MA 351 Section 041/011
Fall 2023

Homework 4

Due on **Sep 20 before 9 am** on gradescope.

To receive full credit, show necessary reasoning unless it's straightforward computation.

1. (20 pts) True or false (you can simply answer T or F, no need to give justification; but try to think about why).

   (a) For $A \in \mathbb{R}^{m\times n}$, its column rank (dimension of its column space) cannot be larger than m.

   (b) For $A \in \mathbb{R}^{m\times n}$, its nullity (dimension of its null space) cannot be larger than m.

   (c) For $A \in \mathbb{R}^{m\times n}$, its nullity is equal to n minus number of pivots in its RREF.

   (d) For $A \in \mathbb{R}^{m\times n}$, its row space can be spanned by rows in its RREF.

   (e) For $A \in \mathbb{R}^{m\times n}$, its column space can be spanned by columns with pivots in its RREF.

   (f) For $A \in \mathbb{R}^{m\times n}$, the column space of A^T has the same dimension as the row space of A.

   (g) For $A \in \mathbb{R}^{m\times n}$, the rank of A^T is equal to the rank of A.

   (h) For a square matrix A, the nullity of A^T is equal to the nullity of A.

   (i) For a square matrix A, if its rows are linearly independent, then so are its columns.

   (j) If V and W are subspaces of $\mathbb{R}^3$, and $dim(V) + dim(W) = 4$, then there must be some nonzero vector in both V and W.

2. (20 pts) Find the rank and the nullity of the matrix

$$A = \begin{pmatrix} 1 & 0 & 2 & -1 & 2 \\ -2 & 3 & 1 & -1 & 2 \\ -1 & 0 & -1 & 1 & 0 \\ -1 & 3 & 3 & -2 & 4 \end{pmatrix}$$

3. (20 pts) Consider the vector space $V = P_3(\mathbb{R})$ which consist of polynomials of degree at most 3. Then the following polynomials are abstract vectors in V:

$$p_1(x) = 1 + x, p_2(x) = 1 + x^3, p_3(x) = 1 + x + x^2.$$

Determine the linear independence of these vectors.

4. (20 pts) Find a basis for the column space of the matrix

$$A = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 1 & 2 & 3 & 0 \\ 0 & 3 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

by the following procedure: 1) solve $A\vec{x} = \vec{0}$, if there are free parameters in the solution, then columns are linearly dependent; 2) remove columns corresponding to free parameters, verify that the remaining columns are linearly independent; 3) verify that the remaining columns can span all other columns in A.

5. (20 pts) Find the dimension and a basis for the hyperplane described by

$$\begin{cases} 2y + 4z + s + 2t = 0 \\ -2x + 3y + z - s + 2t = 0 \end{cases}$$