MA 351 Section 041/011
Fall 2023

Homework 3

Due on **Sep 13 before 9am** on gradescope.

To receive full credit, show necessary reasoning unless it's straightforward computation.

1. (20 pts) True or false (you can simply answer T or F, no need to give justification; but try to think about why).

   (a) A square matrix A is invertible if and only if it can be written as a product of some elementary matrices.

   (b) A square matrix A is invertible if and only if its RREF is an identity matrix.

   (c) A square linear system $A\vec{x} = \vec{b}$ has a unique solution if A is invertible.

   (d) A linear system $A\vec{x} = \vec{b}$ has a solution if and only if $\vec{b}$ is a linear combination of columns of A.

   (e) For $A \in \mathbb{R}^{m\times n}$, its column space is a subspace of $\mathbb{R}^n$.

   (f) For $A \in \mathbb{R}^{n\times n}$, if $AA = I$, then A must be either I or $-I$.

   (g) For $A \in \mathbb{R}^{m\times n}$, its null space is a subspace of $\mathbb{R}^m$.

   (h) For $A \in \mathbb{R}^{m\times n}$, the homogeneous system $A\vec{x} = \vec{0}$ has as many solutions as the nonhomogeneous system $A\vec{x} = \vec{b}$ for any $\vec{b} \in \mathbb{R}^m$ and $\vec{b} \neq \vec{0}$.

   (i) For $A \in \mathbb{R}^{n\times n}$, if $\vec{b} \in Col(A)$, all solutions to $A\vec{x} = \vec{b}$ form a subspace of $\mathbb{R}^n$.

   (j) If $A\vec{x} = \vec{b}$ has no solutions at all, then $\vec{b}$ cannot be a linear combination of columns of A.

2. (20 pts) For the invertible matrix

$$A = \begin{pmatrix} 0 & 1 & 1 \\ -1 & 3 & 1 \\ -1 & 2 & 1 \end{pmatrix},$$

find suitable elementary matrices so that A^{-1} can be written as a product of them.

3. (20 pts) Let $A = \begin{pmatrix} 0 & 2 & 4 \\ -2 & 3 & 1 \\ -4 & 4 & 2 \end{pmatrix}$.

   (a) Determine whether columns of A are linearly independent as follows: assume there are numbers a, b, c s.t.

   $$a \begin{pmatrix} 0 \\ -2 \\ -4 \end{pmatrix} + b \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} + c \begin{pmatrix} 4 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

   which gives three equations about a, b, c. Solve the linear system about a, b, c. If there are nonzero solutions, then the column vectors are linearly dependent. Otherwise, they are linearly independent.

   (b) Determine whether rows of A are linearly independent as follows: assume there are numbers a, b, c s.t.

   $$a\,(0 \;\; 2 \;\; 4) + b\,(-2 \;\; 3 \;\; 1) + c\,(-4 \;\; 4 \;\; 2) = (0 \;\; 0 \;\; 0)\,,$$

   which gives three equations about a, b, c. Solve the linear system about a, b, c. If there are nonzero solutions, then the row vectors are linearly dependent. Otherwise, they are linearly independent.

4. (20 pts)

Definition 1 (Transpose matrix)**.** For a matrix A of size $m \times n$, its transpose matrix A^T has size $n \times m$, and the j-th column of A^T is obtained by converting the j-th row of A to a column. For example,

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}, \quad A^T = \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix}.$$

For a square matrix A, A^T can also be viewed as flipping non-diagonal entries with respect to the diagonal entries. For example:

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad A^T = \begin{pmatrix} a & c \\ b & d \end{pmatrix}.$$

(a) Find the transpose for $A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$ and $B = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \\ j & k & l \end{pmatrix}$.

(b) The transpose for product of matrices is similar to inverse [recall that we have a formula $(AB)^{-1} = B^{-1}A^{-1}$]:

$$(AB)^T = B^T A^T. \tag{1}$$

Verify the formua (1) for the following specific matrices:

$$A = \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}, \quad B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

In other words, compute $(AB)^T$ and $B^T A^T$ to show they are the same.

5. (20 pts) If a square matrix A is invertible, then its transpose A^T is also invertible and the inverse of A^T is equal to the transpose of A^{-1}:

$$(A^T)^{-1} = (A^{-1})^T.$$

We can verify this formula on a specific matrix

$$A = \begin{pmatrix} 0 & 1 & 1 \\ -1 & 3 & 1 \\ -1 & 2 & 1 \end{pmatrix}$$

as follows:

(a) Find A^T and A^{-1}.

(b) Compute the product of A^T and $(A^{-1})^T$ to that show it is identity.