Review:

① $T: V \to V$ is diagonalizable $\Leftrightarrow \exists\, \beta$ s.t. $[T]_\beta$ is diagonal.

② $T: \overset{\gamma}{V} \longrightarrow \overset{\gamma}{V}$ $\quad [T]_\gamma$

$T: \overset{\beta}{V} \longrightarrow \overset{\beta}{V}$ $\quad [T]_\beta$

$I: \overset{\beta}{V} \longrightarrow \overset{\gamma}{V}$ $\quad Q = [I]_\beta^\gamma \qquad [v]_\gamma = [I]_\beta^\gamma [v]_\beta = Q[v]_\beta$

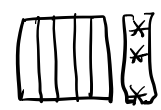
Theorem 2.23 $\Rightarrow [T]_\beta = Q^{-1}[T]_\gamma Q$

③ $A \in F^{n\times n}$ is diagonalizable

$\Leftrightarrow \exists$ an invertible matrix $Q = [v_1\, v_2 \cdots v_n]$

s.t. $Q^{-1}AQ$ is diagonal.

$\Leftrightarrow \quad Q^{-1}AQ = D = \begin{pmatrix} d_1 & & & 0 \\ & d_2 & & \\ & & \ddots & \\ 0 & & & d_n \end{pmatrix}$



$\Leftrightarrow \quad AQ = QD$

$\Leftrightarrow \quad A[v_1\, v_2 \cdots v_n] = [v_1\, v_2 \cdots v_n]\begin{bmatrix} d_1 & 0 & 0 & & \\ 0 & d_2 & 0 & & \\ 0 & 0 & d_3 & & \\ \vdots & \vdots & & \ddots & \\ 0 & 0 & & & d_n \end{bmatrix}$

$\Leftrightarrow \quad Av_i = d_i v_i$

$\Leftrightarrow$ n linearly independent eigenvectors v_i.

How to $\underbrace{\text{diagonalize}}_{\text{diagonalization}}$ a matrix?

Find all eigenvalues and eigenspaces.

Ex: $A = \begin{bmatrix} 4 & 0 & 1 \\ 2 & 3 & 2 \\ 1 & 0 & 4 \end{bmatrix} \in \mathbb{R}^{3\times 3}$

① $\det(A - \lambda I) = \det\begin{pmatrix} 4-\lambda & 0 & 1 \\ 2 & 3-\lambda & 2 \\ 1 & 0 & 4-\lambda \end{pmatrix}$

$$= (4-\lambda)(3-\lambda)(4-\lambda) - (3-\lambda)$$

$$= (3-\lambda)\left[(4-\lambda)^2-1\right]=0$$

$$=-(\lambda-3)^2(\lambda-5)$$

$$\Rightarrow \begin{cases} \lambda=3 \\ \text{or} \\ (4-\lambda)^2-1=0 \Leftrightarrow (\lambda-4)^2=1 \Leftrightarrow \lambda-4=\pm 1 \Rightarrow \lambda=3, \text{ or } 5 \end{cases}$$

$$\Rightarrow \lambda_1=\lambda_2=3,\ \lambda_3=5$$

② Plug $\lambda=3$ in $(A-\lambda I)v=\vec{0}$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 2 & 0 & 2 & 0 \\ 1 & 0 & 1 & 0 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$v_2=s,\ v_3=t$

$\Rightarrow v_1=-v_3=-t$

$$\Rightarrow v=\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}=\begin{pmatrix} -t \\ s \\ t \end{pmatrix}=\begin{pmatrix} 0 \\ s \\ 0 \end{pmatrix}+\begin{pmatrix} -t \\ 0 \\ t \end{pmatrix}=s\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}+t\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

$\forall s,t \in \mathbb{R}$.

Eigenspace for $\lambda=3$ is $\text{Span}\left\{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}\right\}$.

③ Plug $\lambda=5$ in $(A-\lambda I)v=\vec{0}$

$$\left[\begin{array}{ccc|c} -1 & 0 & 1 & 0 \\ 2 & -2 & 2 & 0 \\ 1 & 0 & -1 & 0 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$v_3=s \Rightarrow \begin{cases} v_2=2v_3=2s \\ v_1=v_3=s \end{cases}$$

$$\Rightarrow v=\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}=\begin{pmatrix} s \\ 2s \\ s \end{pmatrix}=s\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$$

Eigenspace for $\lambda=5$ is $\text{Span}\left\{\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}\right\}$

$$Q = \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & 2 \\ 0 & 1 & 1 \end{bmatrix}, \quad Q^{-1}AQ = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

$$Q = \begin{bmatrix} 0 & 1 & -1 \\ 1 & 2 & 0 \\ 0 & 1 & 1 \end{bmatrix}, \quad Q^{-1}AQ = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

Ex: $T: \mathbb{R}^3 \to \mathbb{R}^3$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \mapsto \begin{bmatrix} 4x+z \\ 2x+3y+2z \\ x+4z \end{bmatrix}$$

Find a basis β s.t. $[T]_\beta$ is diagonal.

