• Definition (Abstract Vectors)

A Vector Space over real numbers is a set S and

① addition is defined for any two elements in the set

② scalar multiplication is defined for a scalar in $\mathbb{R}$ and any element in the set. multiply a real number

③ the set is closed under these two operations,

meaning that

1) the sum of two element is still in the set S

2) scalar multiplication is still in the set S

④ elements in this set is called (abstract) vectors.

Example: the following are all (abstract) vector spaces

$\mathbb{R} = \{\text{all real numbers}\}$

elementary vectors

$\mathbb{R}^2 = \left\{ \begin{bmatrix} x \\ y \end{bmatrix}, \forall x, y \in \mathbb{R} \right\}$

$\mathbb{R}^3 = \left\{ \begin{bmatrix} a \\ b \\ c \end{bmatrix}, \forall a, b, c \in \mathbb{R} \right\}$

$\mathbb{R}^4 = \left\{ \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}, \forall a, b, c, d \in \mathbb{R} \right\}$

$\mathbb{R}^{1\times 3} = \left\{ \begin{bmatrix} x & y & z \end{bmatrix} : \forall x, y, z \in \mathbb{R} \right\}$

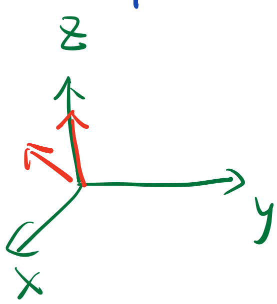
$\mathbb{R}^{2\times 2} = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \forall a, b, c, d \in \mathbb{R} \right\}$

$\mathbb{R}^{m\times n}$

$P_2(x) = \{ax^2+bx+c : \forall a, b, c \in \mathbb{R}\}$

S is a set of vectors, Span(S) denotes the set of all possible linear combination of vectors in S.

Example: ① $\text{Span}\left\{\begin{bmatrix}1\\2\\3\end{bmatrix}\right\} = \left\{a\begin{bmatrix}1\\2\\3\end{bmatrix}, \forall a\in\mathbb{R}\right\}$ is a line.



② $\text{Span}\left\{\begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}\right\} = \left\{a\begin{bmatrix}1\\0\\1\end{bmatrix} + b\begin{bmatrix}0\\0\\1\end{bmatrix}, \forall a,b\in\mathbb{R}\right\}$

$\begin{bmatrix}a\\0\\a+b\end{bmatrix}$

③ $\text{Span}\left\{\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}\right\} = \mathbb{R}^3$

Ex: $\text{Span}\left\{\begin{bmatrix}1\\2\\3\end{bmatrix}\right\}$ is a Vector Space

- There is always a zero vector $\vec{0}$ s.t. $\vec{v}+\vec{0}=\vec{v}$

  1) For $\mathbb{R}^n$, $\vec{0} = \begin{pmatrix}0\\0\\\vdots\\0\end{pmatrix}$

  2) For $\mathbb{R}^{m\times n}$, $\vec{0}$ is the zero matrix of size $m\times n$.

  3) For $P_2(\mathbb{R})$, $\vec{0}$ is the zero polynomial $p(x)=0$.

- Theorem: Let V be an abstract vector space.

  1) $0\cdot\vec{v} = \vec{0}$, $\forall \vec{v}\in V$

  2) Closedness $\Rightarrow \vec{0} = 0\cdot\vec{v} \in V$.

- Definition: If V is a vector space, $W \subseteq V$, and W is also a vector space, then W is called a subspace of V.

  is a subset of

- Example: ① $\text{Span}\left\{\begin{bmatrix}1\\2\\3\end{bmatrix}\right\} = \left\{a\begin{bmatrix}1\\2\\3\end{bmatrix},\ \forall a \in \mathbb{R}\right\}$ is a subspace of $\mathbb{R}^3$.

  ② $\text{Span}\left\{\begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}\right\}$ is a subspace of $\mathbb{R}^3$.

