MA 351 Section 041/011
Fall 2023

Homework 5

Due on **Oct 11 before 9am** on gradescope.

To receive full credit, show necessary reasoning unless it's straightforward computation.

1. (20 pts) True or false (you can simply answer T or F, no need to give justification; but try to think about why).

   (a) For $A \in \mathbb{R}^{m\times n}$ with $m > n$, $A^T A$ is invertible if and only if columns of A are linearly independent.

   (b) Orthonormal column vectors must be linearly independent.

   (c) Orthogonal nonzero column vectors must be linearly independent.

   (d) For $A \in \mathbb{R}^{m\times n}$, $\vec{x} \in \mathbb{R}^n$, $\vec{y} \in \mathbb{R}^m$, if $\vec{y} \in Null(A^T)$, then $\vec{y}^T A\vec{x} = 0$.

   (e) The projection matrix $P = A(A^T A)^{-1}A^T$ satisfies $P^2 = P$ and $P^T = P$.

   (f) If a square matrix A satisfies $A^T = A^{-1}$, then columns of A are orthonormal.

   (g) If a square matrix A has orthonormal columns, then all its rows are also orthonormal vectors.

   (h) If $A\vec{x} = \vec{b}$ is an overdetermined linear system with a least square solution $\hat{x}$, then $\vec{b} - A\hat{x}$ is in the left null space of A.

   (i) Assume that $A \in \mathbb{R}^{m\times n}$ with $m > n$ has linearly independent columns and $\vec{b}$ belongs to the column space of A, then the overdetermined linear system $A\vec{x} = \vec{b}$ has infinitely many solutions.

   (j) For any $A \in \mathbb{R}^{m\times n}$, the intersection of its left null space and its column space has dimension 0.

2. (20 pts) Use the least square approximation to find the best fits with (i) a linear function and (ii) a quadratic function, for the following data on x-y plane
$$\{(-3,9),(-2,6),(0,2),(1,1)\}.$$

3. (20 pts) Find the projection of $\begin{pmatrix}1\\0\\0\\0\end{pmatrix}$ onto the hyperplane in the x-y-z-t space described by $x+y+z=0, x+y+t=0$.

4. (20 pts) The Gram-Schmidt process has another version: first find orthogonal vectors then normalize them to unit vectors. The formula of this version is given as follows: given three vectors w_1, w_2, w_3, first compute
$$v_1 = w_1.$$
$$v_2 = w_2 - \frac{\langle w_2, v_1\rangle}{\|v_1\|^2} v_1$$
$$v_3 = w_3 - \frac{\langle w_3, v_1\rangle}{\|v_1\|^2} v_1 - \frac{\langle w_3, v_2\rangle}{\|v_2\|^2} v_2$$
The vectors v_1, v_2, v_3 are orthogonal. The orthonormal vectors are obtained by normalizing v_1, v_2, v_3:
$$u_1 = \frac{v_1}{\|v_1\|}$$
$$u_2 = \frac{v_2}{\|v_2\|}$$
$$u_3 = \frac{v_3}{\|v_3\|}.$$
Apply this formula to the given set S to obtain an orthonormal basis for $Span(S)$ where $S = \left\{ \begin{pmatrix}1\\-2\\-1\\3\end{pmatrix}, \begin{pmatrix}3\\6\\3\\-1\end{pmatrix}, \begin{pmatrix}1\\4\\2\\8\end{pmatrix} \right\}$.

5. (20 pts) Find an orthonormal basis for the column space of $A = \begin{pmatrix}1 & -2\\1 & 0\\1 & 1\\1 & 3\end{pmatrix}$.

Then compute the projection of $b = \begin{pmatrix} -4 \\ -3 \\ 3 \\ 0 \end{pmatrix}$ onto the column space.