Sol: ① Let $\gamma = \{e_1, e_2, e_3\}$

$\beta = \{v_1, v_2, v_3\}$

$$T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4x+z \\ 2x+3y+2z \\ x+4z \end{pmatrix} = \begin{pmatrix} 4 & 0 & 1 \\ 2 & 3 & 2 \\ 1 & 0 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$[T]_\gamma = \begin{pmatrix} 4 & 0 & 1 \\ 2 & 3 & 2 \\ 1 & 0 & 1 \end{pmatrix}$$

② Diagonalization of $[T]_\gamma$

$$\Rightarrow Q = \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & 2 \\ 0 & 1 & 1 \end{bmatrix}, \quad Q^{-1}[T]_\gamma Q = \begin{bmatrix} 3 & & \\ & 3 & \\ & & 5 \end{bmatrix}$$

③ Let β be the basis s.t. $Q = [I]_\beta^\gamma$,

then $[T]_\beta = Q^{-1}[T]_\gamma Q$ is diagonal.

$$[v]_\gamma = Q[v]_\beta, \quad \forall v \in \mathbb{R}^3.$$

$$\Rightarrow v_i = [v_i]_\gamma = Q\underbrace{[v_i]_\beta}_{e_i} = Qe_i = i\text{-th col of } Q$$

$$\Rightarrow v_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad v_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}.$$

Ex: $T: P_2(\mathbb{R}) \longrightarrow P_2(\mathbb{R})$

$f(x) \longmapsto f(1) + f'(0)x + (f'(0) + f''(0))x^2$

① Determine whether T is diagonalizable

② If yes, find β s.t. $[T]_\beta$ is diagonal.

Sol: ① $\gamma = \{1, x, x^2\}$

$T(1) = 1 + 0\cdot x + 0\cdot x^2$

$T(x) = 1 + 1\cdot x + (1+0)\cdot x^2 = 1 + x + x^2$

$T(x^2) = 1 + (2\cdot 0)\cdot x + (2\cdot 0 + 2)x^2 = 1 + 0\cdot x + 2\cdot x^2$

$$A = [T]_\gamma = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 2 \end{bmatrix}$$

$$\det(A - \lambda I) = \det\begin{pmatrix} 1-\lambda & 1 & 1 \\ 0 & 1-\lambda & 0 \\ 0 & 1 & 2-\lambda \end{pmatrix}$$

$$= (-1)^{1+1}(1-\lambda)\begin{vmatrix} 1-\lambda & 0 \\ 1 & 2-\lambda \end{vmatrix}$$

$$= (1-\lambda)^2(2-\lambda)$$

$\Rightarrow$ $\lambda_1 = \lambda_2 = 1$, $\lambda_3 = 2$

1) Plug $\lambda = 1$ in $(A - \lambda I)v = \vec{0}$

$$\left[\begin{array}{ccc|c} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$v_1 = s$, $v_3 = t$

$\Rightarrow v_2 = -v_3 = -t$

$$\Rightarrow v = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} s \\ -t \\ t \end{pmatrix} = \begin{pmatrix} s \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ -t \\ t \end{pmatrix} = s\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + t\begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$$

$\forall s, t \in \mathbb{R}$

$\Rightarrow$ Eigenspace is $\text{Span}\left\{\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\-1\\1\end{bmatrix}\right\}$

2) Plug in $\lambda = 2$, $(A-\lambda I)v = \vec{0}$

$$\left[\begin{array}{ccc|c} -1 & 1 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$v_3 = s \Rightarrow v_2 = 0, \; v_1 = v_3 = s$

$$\Rightarrow v = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} s \\ 0 \\ s \end{pmatrix} = s\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

Eigenspace is $\text{Span}\left\{\begin{bmatrix}1\\0\\1\end{bmatrix}\right\}$

$$Q = \begin{bmatrix} 1 & 0 & 1 \\ 0 & -1 & 0 \\ 0 & 1 & 1 \end{bmatrix}, \quad Q^{-1}AQ = \begin{bmatrix} 1 & & \\ & 1 & \\ & & 2 \end{bmatrix}$$

② Let $\beta = \{v_1, v_2, v_3\}$ be the basis s.t.

$$Q = [I]_\beta^\gamma .$$

$[v]_\gamma = Q[v]_\beta, \quad \forall v \in P_2(\mathbb{R})$

$\Rightarrow [v_i]_\gamma = Q[v_i]_\beta = Qe_i = i\text{th col of } Q$

$\Rightarrow [v_1]_\gamma = \begin{bmatrix}1\\0\\0\end{bmatrix}, \; [v_2]_\gamma = \begin{bmatrix}0\\-1\\1\end{bmatrix}, \; [v_3]_\gamma = \begin{bmatrix}1\\0\\1\end{bmatrix}$

$$\Rightarrow \begin{cases} v_1 = 1\cdot 1 + 0\cdot x + 0\cdot x^2 = 1 \\ v_2 = 0\cdot 1 + (-1)x + 1\cdot x^2 = x^2 - x \\ v_3 = 1\cdot 1 + 0\cdot x + 1\cdot x^2 = x^2 + 1 \end{cases}$$

Def: The largest positive integer k s.t. $(t-\lambda)^k$ is a factor of characteristic polynomial $f(t)$ is called algebraic multiplicity of eigenvalue λ.

