1 \end{pmatrix} + (-1) \cdot \begin{pmatrix} 3 \\ 0 \end{pmatrix} + 1 \cdot \begin{pmatrix} 4 \\ 2 \end{pmatrix}$

$(2 \ 1 \ 1) = (-1) \cdot (1 \ 0 \ 1) + 1 \cdot (-1 \ 1 \ 0) + 0 \cdot (0 \ 1 \ -1) + 2 \cdot (2 \ 0 \ 1)$

- Identity Matrix : $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

  $AI = A$, $IA = A$.

- Inverse Matrix : $A \in \mathbb{R}^{n \times n}$, $AA^{-1} = I$

  $(A^{-1})^{-1} = A$ $\qquad A^{-1}A = I$

  Ex: $\underset{A}{\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}} \underset{A^{-1}}{\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

- Gaussian Elimination for finding A^{-1} :

  Use row ops on $[A \mid I]$ to find its RREF

  1) If RREF is $\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & a & b & c & d \\ 0 & 1 & 0 & 0 & e & f & g & h \\ 0 & 0 & 1 & 0 & i & j & k & l \\ 0 & 0 & 0 & 1 & m & n & o & p \end{array}\right]$,

  then $A^{-1} = \begin{pmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \\ m & n & o & p \end{pmatrix}$

  2) If not enough pivots, then not invertible

non singular means invertible $\qquad$ also called singular.

- A square linear system $A\vec{x} = \vec{b}$, means that

  A is square.

  If A^{-1} exists $\Rightarrow$ $\vec{x} = A^{-1}\vec{b}$ is the only sol.

  If many/no sols $\Rightarrow$ A is singular.

  not inveritble

Ex: $A\vec{x}=\vec{b}$

$$\begin{pmatrix} 2 & 4 & -2 \\ 4 & 9 & -3 \\ -2 & -3 & 7 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}=\begin{pmatrix} 2 \\ 8 \\ 10 \end{pmatrix}$$

To find A^{-1}:

Step 0: set up $[A \mid I]$

$$\left[\begin{array}{ccc|ccc} 2 & 4 & -2 & 1 & 0 & 0 \\ 4 & 9 & -3 & 0 & 1 & 0 \\ -2 & -3 & 7 & 0 & 0 & 1 \end{array}\right]$$

Step I: use row ops to transform it to RREF

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 27/4 & -11/4 & 3/4 \\ 0 & 1 & 0 & -11/4 & 5/4 & -1/4 \\ 0 & 0 & 1 & 3/4 & -1/4 & 1/4 \end{array}\right]$$

$$\Rightarrow A^{-1}=\frac{1}{4}\begin{bmatrix} 27 & -11 & 3 \\ -11 & 5 & -1 \\ 3 & -1 & 1 \end{bmatrix}$$

$$\Rightarrow \vec{x}=A^{-1}\vec{b}=\frac{1}{4}\begin{bmatrix} 27 & -11 & 3 \\ -11 & 5 & -1 \\ 3 & -1 & 1 \end{bmatrix}\begin{pmatrix} 2 \\ 8 \\ 10 \end{pmatrix}$$

$$=\begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix}$$

- Elementary Matrices: generated by one row/col operation on the identity matrix

E is always invertible. E^{-1} is also an elementary matrix, generated by the inverse operation.

Type I $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ $E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$, $E^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$

Type II $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ $E = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$, $E^{-1} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{pmatrix}$

Type III $E = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $E^{-1} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$

- $\boxed{EA \text{ is equivalent to row op on } A}$

Ex: $\boxed{\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a+c & b+d \\ c & d \end{pmatrix}}$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} a & d & g \\ b & e & h \\ c & f & i \end{pmatrix} = \begin{pmatrix} a & d & g \\ c & f & i \\ b & e & h \end{pmatrix}$$

If we use elementary matrices to represent the Gaussian Elimination, then every time we perform a row op, it's the same as multiplying an E matrix from the left.

Assume we need m row ops to get RREF of $[A|I]$

1st row op: $E_1[A \mid I] = [E_1 A \mid E_1 I] = [E_1 A \mid E_1]$

2nd row op: $E_2[E_1 A \mid E_1] = [E_2 E_1 A \mid E_2 E_1]$

3rd row op: $E_3[E_2 E_1 A \mid E_2 E_1] = [E_3 E_2 E_1 A \mid E_3 E_2 E_1]$

$\vdots$

m-th row op: $E_m[E_{m-1} E_{m-2} \cdots E_1 A \mid E_{m-1} E_{m-2} \cdots E_1]$

$$= [E_m E_{m-1} \cdots E_1 A \mid E_m E_{m-1} \cdots E_1]$$

$$= [I \mid E_m E_{m-1} \cdots E_1]$$

① This means $E_m E_{m-1} \cdots E_1 A = I$

$$\Rightarrow A^{-1} = E_m E_{m-1} \cdots E_1$$

② We know that $\begin{cases} E_1 E_1^{-1} = I \\ E_2 E_2^{-1} = I \\ \vdots \\ E_m E_m^{-1} = I \end{cases}$

