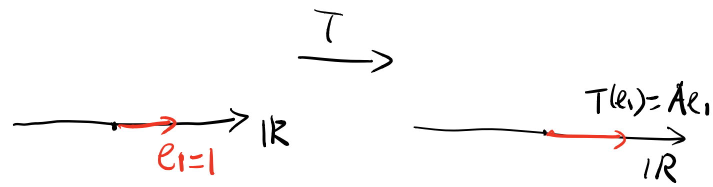
Det is a quantity in 1D/2D/3D of how much T changes
length / area / vol

① $A \in \mathbb{R}^{1\times 1}$, $\det(A) = A$

$L_A : \mathbb{R}^1 \to \mathbb{R}^1$

$x \mapsto Ax$



② 2D : Area

③ 3D : Volume

④ In 2D/3D, det is not linear in matrix

$$\det(A+B) \neq \det(A) + \det(B)$$

⑤ In 2D/3D, det is linear in any row/col.

Example: $\det\begin{pmatrix} a_1+a_2 & c \\ b_1+b_2 & d \end{pmatrix} = \det\begin{pmatrix} a_1 & c \\ b_1 & d \end{pmatrix} + \det\begin{pmatrix} a_2 & c \\ b_2 & d \end{pmatrix}$

$$\det\begin{pmatrix} ka & kb & kc \\ e & d & f \\ g & h & i \end{pmatrix} = k \det\begin{pmatrix} a & b & c \\ e & d & f \\ g & h & i \end{pmatrix}$$

⑥ In 1D/2D/3D, $\det(I) = 1$.

⑦ In n-dim, want to define a function $\delta : \mathbb{R}^{n\times n} \to \mathbb{R}$

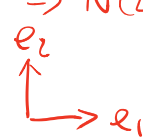
$A \mapsto \delta(A)$

satisfying
$$\begin{cases} ① \text{ it is linear in any row/col} \\ ② \ \delta(I) = 1 \\ ③ \ \delta(A) = 0 \text{ if } A \text{ has two same rows/cols.} \end{cases}$$

Ex: $A = \begin{pmatrix} a & a \\ b & b \end{pmatrix}$ 1) $\text{Rank}(A) < 2$

$\Rightarrow N(L_A)$ is nontrivial

2)



$L_A(e_1) = Ae_1 = \begin{pmatrix} a \\ b \end{pmatrix}$

$L_A(e_2) = Ae_2 = \begin{pmatrix} a \\ b \end{pmatrix}$

then $\delta(A) = \det(A)$, defined by cofactor expansion.

Theorems/Facts ① $\det(A)$ is linear in any row/col

② A has a zero row/col $\Rightarrow \det(A)=0$.

③ Two same rows $\Rightarrow \det(A)=0$.

④ Two same cols $\Rightarrow \det(A)=0$.

⑤ $\operatorname{rank}(A)<n \Rightarrow \det(A)=0$

Proof: $\operatorname{rank}(A)<n \Rightarrow$ its echelon form has a zero row

$\Rightarrow \det(\text{echelon form})=0$

We can obtain A by row/col ops on echelon form $\Rightarrow \det(A)=0$.

⑥ $\operatorname{rank}(A)=n \Leftrightarrow \det(A)\neq 0$

Theorem 4.7 $\det(AB)=\det(A)\det(B)$

Proof: ① If $\operatorname{rank}(A)<n$, then $\det(A)=0$.

Theorem 3.7 $\Rightarrow \operatorname{rank}(AB)\le \operatorname{rank}(A)<n \Rightarrow \det(AB)=0$

$$\dim(R(L_{AB}))\le \dim(R(L_A))$$
$$\Uparrow$$
$$R(L_A L_B)\subseteq R(L_A)$$

② If $\operatorname{rank}(A)=n \Rightarrow A$ is invertible $\Rightarrow A=E_1\cdots E_n$

$$\begin{aligned}\Rightarrow \det(AB) &= \det(E_1\cdots E_n B)\\ &= \det(E_1)\cdot\det(E_2\cdots E_n B)\\ &= \det(E_1)\det(E_2)\det(E_3\cdots E_n B)\\ &\;\vdots\\ &= \underbrace{\det(E_1)\cdots\det(E_n)}_{\det(A)}\det(B)\end{aligned}$$

$$\Uparrow$$

$$A=E_1\cdots E_n I \Rightarrow \det(A)=\det(E_1\cdots E_n I)$$
$$=\det(E_1)\cdots\det(E_n)\det(I).$$

