Homework 8

Due on **Nov 1 before 9am** on gradescope.

To receive full credit, show necessary reasoning unless it's straight-forward computation.

1. (20 pts) True or false (you can simply answer T or F, no need to give justification; but try to think about why).

   (a) A square matrix of size $n \times n$ is diagonalizable if and only if it has n linearly independent eigenvectors.

   (b) The matrix $A = \begin{pmatrix} 1 & 0 & -2 & 1 \\ 0 & 2 & 1 & -2 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix}$ has an eigenvalue $\lambda = 4$, and this eigenvalue $\lambda = 4$ has only one eigenvector.

   (c) All eigenvectors of a square matrix form a vector space.

   (d) Any $A \in \mathbb{R}^{n\times n}$ always has n complex eigenvalues including repeated ones.

   (e) If $Span\left\{\begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}\right\}$ is an eigenspace for a matrix $A \in \mathbb{R}^{3\times 3}$ with eigenvalue $\lambda = 3$. Then
   $$A\begin{pmatrix}1\\1\\1\end{pmatrix} = \begin{pmatrix}3\\3\\3\end{pmatrix}.$$

   (f) For $A \in \mathbb{R}^{n\times n}$, if $AV = VD$ for some invertible $V \in \mathbb{R}^{n\times n}$ and a diagonal $D \in \mathbb{R}^{n\times n}$, then A has n eigenvalues including repeated ones.

   (g) For $A \in \mathbb{R}^{n\times n}$, if $AV = VD$ for some invertible $V \in \mathbb{R}^{n\times n}$ and a diagonal $D \in \mathbb{R}^{n\times n}$, then the geometrical multiplicity is equal to the algebraic multiplicity for any eigenvalue of A.

(h) Any noninvertible $A \in \mathbb{R}^{n\times n}$ has at least one real eigenvalue.

(i) If $A \in \mathbb{R}^{n\times n}$ is singular (not invertible), then $n - rank(A)$ is equal the dimension of one of its eigenspaces.

(j) The dimensions of all eigenspaces of a matrix $A \in \mathbb{R}^{n\times n}$ must sum to n.

2. (20 pts) Determine whether the matrix is diagonalizable or not. If yes, find the diagonalizable in the form $A = VDV^{-1}$. Use complex matrices if needed.

$$A = \begin{pmatrix} 0 & 2 & 2 \\ 2 & 0 & 2 \\ 2 & 2 & 0 \end{pmatrix}.$$

3. (20 pts) Determine whether the matrix is diagonalizable or not. If yes, find the diagonalizable in the form $A = VDV^{-1}$. Use complex matrices if needed.

$$A = \begin{pmatrix} 0 & 1 \\ -2 & 0 \end{pmatrix}.$$

4. (20 pts) Find e^A for $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Multiply all matrices to find e^A as a single matrix. Use Euler's formula $e^{i\theta} = \cos\theta + i\sin\theta$ to simplify your answer if you feel like to.

5. (20 pts) We can find solutions to an ordinary differential equation

$$y'''(t) + y''(t) - y'(t) - y(t) = 0$$

following these steps:

(a) Write it as a first order system $\frac{d}{dt}\vec{u}(t) = A\vec{u}(t)$ with

$$\vec{u}(t) = \begin{pmatrix} y(t) \\ y'(t) \\ y''(t) \end{pmatrix}.$$

Find what A is.

(b) Find basis vectors for each eigenspace of A.

(c) Let $\vec{v}$ be an eigenvector with eigenvalue λ, then $\vec{u}(t) = e^{\lambda t}\vec{v}$ is a solution. Use all the basis vectors you can find to write the solutions for $\vec{u}$, and plug it into both sides of $\frac{d}{dt}\vec{u}(t) = A\vec{u}(t)$ to verify that they are solutions.

(d) Obtain $y(t)$ from $\vec{u}(t)$, and plug $y(t)$ back into the orginal third order differential equation to verify that it is a solution.