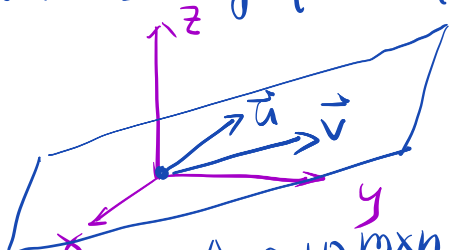
- Theorem: $\forall S \subseteq V$, $\text{Span}(S)$ is a subspace of V.

  $\text{Span}(S) \subseteq V$?

- $\boxed{\text{If } \vec{0} \notin W}$, then W is not a subspace.

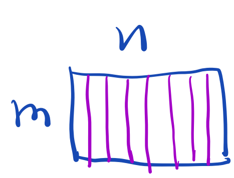
  Example: any plane that does not pass the origin cannot be a subspace of $\mathbb{R}^3$.

  Ex: Is any plane passing the origin a subspace?



$= \text{Span}\{\vec{u}, \vec{v}\}$

- Definition. $A \in \mathbb{R}^{m\times n}$

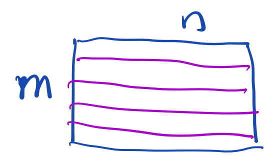


① Span{all cols of A} is called the column space of A, denoted as $\text{Col}(A) \subseteq \mathbb{R}^m$

Column Space of A

② Span{all rows of A} is called the row space of A, denoted as $Row(A) \subseteq \mathbb{R}^{1\times n}$

Row space of A



Example: $\begin{pmatrix} 2 & 4 & -2 \\ 4 & 9 & -3 \\ -2 & -3 & 7 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 8 \\ 10 \end{pmatrix}$

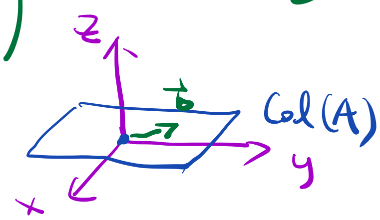
$A \quad \vec{x} \; = \; \vec{b}$

$$Col(A) = \left\{ a\begin{bmatrix} 2 \\ 4 \\ -2 \end{bmatrix} + b\begin{bmatrix} 4 \\ 9 \\ -3 \end{bmatrix} + c\begin{bmatrix} -2 \\ -3 \\ 7 \end{bmatrix},\ \forall a,b,c \in \mathbb{R} \right\} \subseteq \mathbb{R}^3$$

$$Row(A) = \left\{ a[2\ 4\ -2] + b[4\ 9\ -3] + c[-2\ -3\ 7],\ \forall a,b,c \in \mathbb{R} \right\} \subseteq \mathbb{R}^{1\times 3}$$

- $A\vec{x} = \vec{b}$ has at least one sol if and only if $\vec{b} \in Col(A) \subseteq \mathbb{R}^3$

$$\begin{pmatrix} 2 & 4 & -2 \\ 4 & 9 & -3 \\ -2 & -3 & 7 \end{pmatrix}\begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} = \vec{b} \Longleftrightarrow \vec{b} = x_0\begin{pmatrix} 2 \\ 4 \\ -2 \end{pmatrix} + y_0\begin{pmatrix} 4 \\ 9 \\ -3 \end{pmatrix} + z_0\begin{pmatrix} -2 \\ -3 \\ 7 \end{pmatrix}$$



Proof: "if" Assume $\vec{b} \in Col(A)$, then there are $a_0, b_0, c_0 \in \mathbb{R}$ s.t.

$$\vec{b} = a_0 \begin{bmatrix} 2 \\ 4 \\ -2 \end{bmatrix} + b_0 \begin{bmatrix} 4 \\ 9 \\ -3 \end{bmatrix} + c_0 \begin{bmatrix} -2 \\ -3 \\ 7 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2 & 4 & -2 \\ 4 & 9 & -3 \\ -2 & -3 & 7 \end{bmatrix} \begin{bmatrix} a_0 \\ b_0 \\ c_0 \end{bmatrix} = \vec{b}$$

$\Rightarrow \begin{bmatrix} a_0 \\ b_0 \\ c_0 \end{bmatrix}$ is a sol to $A\vec{x} = \vec{b}$