$\begin{cases} (A^{-1})^{-1} = A \\ A^{-1} \cdot A = I \end{cases}$

So $\underbrace{(E_m E_{m-1} \cdots E_1)}_{A^{-1}} \underbrace{(E_1^{-1} E_2^{-1} \cdots E_{m-1}^{-1} E_m^{-1})}_{A}$

$$= E_m E_{m-1} \cdots E_2 \underbrace{E_1 E_1^{-1}}_{I} E_2^{-1} \cdots E_{m-1}^{-1} E_m^{-1}$$

$$= E_m E_{m-1} \cdots \underbrace{E_2 E_2^{-1}}_{I} \cdots E_{m-1}^{-1} E_m^{-1}$$

$$\vdots$$

$$= E_m E_{m-1} E_{m-1}^{-1} E_m$$

$$= E_m E_m^{-1} = I$$

$\Rightarrow$ The inverse of $\underbrace{E_m E_{m-1} \cdots E_1}_{= A^{-1}}$ is $E_1^{-1} E_2^{-1} \cdots E_{m-1}^{-1} E_m^{-1}$,

The inverse of A^{-1} is A

The inverse of A^{-1} is also A, and inverse matrix is unique $\Rightarrow A = E_1^{-1} E_2^{-1} \cdots E_{m-1}^{-1} E_m^{-1}$.

Fact: Any invertible matrix A is a product of some elementary matrices.

A is invertible if and only if A is a product of some elementary matrices

Example:

$$\left[\begin{array}{ccc|ccc} 2 & 4 & -2 & 1 & 0 & 0 \\ 4 & 9 & -3 & 0 & 1 & 0 \\ -2 & -3 & 7 & 0 & 0 & 1 \end{array}\right]$$

① $E_1 = \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ $(\frac{1}{2}r1 \to r1)$ $\left[\begin{array}{ccc|ccc} 1 & 2 & -1 & 1/2 & 0 & 0 \\ 4 & 9 & -3 & 0 & 1 & 0 \\ -2 & -3 & 7 & 0 & 0 & 1 \end{array}\right]$

② $E_2 = \begin{pmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ $(-4r1 + r2 \to r2)$ $\left[\begin{array}{ccc|ccc} 1 & 2 & -1 & 1/2 & 0 & 0 \\ 0 & 1 & 1 & -2 & 1 & 0 \\ -2 & -3 & 7 & 0 & 0 & 1 \end{array}\right]$

③ $E_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix}$ $(2r1 + r3 \to r3)$ $\left[\begin{array}{ccc|ccc} 1 & 2 & -1 & 1/2 & 0 & 0 \\ 0 & 1 & 1 & -2 & 1 & 0 \\ 0 & 1 & 5 & 1 & 0 & 1 \end{array}\right]$

$$E_4 = \begin{pmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

④ $(-2r_2 + r_1 \rightarrow r_1)$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -3 & 9/2 & -2 & 0 \\ 0 & 1 & 1 & -2 & 1 & 0 \\ 0 & 1 & 5 & 1 & 0 & 1 \end{array}\right]$$

$$E_5 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}$$

⑤ $(-r_2 + r_3 \rightarrow r_3)$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -3 & 9/2 & -2 & 0 \\ 0 & 1 & 1 & -2 & 1 & 0 \\ 0 & 0 & 4 & 3 & -1 & 1 \end{array}\right]$$

$$E_6 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/4 \end{pmatrix}$$

⑥ $(\frac{1}{4} r_3 \rightarrow r_3)$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -3 & 9/2 & -2 & 0 \\ 0 & 1 & 1 & -2 & 1 & 0 \\ 0 & 0 & 1 & 3/4 & -1/4 & 1/4 \end{array}\right]$$

$$E_7 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$$

⑦ $(r_2 - r_3 \rightarrow r_3)$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -3 & 9/2 & -2 & 0 \\ 0 & 1 & 0 & -11/4 & 5/4 & -1/4 \\ 0 & 0 & 1 & 3/4 & -1/4 & 1/4 \end{array}\right]$$

$$E_8 = \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

⑧ $(3r_3 + r_1 \rightarrow r_1)$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 27/4 & -11/4 & 3/4 \\ 0 & 1 & 0 & -11/4 & 5/4 & -1/4 \\ 0 & 0 & 1 & 3/4 & -1/4 & 1/4 \end{array}\right]$$

$$A = E_1^{-1} E_2^{-1} \cdots E_8^{-1}$$

$$[E \mid I]$$

$$A^{-1} = E_8 E_7 \cdots E_2 E_1$$