Ex: $A \in \mathbb{R}^{6\times 6}$, $f(t) = \det(A - tI) = (t-1)^3 (t-2)^2 (t-3)^1$

$\lambda = 1$ as alg mul 3

$\lambda = 2$ ------- 2

$\lambda = 3$ ------- 1

Def Dimension of eigenspace for an eigenvalue λ E_λ is called geometrical multiplicity of λ

Theorem: For an operater T,

$$1 \le \dim(E_\lambda) \le \text{Alg Mul of } \lambda.$$

Proof: $E_\lambda = \{v \in V : T(v) = \lambda v\}$ is a subspace

Let $\{v_1, \dots, v_p\}$ be a basis of E_λ.

Extend it to $\underbrace{\{v_1, \dots, v_p, v_{p+1}, \dots, v_n\}}_{\beta}$ a basis of V.

$A = [T]_\beta$, then $T(v_i) = \lambda v_i$, $1 \le i \le p$

$$\Rightarrow A = [T]_\beta = \left[\begin{array}{c|c} \begin{matrix} \lambda & & 0 \\ & \ddots & \\ 0 & & \lambda \end{matrix} & B \\ \hline 0 & C \end{array}\right]$$

$$= \left[\begin{array}{c|c} \lambda I_p & B \\ \hline 0 & C \end{array}\right]_{n\times n}$$

B: $p \times (n-p)$, 0: $(n-p)\times p$, C: $(n-p)\times(n-p)$

$$\Rightarrow \det(A - tI_n) = \det \left[\begin{array}{c|c} \lambda I_p - tI_p & B \\ \hline 0 & C - tI_{(n-p)} \end{array}\right]$$

(Lemma) $= \det(\lambda I_p - tI_p)\det(C - tI_{(n-p)})$

$= \det[(\lambda - t)I_p]\det(\underbrace{C - tI_{(n-p)}}_{g(t)})$

$= (\lambda - t)^p g(t)$

$\Rightarrow \lambda$ is repeated at least p times.

Lemma: $\det\begin{pmatrix} A & B \\ 0 & C \end{pmatrix}_{n\times n} = \det(A)\det(C)$, A: $p\times p$, C: $(n-p)\times(n-p)$

Proof: 1) $\operatorname{rank}(C) < n-p$, $\begin{cases} LHS = 0 \\ RHS = 0 \end{cases}$

2) $\operatorname{rank}(C) = n-p$,

$$\begin{pmatrix} I & 0 \\ 0 & C^{-1} \end{pmatrix}\begin{pmatrix} A & B \\ 0 & C \end{pmatrix} = \begin{pmatrix} A & B \\ 0 & I \end{pmatrix}$$

$$\Rightarrow \det\begin{pmatrix} I & 0 \\ 0 & C^{-1} \end{pmatrix}\det\begin{pmatrix} A & B \\ 0 & C \end{pmatrix} = \det\begin{pmatrix} A & B \\ 0 & I \end{pmatrix}$$

HW #5 $\Rightarrow \begin{cases} \det\begin{pmatrix} A & B \\ 0 & I \end{pmatrix} = \det(A) \\ \det\begin{pmatrix} I & 0 \\ 0 & C^{-1} \end{pmatrix} = \det(C^{-1}) = \dfrac{1}{\det(C)} \end{cases}$

$$\Rightarrow \frac{1}{\det(C)}\det\begin{pmatrix} A & B \\ 0 & C \end{pmatrix} = \det(A)$$

Theorem: An operator T (or a matrix A) is diagonalizable if and only if

$$\left\{\begin{array}{l} 1)\ \text{All Alg Mul sum to } n. \\ 2)\ \text{Geo Mul} = \text{Alg Mul for all eigenvalues.} \end{array}\right.$$

Theorem / Fact: $\left.\begin{array}{l}\text{real symmetric matrices} \\ \underline{\text{complex Hermitian}}\text{ matrices}\end{array}\right\}$ are always diagonalizable with real eigenvalues.

$A^* = A$

$\hookrightarrow \underline{A^* = (\bar{A})^T}$

Ex: $\begin{pmatrix} 1 & 1-i \\ 1+i & 2 \end{pmatrix}^* = \begin{bmatrix} 1 & 1+i \\ 1-i & 2 \end{bmatrix}^T = \begin{pmatrix} 1 & 1-i \\ 1+i & 2 \end{pmatrix}$

is Hermitian

$\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}^* = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}$ is NOT Hermitian.

$$\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} = i\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = i\,QDQ^{-1} = i\,Q\begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix}Q^{-1}$$

$$= Q\begin{pmatrix} id_1 & 0 \\ 0 & id_2 \end{pmatrix}Q^{-1}$$ is still diagonalizable.