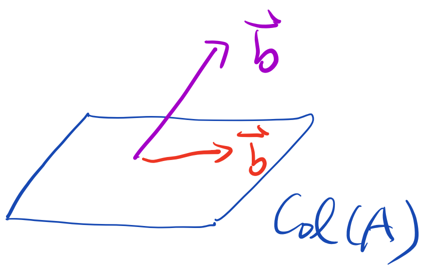
"only if" Assume $A\vec{x} = \vec{b}$ has one sol $\begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix}$

$$\Rightarrow \begin{pmatrix} 2 & 4 & -2 \\ 4 & 9 & -3 \\ -2 & -3 & 7 \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} = \begin{pmatrix} 2 \\ 8 \\ 10 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 2 \\ 8 \\ 10 \end{pmatrix} = x_0 \begin{pmatrix} 2 \\ 4 \\ -2 \end{pmatrix} + y_0 \begin{pmatrix} 4 \\ 9 \\ -3 \end{pmatrix} + z_0 \begin{pmatrix} -2 \\ -3 \\ 7 \end{pmatrix}$$

$\Rightarrow \vec{b}$ is spanned by cols of A.

- For $A \in \mathbb{R}^{3\times 3}$, if $Col(A) = \mathbb{R}^3$, $A\vec{x} = \vec{b}$ always has at least one sol for any $\vec{b}$.



- Definition: all solutions to $A\vec{x} = \vec{0}$ form a subspace in $\mathbb{R}^n$, called null space of A, denoted as $Null(A)$.

$A \in \mathbb{R}^{m\times n}$
$\vec{x} \in \mathbb{R}^n$

Check closedness : $\forall \vec{u}, \vec{v} \in Null(A)$, $a \in \mathbb{R}$

$$\left.\begin{array}{l}\vec{u} \in Null(A) \Rightarrow A\vec{u} = \vec{0} \\ \vec{v} \in Null(A) \Rightarrow A\vec{v} = \vec{0}\end{array}\right\} \Rightarrow A\vec{u} + A\vec{v} = \vec{0} + \vec{0}$$

$$\Rightarrow A(\vec{u} + \vec{v}) = \vec{0}$$

$$\Rightarrow \vec{u} + \vec{v} \in Null(A)$$

$$A\vec{u} = 0 \Rightarrow aA\vec{u} = a\vec{0} \Rightarrow A(a\vec{u}) = \vec{0}$$

$$\Rightarrow a\vec{u} \in Null(A).$$

- Example: Matrix Form $A\vec{x} = \vec{0}$

$$\begin{pmatrix} 2 & 4 & -2 \\ 4 & 9 & -3 \\ -2 & -3 & 7 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Augmented Matrix $[A \mid \vec{0}]$

$$\left(\begin{array}{ccc|c} 2 & 4 & -2 & 0 \\ 4 & 9 & -3 & 0 \\ -2 & -3 & 7 & 0 \end{array}\right)$$

RREF is

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right)$$

$\Rightarrow \vec{0}$ is the only sol

$$\Rightarrow Null(A) = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}$$

- Example: if RREF of $[A \mid \vec{0}]$ is

$$\left[\begin{array}{ccc|c} 1 & \boxed{2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow y = t, \quad \forall t \in \mathbb{R}$$

Solve it backwards

① $z=0$

② $x+2y=0 \Rightarrow x=-2t$

③ $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -2t \\ t \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} -2t \\ t \\ 0 \end{pmatrix}$

$$= \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + t\begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} = t\begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}, \ \forall t \in \mathbb{R}$$

$$\Rightarrow \text{Null}(A) = \text{Span}\left\{\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}\right\} = \left\{t\begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}, \forall t \in \mathbb{R}\right\}$$

- Definition (Linear Independence)

A set of (abstract) vectors $S=\{\vec{v}_1, \cdots, \vec{v}_n\}$ is called linearly dependent if there are scalars $a_1, \cdots, a_n$ which are not all zeros, s.t.

$$a_1\vec{v}_1 + a_2\vec{v}_2 + \cdots + a_n\vec{v}_n = \vec{0}.$$

Otherwise, S is linearly independent.

