Def $T: V \to V$ is linear

If $T(v) = \lambda v$ for $v \neq \vec{0}$ and $\lambda \in F$, then

v is called eigenvector of T

λ - — - eigenvalue of T.

Example: True or False

For any nonzero $v \in N(T)$, v is an eigenvector.

True: $T(v) = \vec{0} = 0 \cdot v$.

Def $A \in F^{n\times n}$, if $Av = \lambda v$ for nonzero vector $v \in F^n$ and some scalar $\lambda \in F$,

then v is called eigenvector of A

λ - - - - eigenvalue of A.

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① Consider $T: V \to V$ with finite dim V

Let $\beta = \{v_1, \dots, v_n\}$ be an ordered basis.

$$\begin{array}{ccc} V & \xrightarrow{T} & V \\ \phi_\beta \downarrow & & \downarrow \phi_\beta \\ F^n & \xrightarrow{[T]_\beta} & F^n \end{array} \qquad \begin{array}{c} v \longmapsto T(v) \\ \\ [v]_\beta \longmapsto [T]_\beta [v]_\beta \end{array}$$

v is an eigenvector of $T \iff [v]_\beta$ is an eigenvector of $[T]_\beta$

$T(v) = \lambda v \iff [T(v)]_\beta = [\lambda v]_\beta \iff [T]_\beta [v]_\beta = \lambda [v]_\beta$

② If V is infinite-dimensional, then no matrices.

Ex: $V = \{\text{smooth functions}\}$

$T: V \to V$

$f(x) \longmapsto \frac{d^2}{dx^2} f(x)$

$f''(x) = \lambda f(x)$

$$\frac{d}{dx}\begin{pmatrix} f(x) \\ f'(x) \end{pmatrix} = \begin{pmatrix} f' \\ f'' \end{pmatrix} = \begin{pmatrix} f' \\ \lambda f \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 1 \\ \lambda & 0 \end{pmatrix}\begin{pmatrix} f \\ f' \end{pmatrix}$$

$$\frac{d}{dx}\vec{v} = \begin{pmatrix} 0 & 1 \\ \lambda & 0 \end{pmatrix}\vec{v}$$

$$\begin{cases} \frac{d^2}{dx^2}\sin(kx) = -k^2\sin(kx) \\ \frac{d^2}{dx^2}\cos(kx) = -k^2\cos(kx) \\ \frac{d^2}{dx^2}e^{ax} = a^2 e^{ax} \\ \frac{d^2}{dx^2}(ax+b) = 0\cdot(ax+b) \end{cases}$$

Facts / Theorems $\qquad A \in F^{n\times n}$

① λ is an eigenvalue of $A \iff \det(A-\lambda I) = 0$

② $f(t) = \det(A - tI)$ is a polynomial of degree n in t with leading coef $(-1)^n$

It is called characteristic polynomial of A.

③ In principle, roots of $f(t) = \det(A-tI)$ are all eigenvalues of A.

Remark: ① If $A \in F^{n\times n}$, then want roots in F.

② A polynomial degree n with real/complex coefs always has n complex roots.

Ex: $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ $\quad f(t) = \det(A - tI)$

$$= \det\begin{pmatrix} -t & 1 \\ -1 & -t \end{pmatrix}$$

$$= t^2 + 1$$

If $A \in \mathbb{R}^{2\times 2}$, then no real roots thus no eigenvalues

If $A \in \mathbb{C}^{2\times 2}$, then roots are $\pm i$.

④ Abel–Ruffini Theorem, Galois Theory } No root formula for polynomial of degree 5 (and up).

⑤ How does a computer (Matlab) find all roots of $f(t)$?

Consider $p(t) = a_0 + a_1 t + a_2 t + \cdots + a_{n-1} t^{n-1} + t^n$

$$A = \begin{pmatrix} 0 & \cdots & \cdots & 0 & a_0 \\ -1 & 0 & \cdots & 0 & a_1 \\ 0 & \ddots & \ddots & \vdots & \vdots \\ \vdots & & \ddots & 0 & \\ 0 & \cdots & 0 & -1 & a_{n-1} \end{pmatrix}$$ is called companion matrix of $p(t)$

because $\det(-A - tI) = (-1)^n \det(A + tI) \overset{(HW\#5)}{=} (-1)^n [a_0 + a_1 t + \cdots + a_{n-1} t^{n-1} + t^n]$

Matlab find roots of $p(t)$ by finding approximations of eigenvalues of companion matrix.

Here is why: for example, the smallest eigenvalue of a real symmetric matrix B is also the minimal value for $\min\limits_{x \in \mathbb{R}^n} \frac{x^T B x}{x^T x}$, which can be efficiently approximated by algorithms.

Releigh Quotient.

Ex: $A = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 3 \end{pmatrix}$ real eigenvalue/vector

$$\det(A - \lambda I) = \det \begin{pmatrix} 1-\lambda & 2 & 0 \\ 0 & 1-\lambda & 0 \\ 1 & 1 & 3-\lambda \end{pmatrix}$$

$$= (-1)^{3+3} (3-\lambda) \begin{vmatrix} 1-\lambda & 2 \\ 0 & 1-\lambda \end{vmatrix}$$

$$= (3-\lambda)(1-\lambda)^2 = 0$$

$$\Rightarrow \lambda = 1, 3$$

① Plug in $\lambda = 1$ in $(A-\lambda I)v = 0$

$$\left(\begin{array}{ccc|c} 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{array}\right)$$

$$\longrightarrow \left(\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\Rightarrow \quad v_3 = s, \quad v_2 = 0, \quad v_1 = -2v_3 = -2s$$

$$\Rightarrow \quad v = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} -2s \\ 0 \\ s \end{pmatrix} = s\begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}, \quad \underline{s \in \mathbb{R}}$$

<u>Def</u> The sol set to $(A-\lambda I) = \vec{0}$ is called $\boxed{\text{eigen-space}}$ of λ consisting of $\left\{ \begin{array}{l} \text{all eigen vectors} \\ \vec{0} \end{array} \right.$

subspace

Eigenspace for $\lambda = 1$ is $\text{Span}\left\{ \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} \right\}$

② Plug in $\lambda = 3$