Theorem 4.8 $\det(A^T) = \det(A)$

Proof: ① If $\operatorname{rank}(A) < n \Rightarrow \operatorname{rank}(A^T) = \operatorname{rank}(A) < n$

$$\Rightarrow \begin{cases} \det(A^T) = 0 \\ \det(A) = 0 \end{cases}$$

② If $\operatorname{rank}(A) = n \Rightarrow A$ is invertible

$$\Rightarrow A = E_1 \cdots E_n$$

$$\Rightarrow \det(A) = \det(E_1) \cdots \det(E_n)$$

$$A^T = (E_1 \cdots E_n)^T$$

$$= E_n^T \cdots E_1^T$$

$$\Rightarrow \det(A^T) = \det(E_n^T) \cdots \det(E_1^T)$$

$$\det(E_i^T) = \det(E_i)$$

Theorem 4.9 (Cramer's Rule)

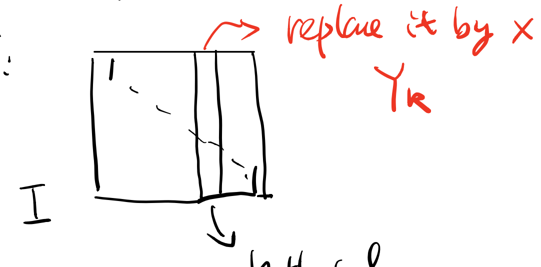
$Ax = b$ A is invertible

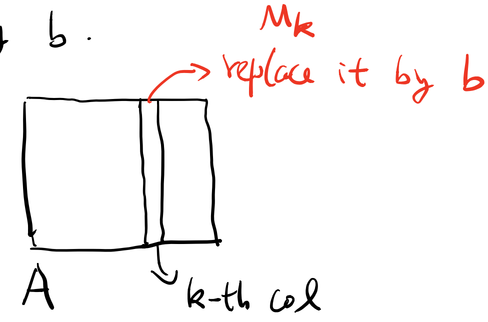
Let the unique sol be $x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$,

then $x_k = \dfrac{\det(M_k)}{\det(A)}$ where M_k is obtained by

replacing k-th col of A by b.

Proof:

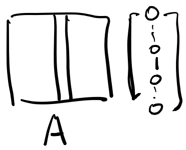




$$Y_k = \begin{bmatrix} 1 & & x_1 & & \\ & \ddots & \vdots & & \\ & & x_n & \ddots & \\ & & & & 1 \end{bmatrix} = [e_1 \ e_2 \cdots e_{k-1} \ x \ e_{k+1} \cdots e_n]$$

$$AY_k = [Ae_1 \; Ae_2 \; \cdots \; Ae_{k-1} \; \underset{\substack{\| \\ b}}{Ax} \; Ae_{k+1} \cdots Ae_n]$$

$$\left(Ae_i = i\text{-th col of } A . \right)$$



$$AY_k = \left[\begin{matrix} \text{1st} \\ \text{col} \\ \text{of } A \end{matrix} \quad \begin{matrix} \text{2nd} \\ \text{col} \\ \text{of } A \end{matrix} \quad \cdots \quad \begin{matrix} (k-1)\text{th col} \\ \text{of } A \end{matrix} \quad b \quad \begin{matrix} k\text{-th} \\ \text{col} \\ \text{of } A \end{matrix} \quad \cdots \right]$$

$$= M_k$$

$$\Rightarrow \det(M_k) = \det(AY_k) = \det(A)\det(Y_k)$$

$$\boxed{\det(Y_k) = x_k \det(I_{(n-1)\times(n-1)}) = x_k}$$

$$Y_k = \begin{pmatrix} 1 & & & x_1 & & & \\ & \ddots & & \vdots & & & \\ & & 1 & \vdots & & & \\ & & & x_k & & & \\ & & & \vdots & 1 & & \\ & & & \vdots & & \ddots & \\ & & & x_n & & & 1 \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 & 0 & x_1 & 0 \\ 0 & 1 & 0 & x_2 & 0 \\ & & 1 & x_3 & \\ 0 & 0 & 0 & x_4 & 0 \\ 0 & 0 & 0 & x_5 & 1 \end{pmatrix}$$

cofactor expansion along this col

cofactor matrix for all other entries has zero row

$$\Rightarrow \det(M_k) = \det(A)\, x_k$$

$$\Rightarrow x_k = \frac{\det(M_k)}{\det(A)} .$$

Example: (P225)

$$\begin{cases} x_1 + 2x_2 + 3x_3 = 2 \\ x_1 + x_3 = 3 \\ x_1 + x_2 - x_3 = 1 \end{cases} \qquad \underset{A}{\begin{bmatrix} 1 & 2 & 3 \\ 1 & 0 & 1 \\ 1 & 1 & -1 \end{bmatrix}} \underset{x}{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}} = \underset{b}{\begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}}$$