Remark: As long as one of a_i is not zero, it satisfies the definition

Example: ① $S=\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}$ is linearly independent

$$a\begin{bmatrix}1\\0\end{bmatrix}+b\begin{bmatrix}1\\1\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}$$

$$\Rightarrow \begin{cases} a+b=0 \\ 0\cdot a+b=0 \end{cases}$$

$$\Rightarrow \begin{bmatrix}1 & 1\\0 & 1\end{bmatrix}\begin{bmatrix}a\\b\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}$$

Augmented Matrix is $\left[\begin{array}{cc|c}1 & 1 & 0\\0 & 1 & 0\end{array}\right]$

RREF is $\left[\begin{array}{cc|c}1 & 0 & 0\\0 & 1 & 0\end{array}\right]$

$\Rightarrow \begin{bmatrix}a\\b\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix} \Rightarrow$ only zero sol

$\Rightarrow$ linearly independent.

If $a\vec{u}+b\vec{v}=\vec{0}$
at least one of a, b
is not 0,
assume $a\neq 0$,

$a\vec{u}=-b\vec{v}$

$\vec{u}=-\frac{b}{a}\vec{v}$

② If two vectors $\vec{u}, \vec{v}$ are parallel in $\mathbb{R}^3$,
then $\{\vec{u}, \vec{v}\}$ is dependent

$\vec{u} \parallel \vec{v} \Rightarrow \vec{u}=a\vec{v}$ for some $a\in\mathbb{R}$

$\Rightarrow \vec{u}-a\vec{v}=\vec{0}$

nation of columns of A.

2. (20 pts) For the invertible matrix

$$A = \begin{pmatrix} 0 & 1 & 1 \\ -1 & 3 & 1 \\ -1 & 2 & 1 \end{pmatrix},$$

find suitable elementary matrices so that A^{-1} can be written as a product of them.

$A^{-1} = E_8 E_7 \cdots E_1$

3. (20 pts) Let $A = \begin{pmatrix} 0 & 2 & 4 \\ -2 & 3 & 1 \\ -4 & 4 & 2 \end{pmatrix}$.

   (a) Determine whether columns of A are linearly independent as follows: assume there are numbers a, b, c s.t.

   $$a \begin{pmatrix} 0 \\ -2 \\ -4 \end{pmatrix} + b \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} + c \begin{pmatrix} 4 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

   $\begin{cases} a \cdot 0 + b \cdot 2 + c \cdot 4 = 0 \\ a \cdot (-2) + b \cdot 3 + c \cdot 1 = 0 \end{cases}$

   which gives three equations about a, b, c. Solve the linear system about a, b, c. If there are nonzero solutions, then the column vectors are linearly dependent. Otherwise, they are linearly independent.

   (b) Determine whether rows of A are linearly independent as follows: assume there are numbers a, b, c s.t.

   $$a\,(0 \;\; 2 \;\; 4) + b\,(-2 \;\; 3 \;\; 1) + c\,(-4 \;\; 4 \;\; 2) = (0 \;\; 0 \;\; 0),$$

   $\begin{cases} a \cdot 0 + b \cdot (-2) + c \cdot (-4) \\ a \cdot 2 + b \cdot 3 + c \cdot 4 = 0 \end{cases}$

   which gives three equations about a, b, c. Solve the linear system about a, b, c. If there are nonzero solutions, then the row vectors are linearly dependent. Otherwise, they are linearly independent.

4. (20 pts)

   Definition 1 (Transpose matrix)**.** For a matrix A of size $m \times n$, its transpose matrix A^T has size $n \times m$, and the j-th column of A^T is obtained by converting the j-th row of A to a column. For example,

   $$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}, \quad A^T = \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix}.$$

   For a square matrix A, A^T can also be viewed as flipping non-diagonal entries with respect to the diagonal entries. For example:

   $$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad A^T = \begin{pmatrix} a & c \\ b & d \end{pmatrix}.$$