Type 3

$$\det(A) = \begin{vmatrix} 1 & 2 & 3 \\ 1 & 0 & 1 \\ 1 & 1 & -1 \end{vmatrix} \overset{\text{(row ops)}}{=} \begin{vmatrix} 1 & 2 & 3 \\ 0 & -2 & -2 \\ 0 & -1 & -4 \end{vmatrix}$$

$$= (-1)^{1+1} \cdot 1 \cdot \begin{vmatrix} -2 & -2 \\ -1 & -4 \end{vmatrix}$$

$$= 8 - 2 = 6 \neq 0$$

$$x_1 = \frac{\det(M_1)}{\det(A)} \qquad M_1 = \begin{pmatrix} 2 & 2 & 3 \\ 3 & 0 & 1 \\ 1 & 1 & -1 \end{pmatrix}$$

$$x_2 = \frac{\det(M_2)}{\det(A)} \qquad M_2 = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 1 \\ 1 & 1 & -1 \end{bmatrix}$$

$$x_3 = \frac{\det(M_3)}{\det(A)} \qquad M_3 = \begin{bmatrix} 1 & 2 & 2 \\ 1 & 0 & 3 \\ 1 & 1 & 1 \end{bmatrix}$$

$$\det(M_1) = 15 \Rightarrow x_1 = \frac{15}{6} = \frac{5}{2}.$$

Chapter 5 Eigenvalue / Eigenvector

eigen is a German word for "own"

here it just means sth invariant.

$T: V \longrightarrow V$

$v \longmapsto T(v)$

Want v s.t. $T(v) = av$, $a \in F$.

If there is such a nonzero vector v, we call it eigenvector and a is called eigenvalue

$T = L_A: \mathbb{R}^3 \longrightarrow \mathbb{R}^3$

$v \longmapsto Av$

For what kind of v, $L_A(v)$ is parallel to v.

In other words,

$Av = av$, $a \in \mathbb{R}$.

Eigenvector is the direction which is not changed by T

For $A \in \mathbb{R}^{n\times n}$, if $Av = \lambda v$ for some $\begin{cases} v \in \mathbb{R}^n \\ v \neq \vec{0} \end{cases}$, $\lambda \in \mathbb{R}$.

then $Av - \lambda v = \vec{0}$

$Av - \lambda I v = \vec{0}$

$(A - \lambda I) v = \vec{0}$

$\Rightarrow$ $(A-\lambda I)x = 0$ has a nonzero sol

$\Rightarrow$ $\begin{cases} A-\lambda I \text{ cannot be invertible} \\ \text{rank}(A-\lambda I) < n \\ \det(A-\lambda I) = 0 \Rightarrow \text{an equation of } \lambda \text{ to solve.} \end{cases}$

Example: $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

$$\det(A-\lambda I) = 0$$

$$\Rightarrow \det\left(\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

$$\Rightarrow \det\begin{pmatrix} \cos\theta-\lambda & -\sin\theta \\ \sin\theta & \cos\theta-\lambda \end{pmatrix} = 0$$

$$\Rightarrow (\cos\theta-\lambda)^2 + \sin^2\theta = 0$$

$$\Rightarrow \lambda^2 - 2\cos\theta + \cos^2\theta + \sin^2\theta = 0$$

$$\Rightarrow \lambda^2 - 2\cos\theta\lambda + 1 = 0$$

$$\lambda = \frac{2\cos\theta \pm \sqrt{4\cos^2\theta - 4}}{2}$$

$$= \cos\theta \pm \sqrt{\cos^2\theta - 1}$$

$$= \cos\theta \pm \sqrt{-\sin^2\theta} \qquad i = \sqrt{-1}$$

$$= \cos\theta \pm i\sin\theta$$

$\Rightarrow$ There is no real solution unless $\sin\theta = 0$

$\Updownarrow$

$\theta = 0, \pi$

If $\theta = \pi$, then $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

$= \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

and $\lambda_1 = \lambda_2 = -1$.

For finding eigenvector, solve $(A - \lambda I)V = 0$

Plug in $\lambda = -1$, we get

$A - \lambda I = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} - (-1)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\left[\begin{array}{cc|c} 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]$

$V_1 = s, \quad V_2 = t$

$\begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = \begin{pmatrix} s \\ t \end{pmatrix} = \begin{pmatrix} s \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ t \end{pmatrix}$

$= s\underset{\sim}{\begin{pmatrix} 1 \\ 0 \end{pmatrix}} + t\underset{\sim}{\begin{pmatrix} 0 \\ 1 \end{pmatrix}}$